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Mixed problem for the three-dimensional elliptic equation with the two singular coefficients

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Abstract: In the present work, the mixed problem for a three-dimensional elliptic equation with two singular coefficients is investigated. Using an “abc” method, the uniqueness for the solution of the mixed problem is proved. Applying a method of Green’s functions, we are able to find the solution of the problem in an explicit form, in finding which the properties of the Appel and Gauss hypergeometric functions are essentially used.

Keywords: Mixed problem, three-dimensional singular elliptic equation, Green’s function, Gauss and Appell’s hypergeometric functions.

Аннотация: В настоящей работе исследуется смешанная задача для трехмерного эллиптического уравнения с двумя сингулярными коэффициентами. Методом «abc» доказана единственность решения смешанной задачи. Применяя метод функций Грина, удаётся найти решение задачи в явном виде, при нахождении которого существенно используются свойства гипергеометрических функций Аппеля и Гаусса.

Ключевые слова: смешанная задача, трехмерное сингулярное эллиптическое уравнение, функция Грина, гипергеометрические функции Гаусса и Аппелла.

Annotatsiya: Mazkur ishda ikkita singular koeffisientli uch o’lchovli elliptik tenglama uchun aralash masala tadqiq etilgan. Masala yechimining yagonaligi “abc” usul yordamida isbotlangan. Masalaning yechimi Appell funksiyasi yordamida ifodalanadigan Grin funksiyasi orqali yozilgan. Yechimni oshkor ko’rinishda topishda Appell va Gauss gipergeometrik funksiyalarining xossalaridan foydalanilgan.

Kalit so’zlar: Aralash masala, uch o’lchovli singular elliptik tenglama, Grin finksiyasi, Gauss va Appell gipergeometrik funksiyalari

Introduction

It is known that the theory of boundary value problems for degenerate equations and equations with singular coefficients is one of the rapidly developing parts of the modern theory of partial differential equations, which is encountered in solving many important questions of an applied nature, for example, [1]. Omitting a huge bibliography in which various local and non-local boundary-value problems for mixed-type equations containing elliptic equations with singular coefficients are studied, we note some papers which are close to the present work. In the work [2], fundamental solutions were constructed for the multidimensional elliptic equation with the several singular coefficients, and in [3,4,5] the explicit solutions of the Dirichlet, Holmgren and generalized Holmgren problems in the hyperoctant of the ball were found. In the works [6,7,8], explicit solutions in the infinite domains of similar problems for the same equation were obtained. In the above-mentioned works [3,4,5], the author managed to find solutions to the problems posed when the value of the sought solution on a part of the multidimensional sphere was known in advance

If on the part of the sphere enclosed between the lateral faces of the hyperoctant a normal derivative is specified, on one of the lateral faces a value, and on the other faces either the value itself or the normal derivative of the desired solution, then such a problem is called a mixed problem, the study of which is devoted to relatively few works. Smirnov [9] was one of the first to solve a mixed problem for the equation

$$u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x = 0,\ \ x > 0,\, 0 < 2\alpha < 1.$$

In the recent work [10] a mixed problem for the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x = 0, \quad 0 < 2\alpha < 1.$$

in a hemisphere $R_3^+ = \{(x,y,z): x > 0\}$ of three-dimensional Euclidean space.

In this paper we find explicit solution to the mixed problem for the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = 0, \quad 0 < 2\alpha,\, 2\beta < 1. \tag{1}$$

Note, that for the equation (1) spatial boundary problems with the Dirichlet and Dirichlet-Neumann conditions on the lateral faces of the quarter of the ball are found in [11,12].

2. Formulation of the problem DM and main results

Let $\Omega \subset R^{+} \times R^{+} \times R$ be a finite simple-connected domain bounded by planes $x=0$, $y=0$ and by a smooth surface S. The intersections of this surface with planes $x=0,\ y=0$ are denoted by $\gamma_1,\ \gamma_2$, respectively. Designate as the domain Ω_1, a plane $y-z$, bounded by $x=0\,(0<y<b,\ -c<z<d)$ and by a curve $\gamma_1: x=f_1(y,z)$. Ω_2 is domain in the plane $x-z$ bounded by $y=0\,(0<x<a,\ -c<z<d)$, $\gamma_2: y=f_2(x,z)$. Here a, b, c, d are positive constants.

Problem DM. To find a function $u(x,y,z) \in C(\overline{\Omega}) \cap C^2(\Omega)$, satisfying Eq. (1) conditions

$$u(0,y,z)=\tau_1(y,z).\quad (y,z)\in\Omega_1 \tag{2}$$

$$u(x,0,z)=\tau_2(x,z).\quad (x,z)\in\Omega_2, \tag{3}$$

$$B_n^{\alpha,\beta}\left[u\right]\Big|_S=\varphi(x,y,z),\quad (x,y,z)\in S, \tag{4}$$

where $\tau_1(y,z),\ \tau_2(x,z),\ \varphi(x,y,z)$ are given functions fulfilling the following matching conditions: $\tau_1(0,z)=\tau_2(0,z),\ -c<z<d$;

$$B_n^{\alpha,\beta}\left[\quad\right]=x^{2\alpha}y^{2\beta}\left(\cos(n,x)\cdot\frac{\partial}{\partial x}+\cos(n,y)\cdot\frac{\partial}{\partial y}+\cos(n,z)\cdot\frac{\partial}{\partial z}\right).$$

2.1. The uniqueness of the solution

One can readly check the validity of the following relation

$$x^{2\alpha}y^{2\beta}\left[uH_{\alpha,\beta}(w)-wH_{\alpha,\beta}(u)\right]=y^{2\beta}\frac{\partial}{\partial x}\left[x^{2\alpha}(uw_x-wu_x)\right]+$$

$$+x^{2\alpha}\frac{\partial}{\partial y}\left[y^{2\beta}(uw_y-wu_y)\right]+x^{2\alpha}y^{2\beta}\frac{\partial}{\partial z}\left[uw_z-wu_z\right].$$

Let Ω_ε be a sup-domain of Ω at a distance $\varepsilon > 0$ from its boundary $\partial\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup S$ and $\cos(n,x)d\sigma = dydz,\ \cos(n,y)d\sigma = dxdz,\ \cos(n,z)d\sigma = dxdy,$ n is the outer normal to $\partial\Omega.$

Integrate both sides of above given equlity on the domain Ω_ε and use the classical formula of Gauss-Ostrogradsky:

$$\iiint_{\Omega_\varepsilon} x^{2\alpha} y^{2\beta} \left[uH_{\alpha,\beta}(w) - wH_{\alpha,\beta}(u) \right] dxdydz = \iint_{\partial\Omega_\varepsilon} x^{2\alpha} y^{2\beta} [(uw_x - wu_x)\cos(n,x) +$$

$$+ (uw_y - wu_y)\cos(n,y) + (uw_z - wu_z)\cos(n,z)] d\sigma. \tag{5}$$

Using the equality

$$x^{2\alpha} y^{2\beta} \left[uH_{\alpha,\beta}(u) + u_x^2 + u_y^2 + u_z^2 \right] = (x^{2\alpha} y^{2\beta} uu_x)_x + (x^{2\alpha} y^{2\beta} uu_y)_y + (x^{2\alpha} y^{2\beta} uu_z)_z,$$

we obtain

$$\iiint_{\Omega_\varepsilon} x^{2\alpha} y^{2\beta} uH_{\alpha,\beta}(u) dxdydz + \iiint_{\Omega_\varepsilon} x^{2\alpha} y^{2\beta} [u_x^2 + u_y^2 + u_z^2] dxdydz =$$

$$= \iiint_{\Omega_\varepsilon} [(x^{2\alpha} y^{2\beta} uu_x)_x + (x^{2\alpha} y^{2\beta} uu_y)_y + (x^{2\alpha} y^{2\beta} uu_z)_z] dxdydz.$$

Applying again the formula of Gauss-Ostrogradsky to this equality and letting $\varepsilon \to 0$, we get

$$\iiint_{\Omega} x^{2\alpha} y^{2\beta} \left[u_x^2 + u_y^2 + u_z^2 \right] dxdydz =$$

$$= \iint_{\Omega_1} y^{2\beta} \tau_1 \nu_1 dydz + \iint_{\Omega_2} x^{2\alpha} \tau_2 \nu_2 dxdz + \iint_S u\varphi d\sigma, \tag{6}$$

where

$$x^{2\alpha} u_x(x,y,z)\Big|_{x=0} = \nu_1(y,z) \in \Omega_1, \quad y^{2\beta} u_y(x,y,z)\Big|_{y=0} = \nu_2(x,z) \in \Omega_2.$$

To prove the uniqueness of the solution, as usual, we suppose that the problem has two v, w solutions. Denoting $u = v - w$ we have that u satisfies homogeneous Problem DM $(\tau_1 = 0,\ \tau_2 = 0,\ \varphi = 0).$ Further we have to prove that the homogeneous problem has only trivial solution. In this case from (6) one can easily get

$$\iiint_{\Omega} x^{2\alpha} y^{2\beta} [u_x^2 + u_y^2 + u_z^2] dxdydz = 0.$$

Hence, it follows that $u_x = u_y = u_z = 0,$ which implies that u is a constant function. Considering homogeneous conditions (2)-(4), we conclude that $u(x,y,z) \equiv 0,$ in Ω.

2.2. The existence of the solution

We prove the existence of the solution in a special case of the domain Ω in order to get the solution in an explicit form. Assume $R = a = b = c = d$ and let $\Omega = \{(x,y,z): x^2 + y^2 + z^2 < R^2,\ x > 0, y > 0, -R < z < R\}$. We find a solution of considered problem using method of Green's functions [31]. Therefore, first we give a definition of Green's function for the formulated problem.

Definition. We call the function $G_1(x,y,z;x_0,y_0,z_0)$ as Green's function of the Problem DM, if it satisfies the following conditions:

1. this function is a regular solution of Eq. (1) in the domain Ω, except at the point (x_0,y_0,z_0), which is any fixed point of Ω;
2. it satisfies boundary conditions
$$G_1(x,y,z;x_0,y_0,z_0)\big|_{x=0} = 0, \quad G_1(x,y,z;x_0,y_0,z_0)\big|_{y=0} = 0,$$
$$\left\{ B_n^{\alpha,\beta}\left[G_1(x,y,z;x_0,y_0,z_0)\right]\right\}\Big|_S = 0;$$
3. it can be represented as
$$G_1(x,y,z;x_0,y_0,z_0) = q(x,y,z;x_0,y_0,z_0) + \overline{q}(x,y,z;x_0,y_0,z_0), \tag{7}$$

where
$$q(x,y,z;x_0,y_0,z_0) = k(r^2)^{\alpha+\beta-\frac{5}{2}}(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}F_2\left(\frac{5}{2}-\alpha-\beta;\alpha,1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta\right)$$
is the fundamental solution found earlier [11] (see, also [2]),
$$F_2(a;b_1,b_2;c_1,c_2;x,y) = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$
is a famous Appell function [13] (p.224, Eq.(7)),
$$(\lambda)_m = \lambda(\lambda+1)...(\lambda+m-1),\ m = 1,2,...,\ (\lambda)_0 = 1$$
is a Pochhammer symbol; a function
$$\overline{q}(x,y,z;x_0,y_0,z_0) = -\left(\frac{a}{R_0}\right)^{5-2\alpha-2\beta} q(x,y,z;\overline{x}_0,\overline{y}_0,\overline{z}_0)$$
is a regular solution of Eq.(1) in the domain Ω.

Here,

$$\bar{x}_0 = \frac{a^2}{R_0^2} x_0, \quad \bar{y}_0 = \frac{a^2}{R_0^2} y_0, \quad \bar{z}_0 = \frac{a^2}{R_0^2} z_0, \quad R_0^2 = x_0^2 + y_0^2 + z_0^2.$$

Excise a small ball with its center at (x_0, y_0, z_0) and with radius $\rho > 0$ from the domain Ω. Designate the sphere of the excised ball as C_ρ and by Ω_ρ denote the remaining part of Ω.

Applying formula (5), obtain

$$\iint_{C_\rho} x^{2\alpha} y^{2\beta} \left[u \frac{\partial G}{\partial n} - G_1 \frac{\partial u}{\partial n} \right] dS = \iint_{\Omega_1} y^{2\beta} \tau_1(y,z) G^*(0, y, z; x_0, y_0, z_0) dydz +$$

$$+ \iint_{\Omega_2} x^{2\alpha} \tau_2(x,z) G^{**}(x, 0, z; x_0, y_0, z_0) dxdz +$$

$$+ \iint_S G(x, y, z; x_0, y_0, z_0) \varphi(\sigma) d\sigma, \tag{8}$$

where

$$G^*(0, y, z; x_0, y_0, z_0) = x^{2\alpha} \left. \frac{\partial G(x, y, z; x_0, y_0, z_0)}{\partial x} \right|_{x=0}, \quad (y,z) \in \Omega_1$$

$$G^{**}(x, 0, z; x_0, y_0, z_0) = y^{2\beta} \left. \frac{\partial G(x, y, z; x_0, y_0, z_0)}{\partial y} \right|_{y=0}, \quad (x,z) \in \Omega_2.$$

It is known [11]:

$$\iint_{C_\rho} x^{2\alpha} y^{2\beta} u \frac{\partial G}{\partial n} dS = u(x_0, y_0, z_0), \qquad \iint_{C_\rho} x^{2\alpha} y^{2\beta} G \frac{\partial u}{\partial n} dS = 0. \tag{9}$$

Now from (8) and (9) we can write the solution of the Problem DM as follows:

$$u(x_0, y_0, z_0) = \iint_{\Omega_1} y^{2\beta} \tau_1(y,z) G^*(0, y, z; x_0, y_0, z_0) dydz +$$

$$+ \iint_{\Omega_2} x^{2\alpha} \tau_2(x,z) G^{**}(x, 0, z; x_0, y_0, z_0) dxdz + \iint_S G(x, y, z; x_0, y_0, z_0) \varphi(\sigma) d\sigma. \tag{10}$$

The particular values of Green’s function is given by

$$G^*(0,y,z;x_0,y_0,z_0)=k(1-2\alpha)x_0^{1-2\alpha}(yy_0)^{1-2\beta}\left[\frac{{}_2F_1\left(\frac{5}{2}-\alpha+\beta,1-\beta,2-2\beta;\eta_{0x}\right)}{\left[x_0^2+(y-y_0)^2+(z-z_0)^2\right]^{\frac{5}{2}-\alpha-\beta}}-\right.$$

$$\left.-\left(\frac{a}{R_0}\right)^{4-4\alpha-4\beta}\frac{{}_2F_1\left(\frac{5}{2}-\alpha+\beta,1-\beta,2-2\beta;\bar{\eta}_{0x}\right)}{\left[\left(a-\frac{yy_0}{a}\right)^2+\left(a-\frac{zz_0}{a}\right)^2+\frac{x_0^2+z_0^2}{a^2}y^2+\frac{x_0^2+y_0^2}{a^2}z^2-a^2\right]^{\frac{5}{2}-\alpha-\beta}}\right],$$

$$G^{**}(x,0,z;x_0,y_0,z_0)=k(1-2\beta)(xx_0)^{1-2\alpha}y_0^{1-2\beta}\left[\frac{{}_2F_1\left(\frac{5}{2}-\alpha-\beta,1-\alpha,2-2\alpha;\xi_{0y}\right)}{\left[(x-x_0)^2+y_0^2+(z-z_0)^2\right]^{\frac{5}{2}-\alpha-\beta}}-\right.$$

$$\left.-\left(\frac{a}{R_0}\right)^{4-4\alpha-4\beta}\frac{{}_2F_1\left(\frac{5}{2}-\alpha-\beta,1-\alpha,2-2\alpha;\bar{\xi}_{0y}\right)}{\left[\left(a-\frac{xx_0}{a}\right)^2+\left(a-\frac{zz_0}{a}\right)^2+\frac{y_0^2+z_0^2}{a^2}x^2+\frac{x_0^2+y_0^2}{a^2}z^2-a^2\right]^{\frac{5}{2}-\alpha-\beta}}\right],$$

Here,

$$\xi_{0y}=-\frac{4xx_0}{(x-x_0)^2+y_0^2+(z-z_0)^2},\quad \eta_{0x}=-\frac{4yy_0}{x_0^2+(y-y_0)^2+(z-z_0)^2},$$

$$\bar{\xi}_{0y}=-\frac{4a^2}{R_0^2}\frac{xx_0}{\left(a-\frac{yy_0}{a}\right)^2+\left(a-\frac{zz_0}{a}\right)^2+\frac{y_0^2+z_0^2}{a^2}x^2+\frac{x_0^2+y_0^2}{a^2}z^2-a^2},$$

$$\bar{\eta}_{0x}=-\frac{4a^2}{R_0^2}\frac{yy_0}{\left(a-\frac{yy_0}{a}\right)^2+\left(a-\frac{zz_0}{a}\right)^2+\frac{x_0^2+z_0^2}{a^2}y^2+\frac{x_0^2+y_0^2}{a^2}z^2-a^2},$$

$$k=\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(4-2\alpha-2\beta)}{\Gamma(2-2\alpha)\Gamma(2-2\beta)\Gamma(2-\alpha-\beta)};$$

$F(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}$ is a Gaussian hypergeometric function [13] (p. 68).

Hence, the main result of the paper is formulated as the following theorem:

Theorem. If $\tau_1(y,z)\in C^2(\Omega_1),\ \ \tau_2(x,z)\in C^2(\Omega_2),\ \varphi(x,y,z)\in C^1(S)$ then the Problem DM has unique solution represented by formula (10).

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