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I_3 TRAYEKTORIYALARINING MAXSUS KLASSLARI

Annotatsiya: Idempotent matematika bugungi kunda dolbzar bo'lgan iqtisodiy masalalarni optimallashtirishda, biologik va genetikaga oid masalalarni yechishda qo'llaniladi. Idempotent matematika asosida oddiy arifmetik amallarni bazaviy amallarning (maksimum yoki minimum kabi) yangi to'plami bilan almashtirish yotadi, bunda sonlar maydoni idempotent yarim xalqalar va yarim maydonlar bilan almashadi.

Kalit sozlar: Idempotent, operator, simpleks, chiziqli, qo'zg'almas nuqta, yechimlar to'plami, matritsa, matematika, trayektoriya, dinamika, idempotent o'lchovlar simpleksini o'zini-o'ziga akslantiruvchi.

СПЕЦИАЛЬНЫЕ КЛАССЫ ТРАЕКТОРИЙ I_3

Аннотация: Идемпотентная математика сегодня используется при оптимизации экономических задач, а также при решении задач, связанных с биологией и генетикой. В основе идемпотентной математики лежит замена обычных арифметических операций новым набором основных операций (таких как максимумы или минимумы), в которых поле чисел заменяется идемпотентными полукругами и полу полями.

Ключевые слова: Идемпотент, оператор, симплекс, линейный, неподвижная точка, множество решений, матрица, математика, траектория,

динамика, отображающий саму на себя симплкс идемпотичных мер.

SPECIAL CLASSES OF I_3 TRAJECTORIES

Annotation: Idempotent mathematics is used today in the optimization of economic problems, as well as in solving problems related to biology and genetics. The basis of idempotent mathematics is the replacement of ordinary arithmetic operations with a new set of basic operations (such as maxima or minima) in which the field of numbers is replaced by idempotent semicircles and semifields.

Abstract: Idempotent, operator, simplex, linear, fixed point, solution set, matrix, mathematics, trajectory, dynamics, self-reflecting complex of idempotic measures.

In this article, the dynamics of linear operators reflecting a simplex of idempotent measures is considered. Six different types of representations of the linear operator and the fixed points (or invariant points) and the trajectories of the points for each are extensively analysed. This analysis examines how systems approach their fixed points under different initial conditions and how these trajectories can be generalized. Determining the motion trajectory and fixed points for each representation of the linear operator allows us to understand how the system evolves given the initial conditions and parameters affecting the system. Such studies provide basic analytical tools for determining the stability, reversibility, and other properties of systems.

1 Introduction

Dynamical systems and behaviour of trajectories were studied in detail in [1, 2, 3, 4, 5]. In [6] studied quadratic stochastic operators and their dynamics were constructed. In [7, 8, 9] idempotent analysis was introduced in science as simplex of idempotent measures. As typical examples of simplex idempotent measures we can take algebras Maks-plus $R_{\max}$ and min-plus $R_{\min}$.

Let R - be a field of real numbers. Then The operations in $R_{\max} = R \cup \{-\infty\}$ are in the following:

$$x \oplus y = \max\{x, y\} \text{ and } x \otimes y = x + y .$$

Similarly, The operations in $R_{\min} = R \cup \{+\infty\}$ are in the following:

$$\oplus = \min, \otimes = +$$

We consider with $R_{\max} = R \cup \{-\infty\}$ idempotent addition and multiplication operations .

We define a simlex of idempotent measures with $(n-1)$- dimension in the following case:

$$I_n = \left\{ (x_1,...x_n) \in R^n_{\max} : \max_{1\le i\le n} x_i = 0 \right\} =$$
$$= \left\{ (x_1,...x_n) \in R^n_{\max} : x_1 \oplus ... \oplus x_n = 1 \right\}.$$

The first work on linear operators which map $(n-1)$-dimensional simplex of idempotent measures into itself was studied in work [10]. In [10]defined two classes of linear operators and their fixed points are constructed and studied. Their dynamics were also studied for the case n=2. The first works to obtain a general class of quadratic operators which map the $(n-1)$-dimensional simplex of idempotent measures into itself were done in [11]. In [11] quadratic operators were studied by dividing into two classes and fixed points of these types of quadratic operators were obtained in the case n=2. Certain quadratic operators in [11] are idempotent analogies of quadratic stochastic operators studied in [6]. In [12] fixed points were obtained for both types of quadratic operators defined in [11]. Also characteristics and properties of the obtained fixed points were studied in the case of n=3. In [13] trajectories of non-fixed points under quadratic operators are constructed in the case n=2. In [14] trajectories (dynamical system) of some quadratic operators were described in case n=3.

2 Linear operators and their trajectories

In this article we will continue our research outlined in [10], i.e. Let us study the dynamical system of linear operators of idempotent measures, defined in [10] for the case n=3. We will find the i-th element of I_n using all the elements of the i-th row of the corresponding quadratic matrix. We present this theorem without proof. Following theorem shows the types of linear operations which map I_n to itself.

Teorema1. [10]. A linear operator $A = (a_{ij})_{i,j=\overline{1,n}}$ maps I_n to itself, if and only it satisfies one of following conditions:

a) A has at least one zero row ($\exists m \le n$ such that all of elements of m-th row of the matrix A consists of only zeroes).

b) Each k-th row of the matrix A contains exactly one non-zero element: In each k-th row of the matrix A all of elements except a_{k,m_k} are zeroes. Where $m_q \ne m_r$ if $q \ne r$ ($q = \overline{1,n}$, $r = \overline{1,n}$, $k = \overline{1,n}$);

Consider b) condition of the theorem is satisfied. i.e. $a_{ij} \ge 0$ and A matrix has a not more than one non zero element in each row and in each column. In the case n=3 we obtain 6 linear operator which satisfy b) condition of the theorem:

a) $A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$ b) $A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}$ c) $A=\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$

d) $A=\begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$ e) $A=\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$ f) $A=\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix}$

Let's use denotes which given in [12], i.e. denote by $A_{(n;m_1,m_2,\dots m_n)}$ operator **A** as

$$A_{(n;m_1,m_2,\dots m_n)}=\left(a_{i,j}\right)_{i,j=1}^{n} \tag{1}$$

where n - order of the matrix and m_k - is the number mentioned in Theorem 1 which corresponding to the k - th row of the matrix A, $1\le m_k \le n$, $m_k \ne m_q$, $k\ne q$.

According to (1) we have there are n! cases possible. It is very hard to analysis all of these cases. That's why due to to the form (1) we can make the set A as (see [12]):

$$\Omega=\left\{ k\in\{1,...,n\} \mid m_k = k\right\} \tag{2}$$

For $n=3$ from (1) and (2) we obtain results:

1) If $|\Omega|=3$ we have only one operator: $A_{(3;1,2,3)}$;

$$A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

2) If $|\Omega|=1$ we get three operators: $A_{(3;1,3,2)}$, $A_{(3;2,1,3)}$ and $A_{(3;3,2,1)}$;

$$A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix} A=\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} A=\begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

3) If $|\Omega|=0$ have two operators as: $A_{(3;3,1,2)}$ and $A_{(3;2,3,1)}$.

$$A=\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} A=\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$$

Assume linear operator A satisfies b) condition of the theorem1 and $|\Omega|=3$. Let's find n-th iteration of the point $x^0=(x_1^0,x_2^0,\ x_3^0)\in I_3$ under linear operator $A_{(3;1,2,3)}$ as:

$$A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

By using mathematical induction we can describe n-th degree of the matrix A:

$$A\times A=\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{pmatrix}\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}^2&0&0\\0&a_{22}^2&0\\0&0&a_{33}^2\end{pmatrix},$$

$$A\times A\times A=\begin{pmatrix}a_{11}^2&0&0\\0&a_{22}^2&0\\0&0&a_{33}^2\end{pmatrix}\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}^3&0&0\\0&a_{22}^3&0\\0&0&a_{33}^3\end{pmatrix},$$

$$A^n=\begin{pmatrix}a_{11}^{n-1}&0&0\\0&a_{22}^{n-1}&0\\0&0&a_{33}^{n-1}\end{pmatrix}\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}^n&0&0\\0&a_{22}^n&0\\0&0&a_{33}^n\end{pmatrix}$$

In this case we get n-th iteration of the point $x^0=(x_1^0,x_2^0,\ x_3^0)\in I_3$ as:

$$A^n x^0=\begin{pmatrix}a_{11}^n&0&0\\0&a_{22}^n&0\\0&0&a_{33}^n\end{pmatrix}\begin{pmatrix}x_1^0\\x_2^0\\x_3^0\end{pmatrix}=\begin{pmatrix}a_{11}^n x_1^0\\a_{22}^n x_2^0\\a_{33}^n x_3^0\end{pmatrix}.$$

So, for every point $x^0=(x_1^0,x_2^0,\ x_3^0)\in I_3$ we get n-th iteration as:

$$x^{(n)}=A^n x^0=(a_{11}^n x_1^0,\ a_{22}^n x_2^0,\ a_{33}^n x_3^0). \qquad (3)$$

We know that, the point $x^0=(x_1^0,x_2^0,\ x_3^0)\in I_3$ can be defined by one of following cases:

$$1.(x_1^0;x_2^0;0)\ 2.(x_1^0;0;x_3^0)\ 3.(0;x_2^0;x_3^0)\ 4.(x_1^0;0;0)$$

$$5.(0;x_2^0;0)\ 6.(0;0;x_3^0)\ 7.(0;0;0)$$

Let's study trajectories of points as $x^0=(x_1^0;x_2^0;0)\in I_3$. Using formula (3) we get following trajectories for the point $x^0=(x_1^0;x_2^0;0)\in I_3$:

$$\lim_{n\to\infty}A^n x^0=\begin{cases}(0;0;0)\ \ if\ \ a_{22}<1,\ a_{33}<1;\\(0;x_2^0;0)\ \ if\ \ a_{22}=1,\ a_{33}<1;\\(0;0;x_3^0)\ \ if\ \ a_{22}<1,\ a_{33}=1;\\(0;x_2^0;x_3^0)\ \ if\ \ a_{22}=1,\ a_{33}=1;\\(0;x_2^0;-\infty)\ \ if\ \ a_{22}=1,\ a_{33}>1;\\(0;-\infty;x_3^0)\ \ if\ \ a_{22}>1,\ a_{33}=1;\\(0;-\infty;-\infty)\ \ if\ \ a_{22}>1,\ a_{33}>1;\\(0;-\infty;0)\ \ if\ \ a_{22}>1,\ a_{33}<1;\\(0;0;-\infty)\ \ if\ \ a_{22}<1,\ a_{33}>1.\end{cases}$$

Analogical results of trajectory we can get for other points of I_3.

Definition. Recall that *fixed points* of $A:I_n\to I_n$ are solutions of the equation

$A(x) = x$.

Solving this equation we get that linear operator $A_{(3;1,2,3)} : I_3 \to I_3$ has only one fixed point as $(0;0;0)$. After studying all trajectories of points as $x^0 = (x_1^0, x_2^0,\ x_3^0) \in I_3$ under linear operator $A_{(3;1,2,3)} : I_3 \to I_3$ we can conclude following theorem:

Theorem2. For the linear operator $A_{(3;1,2,3)} : I_3 \to I_3$:

1. the fixed point $(0;0;0)$ is an *attracting fixed point* if $a_{ii} < 1$, for i=1,2,3;

2. the fixed point $(0;0;0)$ is a *repelling fixed point* if $a_{ii} > 1$, for i=1,2,3;

3. the fixed point $(0;0;0)$ is a *neutral* or an *indifferent fixed point* for all other cases.

Consider, linear operator A satisfies b) condition of the theorem1 and $|\Omega| = 1$. Let's research n-th iteration of the point $x^0 = (x_1^0, x_2^0,\ x_3^0) \in I_3$ under linear operators $A_{(3;1,3,2)}$, $A_{(3;3,2,1)}$ and $A_{(3;2,1,3)}$ as:

$$A_{(3;1,3,2)} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix};\ A_{(3;3,2,1)} = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \text{ and } A_{(3;3,2,1)} = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Let's see the operator $A_{(3;3,2,1)}$. By using mathematical induction we can describe n-th degree of the matrix A:

$$A \times A = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{13}a_{31} & 0 & 0 \\ 0 & a_{22}^2 & 0 \\ 0 & 0 & a_{31}a_{13} \end{pmatrix},$$

$$A \times A \times A = \begin{pmatrix} a_{13}a_{31} & 0 & 0 \\ 0 & a_{22}^2 & 0 \\ 0 & 0 & a_{31}a_{13} \end{pmatrix}\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{13}^2a_{31} \\ 0 & a_{22}^3 & 0 \\ a_{31}^2a_{13} & 0 & 0 \end{pmatrix},$$

$$A_{(3;3,2,1)}^n = \begin{cases} \begin{pmatrix} a_{13}^k a_{31}^k & 0 & 0 \\ 0 & a_{22}^{2k} & 0 \\ 0 & 0 & a_{31}^k a_{13}^k \end{pmatrix} & if \quad n = 2k \\ \begin{pmatrix} 0 & 0 & a_{13}^{k+1} a_{31}^k \\ 0 & a_{22}^{2k+1} & 0 \\ a_{31}^{k+1} a_{13}^k & 0 & 0 \end{pmatrix} & if \quad n = 2k+1 \end{cases}$$

After some algebra we get n-th iteration of the point $x^0 = (x_1^0, x_2^0,\ x_3^0) \in I_3$ as:

1. if n=2k so we get:

$$A^n x^0 = \begin{pmatrix} a_{13}^k a_{31}^k & 0 & 0 \\ 0 & a_{22}^{2k} & 0 \\ 0 & 0 & a_{31}^k a_{13}^k \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} = \begin{pmatrix} a_{13}^k a_{31}^k x_1^0 \\ a_{22}^{2k} x_2^0 \\ a_{31}^k a_{13}^k x_3^0 \end{pmatrix},$$

2. if n=2k+1 we get following:

$$A^n x^0 = \begin{pmatrix} 0 & 0 & a_{13}^{k+1} a_{31}^k \\ 0 & a_{22}^{2k+1} & 0 \\ a_{31}^{k+1} a_{13}^k & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} = \begin{pmatrix} a_{13}^{k+1} a_{31}^k x_3^0 \\ a_{22}^{2k+1} x_2^0 \\ a_{31}^{k+1} a_{13}^k x_1^0 \end{pmatrix}.$$

Assume n=2k, then we obtain equality:

$$\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty}((a_{13}a_{31})^k x_1^0, a_{22}^{2k} x_2^0, (a_{31}a_{13})^k x_3^0) = \tag{4}$$

$$= \begin{cases} (0,0,0), & if\ a_{13}a_{31} < 1, a_{22} < 1, \\ (x_1^0, 0, x_3^0), & if\ a_{13}a_{31} = 1, a_{22} < 1, \\ (0, x_2^0, 0), & if\ a_{13}a_{31} < 1, a_{22} = 1, \\ (x_1^0, x_2^0, x_3^0), & if\ a_{13}a_{31} = a_{22} = 1, \\ (-\infty, -\infty, -\infty), & if\ a_{13}a_{31} > 1, a_{22} > 1, \\ (-\infty, 0, -\infty), & if\ a_{13}a_{31} > 1, a_{22} < 1, \\ (0, -\infty, 0), & if\ a_{13}a_{31} < 1, a_{22} > 1, \\ (x_1^0, -\infty, x_3^0), & if\ a_{13}a_{31} = 1, a_{22} > 1, \\ (-\infty, x_2^0, -\infty), & if\ a_{13}a_{31} > 1, a_{22} = 1, \end{cases} \tag{5}$$

Let's n=2k+1 then

$$\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty}((a_{13}a_{31})^k\, a_{13}x_3^0, a_{22}^{2k+1}x_2^0, (a_{31}a_{13})^k\, a_{31}x_1^0) =$$

$$= \begin{cases} (0,0,0), & if\ a_{13}a_{31} < 1, a_{22} < 1, \\ (a_{13}x_3^0, 0, a_{31}x_1^0), & if\ a_{13}a_{31} = 1, a_{22} < 1, \\ (0, x_2^0, 0), & if\ a_{13}a_{31} < 1, a_{22} = 1, \\ (a_{13}x_3^0, x_2^0, a_{31}x_1^0), & if\ a_{13}a_{31} = a_{22} = 1, \\ (-\infty,-\infty,-\infty), & if\ a_{13}a_{31} > 1, a_{22} > 1, \\ (-\infty, 0, -\infty), & if\ a_{13}a_{31} > 1, a_{22} < 1, \\ (0, -\infty, 0), & if\ a_{13}a_{31} < 1, a_{22} > 1, \\ (a_{13}x_3^0, -\infty, a_{31}x_1^0), & if\ a_{13}a_{31} = 1, a_{22} > 1, \\ (-\infty, x_2^0, -\infty), & if\ a_{13}a_{31} > 1, a_{22} = 1, \end{cases}$$

We can get similar results for operators $A_{(3;1,3,2)}$ and $A_{(3;2,1,3)}$ also. Consequently, we have following theorem:

Theorem3. Assume linear operator A satisfies b) condition of the theorem1 and $|\Omega| = 1$. For the linear operator $A_{(3;m_1,m_2,m_3)} : I_3 \to I_3$:

1. the fixed point $(0;0;0)$ is an *attracting fixed point* if $a_{ii} < 1$, $a_{jk} \cdot a_{kj} < 1$, $i \neq j$, $j \neq k$, $i \neq k$ $(i,j,k = \overline{1,3})$;

2. the fixed point $(0;0;0)$ is a *repelling fixed point* if $a_{ii} > 1$, $a_{jk} \cdot a_{kj} > 1$, $i \neq j$, $j \neq k$, $i \neq k$ $(i,j,k = \overline{1,3})$;

3. the fixed point $(0;0;0)$ is a *neutral* or an *indifferent fixed point* for all other cases.

Let linear operator A satisfies b) condition of the theorem1 and $|\Omega| = 0$. There are two types of such linear operators: $A_{(3;3,1,2)}$ and $A_{(3;2,3,1)}$. We do research on n-th iteration of the point $x^0 = (x_1^0, x_2^0, x_3^0) \in I_3$ under linear operators $A_{(3;3,1,2)}$ as:

$$A_{(3;3,1,2)} = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix};$$

By using mathematical induction we can define n-th degree of the matrix $A_{(3;3,1,2)}$:

$$A \times A = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{13}a_{32} & 0 \\ 0 & 0 & a_{21}a_{13} \\ a_{32}a_{21} & 0 & 0 \end{pmatrix},$$

$$A \times A \times A = \begin{pmatrix} 0 & a_{13}a_{32} & 0 \\ 0 & 0 & a_{21}a_{13} \\ a_{32}a_{21} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} a_{21}a_{13}a_{32} & 0 & 0 \\ 0 & a_{21}a_{13}a_{32} & 0 \\ 0 & 0 & a_{21}a_{13}a_{32} \end{pmatrix},$$

$$1)\quad A^{3k-2} = \begin{pmatrix} 0 & 0 & a_{21}^{k-1}a_{13}^{k}a_{32}^{k-1} \\ a_{21}^{k}a_{13}^{k-1}a_{32}^{k-1} & 0 & 0 \\ 0 & a_{21}^{k-1}a_{13}^{k-1}a_{32}^{k} & 0 \end{pmatrix}$$

$$2)\quad A^{3k-1} = \begin{pmatrix} 0 & a_{21}^{k-1}a_{13}^{k}a_{32}^{k} & 0 \\ 0 & 0 & a_{21}^{k}a_{13}^{k}a_{32}^{k-1} \\ a_{21}^{k}a_{13}^{k-1}a_{32}^{k} & 0 & 0 \end{pmatrix}$$

$$3)\quad A^{3k} = \begin{pmatrix} a_{21}^{k}a_{13}^{k}a_{32}^{k} & 0 & 0 \\ 0 & a_{21}^{k}a_{13}^{k}a_{32}^{k} & 0 \\ 0 & 0 & a_{21}^{k}a_{13}^{k}a_{32}^{k} \end{pmatrix}$$

Then these three cases are possible:

Case1. If n=3k-2 then we have:

So, we get $x^{(n)} = A^n x^0 = (a_{21}^{k-1}a_{13}^{k}a_{32}^{k-1}x_3^0, a_{21}^{k}a_{13}^{k-1}a_{32}^{k-1}x_1^0, a_{21}^{k-1}a_{13}^{k-1}a_{32}^{k}x_2^0)$ for all $x^0 = (x_1^0, x_2^0, x_3^0) \in I_3$

Case 2. Assume n=3k-1.

$$A^n x^0 = \begin{pmatrix} 0 & a_{21}^{k-1}a_{13}^{k}a_{32}^{k} & 0 \\ 0 & 0 & a_{21}^{k}a_{13}^{k}a_{32}^{k-1} \\ a_{21}^{k}a_{13}^{k-1}a_{32}^{k} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} = \begin{pmatrix} a_{21}^{k-1}a_{13}^{k}a_{32}^{k}x_2^0 \\ a_{21}^{k}a_{13}^{k}a_{32}^{k-1}x_3^0 \\ a_{21}^{k}a_{13}^{k-1}a_{32}^{k}x_1^0 \end{pmatrix}.$$

We have $x^{(n)} = A^n x^0 = (a_{21}^{k-1}a_{13}^{k}a_{32}^{k}x_2^0, a_{21}^{k}a_{13}^{k}a_{32}^{k-1}x_3^0, a_{21}^{k}a_{13}^{k-1}a_{32}^{k}x_1^0)$ for all $x^0 = (x_1^0, x_2^0, x_3^0) \in I_3$.

Case 3. Let n=3k.

$$A^n x^0 = \begin{pmatrix} a_{21}^{k}a_{13}^{k}a_{32}^{k} & 0 & 0 \\ 0 & a_{21}^{k}a_{13}^{k}a_{32}^{k} & 0 \\ 0 & 0 & a_{21}^{k}a_{13}^{k}a_{32}^{k} \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} = \begin{pmatrix} a_{21}^{k}a_{13}^{k}a_{32}^{k}x_1^0 \\ a_{21}^{k}a_{13}^{k}a_{32}^{k}x_2^0 \\ a_{21}^{k}a_{13}^{k}a_{32}^{k}x_3^0 \end{pmatrix}.$$

We obtain $x^{(n)} = A^n x^0 = (a_{21}^{k}a_{13}^{k}a_{32}^{k}x_1^0, a_{21}^{k}a_{13}^{k}a_{32}^{k}x_2^0, a_{21}^{k}a_{13}^{k}a_{32}^{k}x_3^0)$ for all $x^0 = (x_1^0, x_2^0, x_3^0) \in I_3$

We define limiting under condition $n \to \infty$ and we'll get following theorem:

Theorem4. Assume linear operator A satisfies b) condition of the theorem1 and $|\Omega| = 0$. For the linear operator $A_{(3;m_1,m_2,m_3)} : I_3 \to I_3$:

1. the fixed point $(0;0;0)$ is an *attracting fixed point* if $a_{ij} \cdot a_{jk} \cdot a_{ki} < 1$, $i \neq j$, $j \neq k$, $i \neq k$ $\left(i, j, k = \overline{1,3}\right)$;

2. the fixed point $(0;0;0)$ is a *repelling fixed point* if $a_{ij} \cdot a_{jk} \cdot a_{ki} > 1$, $i \neq j$, $j \neq k$, $i \neq k$ $\left(i, j, k = \overline{1,3}\right)$;

3. the fixed point $(0;0;0)$ is a *neutral* or an *indifferent fixed point* for all other cases.

3 Conclusion

Until now, in [10, 11, 12, 13, 14], representation of linear and quadratic operators in the simplex of idempotent measures has been constructed and the dynamics i.e., the trajectories of the non-fixed points have been studied. However, the problem of studying the dynamics of linear operators with n=3 remained open. In this paper we have researched this open problem and studied all possible dynamics for linear operators of both forms which defined in [10] in the simplex of idempotent measures in the case n=3, i.e., we studied the trajectories of motion of non-fixed points which lying in this simplex under the linear operator. Thus, in this paper we have completed the studying construction and dynamics of linear operators which map simplex of idempotent measures into itself. Therefore we are completely closing the open problem we mentioned above.

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