p-ADIC (2,1)-RATIONAL DYNAMICAL SYSTEM WITH UNIQUE MULTIPLE FIXED POINT

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Annotatsiya. Ushbu maqolada biz $f(x)=\frac{x^2+ax}{bx+a}$ ko‘rinishdagi ratsional funksiyaning diskret vaqtli p-adik dinamik sistemasini o‘rgandik. O‘rganilgan ratsional funksiya yagona $x=0$ qo‘zg‘almas nuqtaga ega hamda uning betaraf qo‘zg‘almas nuqta ekanligini ko‘rsatdik. Bundan tashqari, b parametrning turli qiymatlarida maksimal Zigel disklari topdik.

Kalit so'zlar: p-adik son, qo'zg'almas nuqta, itaruvchi, tortuvchi, betaraf, Zigel diski, maksimal Zigel diski.

Аннотация. В данной статье мы изучили дискретную систему динамики на p-адических числах для рациональной функции вида $f(x)=\frac{x^2+ax}{bx+a}$. Мы показали, что рациональная функция имеет единственную неподвижную точку при $x=0$ и доказали, что эта неподвижная точка является нейтральной. Кроме того, мы нашли максимальные диски Зигеля для различных значений параметра b.

Annotation. In this article, we studied the discrete-time p-adic dynamic system of the rational function of the form $f(x)=\frac{x^2+ax}{bx+a}$. We demonstrated that the rational function has a unique fixed point at $x=0$ and showed that this fixed point is neutral. Additionally, we found the maximal Siegel disks for different values of the parameter b.

Key words: *p-adic number, fixed point, attractor, repeller , indifferent, Siegel disk, maximum Siegel disk.*

In this work we consider the dynamical system generated by a (2,1)-rational function. For motivations and methods of the study see [1-5] and the references their in.

Let Q be the field of rational numbers. The greatest common divisor of the positive integers *n* and *m* is denotes by (*n*,*m*). Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in Z, m$ is a positive integer, (*p*,*n*)=1, (*p*,*m*)=1 and *p* is a fixed prime number. The *p*-adic norm of *x* is given by

$$|x|_p = \begin{cases} p^{-r}, for\ x \neq 0, \\ 0, for\ x = 0. \end{cases}$$

This norm is a non-Archimedean one. The completion of Q with respect to p-adic norm defines the p-adic field which is denoted by Q_p (see [2]). For any $a \in Q_p$ and $r > 0$ denote

$$U_r(a) = \{x \in Q_p : |x-a|_p < r\}, V_r(a) = \{x \in Q_p : |x-a|_p \leq r\},$$

$$S_r(a) = \{x \in Q_p : |x-a|_p = r\}.$$

Let $f : U \to U \subset Q_p$be a function. If $f(x_0) = x_0$ then x_0 is called a ¿ *point*. The set of all fixed points of f is denoted by Fix (f).

Denote $f^n(x) = \underbrace{f \circ \cdots \circ f(x)}_{n}$ A fixed point x_0 is called an *attractor* if there exists a neighborhood $U(x_0)$ of x_0 such that for all points $x \in U(x_0)$ it holds $\lim_{n \to \infty} f^n(x) = ¿ x_0 . ¿$ If x_0 is an attractor then its *basin of attraction*is $A(x ¿¿ 0) = \{x \in Q_p : f^n(x) \to x_0, n \to \infty\}. ¿$

A fixed point x_0 is called *repeller* if there exists a neighborhood $U(x_0)$ of x_0 such that $|f(x) - x_0|_p > |x - x_0|_p$for $x \in U(x_0), x \neq x_0$.

The ball $V_r(x_0)$ is said to be a *Siegel disc* if each sphere $S_p(x_0), p < r$ is an invariant sphere of $f(x)$, i.e. if x$\in S_p(x_0)$then all iterated points $f^n(x) \in S_p(x_0)$ for all $n = 1, 2 \ldots$. The union of all Siegel discs with the center at x_0 is said to *a maximum Siegel disc* and is denoted by $SI(x_0)$.

Let $f : U \to U$ and $g : V \to V$ be two maps. f and g are said to be *topologically conjucate* if there exists a homeomorphism $h : U \to V$such that $h \circ f = g \circ h$. The homeomorphism h is called a *topological conjugacy*. Mappings that are topologically conjugate are

completely equivalent in terms of their dynamics.
In this work, we consider a dynamical system generated by the (2,1)-rational function with unique multiple fixed point.

Proposition 1. Any (2,1)-rational function with unique multiple fixed point is topologically conjugate to the function with following form

$$f(x)=\frac{x^2+ax}{bx+a},a,b\in Q_p,ab(b-1)\neq 0,(1)$$

where $x\neq\hat{x}=\frac{-a}{b}$.

Thus in this paper we study dynamical system generated by the function (1). It is easy to see that function (1) has unique fixed point $x_0=0$.For (1) we have $f'(x¿¿0)=f'(0)=1¿$, i.e., the point x_0 is an indifferent point.
Define the sets $A_1=\{r:r<|a|_p\}$ for $|b|_p<1$, $A_2=\{r:r\neq|a|_p\}$ for $|b|_p=1$, $A_3=\left\{r:r<\frac{|a|_p}{|b|_p}\right\}$ for $|b|_p>1$. Denote $A=A_1\cup A_2\cup A_3$. Then we get the following

Theorem 1. The sphere $S_r(0)$ is invariant for the function (1) if $r\in A$.

Proof: Now we calculate the following. Take an arbitrary point. There can be 3 cases.

I) Case first is: $y\in A_1=\{r:r<|a|_p\}.|b|_p<1.$

$$|f(y)-x_0|_p=\frac{|y|_p|y+a|_p}{|by+a|_p}\leq|a| ¿>¿ y\in A_1=¿f(y)\in A_1$$

II) Case second is given us in 2 different ways:
$y\in A_2=\{r:r\neq|a|_p\}.|b|_p=1.$

2.1) $|y|_p<|a|_p$

$$|f(y)-x_0|_p=\frac{|y|_p|y+a|_p}{|by+a|_p}=\frac{r|a|_p}{|a|_p}=r<|a|_p\in A_2$$

2.2) $|y|_p>|a|_p$

$$|f(y)-x_0|_p=\frac{|y|_p|y+a|_p}{|by+a|_p}=\frac{r|y|_p}{|y|_p}=r=|y|_p>|a|_p\in A_2$$

III) Case third is : $y \in A_3=\left\{r : r<\frac{|a|_p}{|b|_p}\right\}.|b|_p>1.$

$$\left|f(y)-x_0\right|_p=\frac{|y|_p|y+a|_p}{|b|_p\left|y+\frac{a}{b}\right|_p}=\frac{r|a|_p}{|b|_p\left|\frac{a}{b}\right|_p}=r<\frac{|a|_p}{|b|_p}\in A_3$$

The element of $r \in A$. Obtained from the set is invariant for the function.

The theorem 1 is proved (see[1]).

Corollary 1. $SI(0)=U_{|a|_p}(0)$ for $|b|_p\leq 1$ and SI(0)=$U_{\frac{|a|_p}{|b|_p}}(0)$ for $|b|_p>1$.

Now we study the dynamical system on each invariant sphere.

Theorem 2. Let $r \in A$. Then the function $f: S_r(0)\to S_r(0)$ is an isometry.

Proof: Let $r \in A_1$, than A_1note the set of $|b|_p$. Now let`s calculate the following: $\forall x, y \in S_r(0)$, $f(x)=\frac{x^2+ax}{bx+a}$. d(x,y)=$|x-y|_p<r$.

I) $r \in A_1$ $\forall x, y \in S_{|a|_p}(0)$ $r<|a|_p, |b|_p<1$

$$\left|\frac{(bx+a)y+a(a+x)}{(bx+a)(by+a)}\right|_p|x-y|_p=\frac{|a|_p^2}{|a|_p^2}|x-y|_p=|f(x)-f(y)|_p=|x-y|_p$$

It can be seen that the elements of set A_1 satisfy the isometry definition.

so f is an isometry in the set A_1.

II) $r \in A_2$ $\forall x, y \in S_{|a|_p}(0)$ $r\neq|a|_p, |b|_p=1$

Let $r \in A_2$, than A_2note the set of $|b|_p$. Now let`s calculate the following:

2.1) $r \in A_2$ $\forall x, y \in S_{|a|_p}(0)$ $r<|a|_p, |b|_p=1$

$$\left|\frac{(bx+a)y+a(a+x)}{(bx+a)(by+a)}\right|_p|x-y|_p=\frac{r*|a|_p^2}{|a|_p^2}|x-y|_p=|f(x)-f(y)|_p=|x-y|_p$$

2.2) $r \in A_2$ $\forall x, y \in S_{|a|_p}(0)$ $r>|a|_p, |b|_p=1$

$$\left|\frac{(bx+a)y+a(a+x)}{(bx+a)(by+a)}\right|_p|x-y|_p=\frac{r^2}{r^2}|x-y|_p=|f(x)-f(y)|_p=|x-y|_p$$

It can be seen that the elements of set A_2 satisfy the isometry definition.

so f is an isometry in the set A_2.

III) $r \in A_3 \quad \forall x, y \in S_{|a|_p}(0) \quad r > |a|_p, |b|_p > 1$

Let $r \in A_3$, than A_3note the set of $|b|_p$. Now let`s calculate the following:

$$\left|\frac{(bx+a)y+a(a+x)}{(bx+a)(by+a)}\right|_p |x-y|_p = |x-y|_p \frac{(|b|_r r)^2}{(|b|_r r)^2} = |f(x)-f(y)|_p = |x-y|_p$$

It can be seen that the elements of set A_3 satisfy the isometry definition.

so f is an isometry in the set A_3

The theorem 2 is proved (see[2]).

Corollary 2. Let $r \in A$. Then the function $f: S_r(0) \to S_r(0)$ is a bijective.

An invariant measure is a measure that is preserved by some function. Ergodic theory is the study of invariant measures in dynamical systems.

Let (X, Σ) be a measurable space and let $f: X \to X$ be a measurable function from X to itself. A measure μ on (X, Σ) is said to be invariant under f if, for every measurable set V in $\Sigma, \mu(f^{-1}(V)) = \mu(V)$. In this case, the function f is called a measure-preserving.

Consider the dynamical system (X, Σ, μ, f), where $f: X \to X$ is a measure-preserving function, and μ is a probability measure. We say the dynamical system is *ergodic* if for every invariant set $W \in V$ we have $\mu(V) = 0$ or $\mu(V) = 1$ (see[5]).

For each invariant sphere we consider a measurable space $(S_r(0), B)$, here B is the algebra generated by closed subsets of $(S_r(0), r \in A$. Every element of $B$is a union of some balls in $S_r(0)$. A measure $\bar{\mu}: B \to R$ is a *Haar measure* if it is defined by $\bar{\mu}(V_p(s)) = \rho$ for all $s \in S_r(0)$ and $\rho \in p^Z$ such that $V_p(s) \subset S_r(0)$. Note that $S_r(0) =$ $V_r(0) \{ V ¿_{\frac{r}{p}}(0)$. So, we have $\bar{\mu}(S_r(0)) = r\left(1 - \frac{1}{p}\right)$. We consider normalized (probability) Haar measure:

$$\mu ¿ s ¿ ¿ = \frac{\bar{\mu}(V_p(s))}{\bar{\mu}(S_r(0))} = \frac{p\rho}{(p-1)r}.$$

It is easy to see that $|f(x) - x|_p$ depends on $r = |x|_p$, but does not depend on $x \in S_r(0)$, therefore we define $\rho(r) = |f(x) - x|_p$, if $x \in S_r(0)$.

Theorem 3. The dynamical system $(S_r(0), B, \mu, f)$, has the following properties:

- The function f is measure-preserving with respect to Haar measure.
- For any initial point $x \in S_r(0)$ (epect fixed point) the orbit $\{f^n(x) \vee n \in N\}$ is not convergent.
- The ball with radius $\rho(r)$ is minimial invariant ball for f.
- The dynamical system $(S_r(0), B, \mu, f)$ is not ergodic for all $p \geq 3$.
- The dynamical system $(S_r(0), B, \mu, f)$ is ergodic if and only if $r = 2\rho(r)$ for $p = 2$.

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