# CONTROL PROBLEM OF HEAT PROCESS IN A ONE-DIMENSIONAL DOMAIN

F.N. Dekhkonov, N.Y. Tulaboeva
Namangan State University, PhD, Associate Professor
Namangan State University, Master student
E-mail: f.n.dehqonov@mail.ru

**Abstract:** In this paper, we consider the control problem associated with the heat transfer process in a bounded one-dimensional domain. Finding an admissible control function is required to achieve a given average temperature in the domain. The existence of such an admissible control was proved using the Laplace transform method.

**Keywords:** Parabolic equation; integral equation; control problem; boundary value problem; Laplace transform; admissible control; heat process; mathematical model.

# ЗАДАЧА УПРАВЛЕНИЯ ТЕПЛОВЫМ ПРОЦЕССОМ В ОДНОМЕРНОЙ ОБЛАСТИ

**Ф.Н. Дехконов, Н.Й. Тулабоева**
Наманганский государственный университет, кандидат наук, доцент
Наманганский государственный университет, магистрант
E-mail: f.n.dehqonov@mail.ru

**Аннотация:** В данной работе рассматривается задача управления, связанная с процессом теплопередачи в одномерной ограниченной сфере. Требуется найти приемлемую функцию управления для достижения заданной средней температуры в полевых условиях. Существование такого управления было доказано с помощью метода замены Лапласа.

**Ключевые слова:** Параболическое уравнение; интегральное уравнение; задача управления; краевая задача; преобразование Лапласа; допустимый контроль; тепловой процесс; математическая модель.

# BIR O‘LCHOVLI SOHADA ISSIQLIK ALMASHINISHINI BOSHQARISH MASALASI

F.N. Dexqonov, N.Y. To‘laboyeva
Namangan davlat universiteti, fizika-matematika fanlari bo‘yicha falsafa doktori (PhD), dotsent
Namangan davlat universiteti, magistrant
E-mail: f.n.dehqonov@mail.ru

**Annotatsiya:** Ushbu maqolada biz bir o‘lchovli chegaralangan sohada issiqlik almashinish jarayoni bilan bilan bog‘liq boshqaruv masalasini ko‘rib chiqamiz. Sohadagi berilgan o‘rtacha temperaturaga erishish uchun joiz boshqaruv funksiyasini topish talab qilinadi. Laplas almashtirish usuli yordamida bunday boshqaruv funksiyasi mavjudligi isbotlangan.

**Kalit so‘zlar:** Parabolik tenglama; integral tenglama; boshqaruv masalasi; chegaraviy qiymat masalasi; Laplas almashtirishi; joiz boshqaruv; issiqliq almashinish jarayoni; matematik model.

## 1. Introduction

Consider the heat exchange process along the domain $\Omega = \{(x,t):\ 0 < x < \pi,\ \ t > 0\}$:

$$u_t(x,t) = u_{xx}(x,t), \quad (x,t) \in \Omega, \tag{1.1}$$

with boundary value conditions

$$u(0,t) = \mu(t), \qquad u_x(\pi,t) = 0, \qquad t \geq 0, \tag{1.2}$$

and initial value condition

$$u(x,0) = 0, \quad 0 \leq x \leq \pi. \tag{1.3}$$

We say that the function $\mu(t)$ is an *admissible control,* if this function is differentiable on the half-line $t \geq 0$ and satisfies the following constraints

$$\mu(0) = 0, \quad |\mu(t)| \leq 1, \quad t \geq 0.$$

One of the urgent problems for the equations of mathematical physics is the problem of mathematical modeling of processes associated with various partial differential equations. In particular, mathematical modeling of the heat exchange process and the control of this process. Control in this situation is made by changing the heat flux entering to the region under consideration from a part of its boundary. It is natural to achieve an average temperature in the whole area. Therefore, it is important to control the boundary flow to reach the average temperature in any part of the area, and in the case of a delta-like distribution, at a fixed point in the area.

In this article, we consider the control problem associated with a parabolic equation. On part of the boundary of the segment $[0,\pi]$ there is a source with a given flow. It is required to find such a mode of operation of the source so that the average value of the solution in some part of the segment $[0,\pi]$ takes the specified value.

In the present work we consider the following problem:

**Control problem.** *For a given function* $\theta(t)$ *problem consist looking for the admissible control* $\mu(t)$ *such that the solution* $u(x,t)$ *of the initial-boundary value problem (1.1)-(1.3) exists and satisfies the following equation*

$$\int_0^l u(x,t)dx = \theta(t) \tag{1.4}$$

From the increasing interest in physics and mathematics, a lot of effects have been devoted to the studies of boundary control problems for the heat equation in recent years. The optimal control problem for the second order parabolic type equations was studied by Friedman and Fattorini [1, 2]. Time-optimal problems with control on the boundary for the second order parabolic equation have been treated by Egorov [3], and he proved a Bang-bang principle in the special case.

The boundary control problem for a heat equation with a piecewise smooth boundary in a n-dimensional domain was studied by Albeverio and Alimov [4, 5] and an estimate for the minimum time required to reach a given average temperature was found. In [6], the boundary control problem with a positive weight function placed under the integral in the n-dimensional domain for the homogeneous heat conduction equation was studied.

The latest results on boundary control problems for the heat conduction equation are studied in works [7-10], [11-13]. These articles are mainly devoted to the problems of finding the boundary control function for the heat transfer equation in one and two-dimensional domain. In [14], an estimate of the minimum time for a given average temperature in a two-dimensional domain was obtained.

Boundary control problems for pseudo-parabolic equations are studied in works [15-17]. In these works, the control problem is studied for the pseudo-parabolic type equation, but the proof of the control function is proved using the Laplace transform method.

A lot of information about optimal control problems is given in detail in the monographs of Friedman, Fursikov and Lions [18-20]. General numerical optimization and optimal boundary control have been studied in a great number of publications such as [21]. The practical approaches to optimal control of the heat equation are described in publications like [22].

In our work, it is proved that the control function exists. In Section 2, the given problem is reduced to the Volterra integral equation of the first kind. Finally, required estimate for the kernel is obtained and proved the existence of the control function using the Laplace transform method in the section 3.

## 2. Main integral equation

We consider the eigenvalue problem

$$X''(x)+\lambda X(x)=0, \quad 0<x<\pi,$$

with boundary conditions

$$X(0)=0, \quad X'(\pi)=0, \quad 0\le x\le \pi.$$

Then we get

$$\lambda_k=\left(\frac{2k+1}{2}\right)^2, \qquad X_k=\sin\frac{2k+1}{2}x, \quad k=0,1,....$$

**Definition 2.1.** *By the solution of the problem (1.1)-(1.3) we understand the function* $u(x,t)$ *represent in the form*

$$u(x,t)=\mu(t)-w(x,t) \tag{2.1}$$

*where the function* $w(x,t)$ *with the regularity* $w(x,t)\in C^{2,1}_{x,t}(\Omega)\cap C(\overline{\Omega})$ *is the solution to the problem:*

$$w_t(x,t)=w_{xx}(x,t)+\mu'(t),$$

*with boundary condition*

$$w(0,t)=0, \quad w_x(\pi,t)=0,$$

*and initial condition*

$$w(x,0)=0, \qquad 0\le x\le\pi.$$

Consequently, we have (see, [23, 24])

$$w(x,t)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\int_0^t\mu'(s)e^{-\lambda_k(t-s)}ds\right)\sin\frac{2k+1}{2}x. \tag{2.2}$$

According to the solution (2.2) and from (2.1) we have solution of the initial-boundary problem (1.1)-(1.3)

$$u(x,t)=\mu(t)-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\int_0^t\mu'(s)e^{-\lambda_k(t-s)}ds\right)\sin\frac{2k+1}{2}x, \tag{2.3}$$

where $\lambda_k=\left(\frac{2k+1}{2}\right)^2$.

Using the solution (2.3) and the condition (1.6) and the properties of the function $\mu(t)$, we can write the following equation

$$\theta(t)=\int_0^\pi u(x,t)dx=\pi\mu(t)-\frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\int_0^t e^{-\lambda_k(t-s)}\mu'(s)ds=$$

$$=\pi\mu(t)-\frac{8\mu(t)}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}+\frac{2}{\pi}\sum_{k=0}^{\infty}\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds. \tag{2.4}$$

Note that,

$$\frac{\pi^2}{8}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \quad \text{or} \quad \pi=\frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}. \tag{2.5}$$

From (2.4) and (2.5), we get

$$\frac{2}{\pi}\sum_{k=0}^{\infty}\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds=\theta(t).$$

We set

$$B(t)=\frac{2}{\pi}\sum_{k=0}^{\infty}e^{-\lambda_k t}. \tag{2.6}$$

Then we get the main integral equation

$$\int_0^t B(t-s)\mu(s)ds=\theta(t), \qquad t>0. \tag{2.7}$$

**Proposition 2.1.** *For the function* $B(t)$ *defined by (2.6) the following estimate*

$$0<B(t)\le\frac{C}{\sqrt{t}},\ \ 0<t\le 1,$$

*is valid, where* $C=const>0$.

**Proof.** It is clear that $B(t)>0$. Then we obtain

$$B(t)\le const\sum_{k=0}^{\infty}e^{-\lambda_k t}=const\sum_{k=0}^{\infty}e^{-(2k+1/2)^2 t}\le\frac{C}{\sqrt{t}},$$

Proposition 2.1 is proved.

## 3. Solution of the Control problem

Denote by $W(M)$ the set of function $\theta\in W_2^2(-\infty,+\infty)$, $\theta(t)=0$ for $t\le 0$ which satisfies the condition

$$\|\theta(t)\|_{W_2^2(R_+)}\le M$$

**Theorem 3.1.** *There exists* $M>0$ *such that for any function* $\theta\in W(M)$ *the solution* $\mu(t)$ *of the equation (2.7) exists and satisfies condition*

$$|\mu(t)|\le 1.$$

The solve equation (2.7), we use the Laplace transform method. We introduce the notation

$$\tilde{\mu}(p)=\int_0^{\infty}e^{-pt}\mu(t)dt.$$

Using the convolution property from equation (2.7) we obtain

$$\tilde{B}(p)\tilde{\mu}(p)=\tilde{\theta}(p).$$

Consequently,

$$\tilde{\mu}(p) = \frac{\tilde{\theta}(p)}{\tilde{B}(p)}, \text{ where } p = a + i\xi, \quad a > 0,\ \xi \in R,$$

and

$$\mu(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\tilde{\theta}(p)}{\tilde{B}(p)} e^{pt} dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{\theta}(a+i\xi)}{\tilde{B}(a+i\xi)} e^{(a+i\xi)t} d\xi. \tag{3.1}$$

It is clear that

$$\int_0^\infty e^{-\lambda_k t} e^{-pt} dt = \frac{1}{p+\lambda_k}.$$

We can write

$$\tilde{B}(p) = \int_0^\infty B(t) e^{-pt} dt = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{p+\lambda_k},$$

where $B(t)$ defined by (2.6) and we have

$$\tilde{B}(a+i\xi) = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{a+i\xi+\lambda_k} = \operatorname{Re}\tilde{B}(a+i\xi) + \operatorname{Im}\tilde{B}(a+i\xi),$$

where

$$\operatorname{Re}\tilde{B}(a+i\xi) = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{a+\lambda_k}{(a+\lambda_k)^2+\xi^2},$$

$$\operatorname{Im}\tilde{B}(a+i\xi) = -\frac{2\xi}{\pi}\sum_{k=0}^{\infty} \frac{1}{(a+\lambda_k)^2+\xi^2}.$$

We know that

$$(a+\lambda_k)^2 + \xi^2 \le \left((a+\lambda_k)^2+1\right)\left(1+\xi^2\right),$$

or

$$\frac{1}{(a+\lambda_k)^2+\xi^2} \ge \frac{1}{1+\xi^2} \cdot \frac{1}{(a+\lambda_k)^2+1}.$$

Consequently, we get the estimate

$$\left|\operatorname{Re}\tilde{B}(a+i\xi)\right| = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{a+\lambda_k}{(a+\lambda_k)^2+\xi^2} \ge$$

$$\ge \frac{2}{\pi}\frac{1}{1+\xi^2}\sum_{k=0}^{\infty} \frac{a+\lambda_k}{(a+\lambda_k)^2+1} = \frac{C_{1a}}{1+\xi^2}, \tag{3.2}$$

and

$$\left|\operatorname{Im}\tilde{B}(a+i\xi)\right| = \frac{2|\xi|}{\pi}\sum_{k=0}^{\infty} \frac{1}{(a+\lambda_k)^2+\xi^2} \ge$$

$$\geq \frac{2}{\pi}\frac{|\xi|}{1+\xi^2}\sum_{k=0}^{\infty}\frac{1}{(a+\lambda_k)^2+1}=\frac{|\xi| C_{2a}}{1+\xi^2}, \tag{3.3}$$

where

$$C_{1a}=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{a+\lambda_k}{(a+\lambda_k)^2+1}, \qquad C_{2a}=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(a+\lambda_k)^2+1}.$$

From (3.2) and (3.3), we have

$$\left|\tilde{B}(a+i\xi)\right| \geq \frac{C_{3a}}{\sqrt{1+\xi^2}}, \tag{3.4}$$

where $C_{3a}=\min\{C_{1a}, C_{2a}\}$.

Then, when $a \to 0$ from (3.1) we obtain the equality

$$\mu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\theta}(i\xi)}{\tilde{B}(i\xi)}e^{i\xi t}d\xi. \tag{3.5}$$

**Lemma 3.1.** *Let* $\theta(t)\in W(M)$. *Then for the image of the function* $\theta(t)$ *the inequality*

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\xi)\right|\sqrt{1+\xi^2}\,d\xi<\infty$$

*is valid.*

**Proof.** Using the integral representation of the image of a given function $\theta(t)$ and integration by parts we get

$$\tilde{\theta}(a+i\xi)=\int_0^{\infty}e^{-(a+i\xi)t}\theta(t)dt=\theta(t)\frac{e^{-(a+i\xi)t}}{-a-i\xi}\Bigg|_{t=0}^{t=\infty}+\frac{1}{a+i\xi}\int_0^{\infty}e^{-(a+i\xi)t}\theta'(t)dt,$$

then

$$(a+i\xi)\tilde{\theta}(a+i\xi)=\int_0^{\infty}e^{-(a+i\xi)t}\theta'(t)dt.$$

Therefore, when $a \to 0$ we have

$$i\xi\tilde{\theta}(i\xi)=\int_0^{\infty}e^{-i\xi t}\theta'(t)dt,$$

and

$$(i\xi)^2\,\tilde{\theta}(i\xi)=\int_0^{\infty}e^{-i\xi t}\theta''(t)dt.$$

Consequently,

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\xi)\right|\sqrt{1+\xi^2}\,d\xi=\int_{-\infty}^{+\infty}\frac{\left|\tilde{\theta}(i\xi)\right|(1+\xi^2)}{\sqrt{1+\xi^2}}d\xi \leq$$

$$\le \left( \int_{-\infty}^{+\infty} \left| \tilde{\theta}(i\xi) \right|^2 \left(1+\xi^2\right)^2 d\xi \right)^{1/2} \left( \int_{-\infty}^{+\infty} \frac{d\xi}{1+\xi^2} \right)^{1/2} \le C \|\theta\|_{W_2^2(R_+)}.$$

Lemma 3.1 is proved.

**Proof of theorem 3.1.** According to (3.4), (3.5) and lemma 3.1, we can write

$$|\mu(t)| \le \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\left|\tilde{\theta}(i\xi)\right|}{\left|\tilde{B}(i\xi)\right|} d\xi \le \frac{1}{2\pi C_0} \int_{-\infty}^{+\infty} \left|\tilde{\theta}(i\xi)\right| \sqrt{1+\xi^2}\, d\xi < \infty.$$

Then, we get the estimate

$$|\mu(t)| \le \frac{1}{2\pi C_0} C \|\theta\|_{W_2^2(R_+)} \le \frac{CM}{2\pi C_0} = 1,$$

where $M = \dfrac{2\pi C_0}{C}$.

Theorem 3.1 is proved.

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