Surface at a constant distance from the edge of regression on a transfer surface of Type 1 in isotropic space. Monge-Ampere equation.

Xolmurodova Gulnoza Narimon qizi

Tashkent State Transport University, Tashkent, Uzbekistan.

xolmurodovagulnoza3@gmail.com

Annotation: The main goal of this work is to recover surfaces generated from transfer surfaces by their total curvature being a product of two functions with separate variables. This problem was solved for three cases in isotropic space and equations of the surfaces at a constant distance from the edge of regression on a transfer surface of type 1 were found for each case separately. Moreover, the surface equation was found by solving the parabolic Monge-Ampere equation for these surfaces.

Keywords: Isotropic space, transfer surfaces, surfaces at a constant distance from the edge of regression on a transfer surface, total and mean curvatures, parabolic Monge-Ampere equation.

Izotrop fazoda 1-turdagi ko'chiruvchi sirt regressiya chetidan doimiy masofada joylashgan sirt. Monge-Amper tenglamasi.

Xolmurodova Gulnoza Narimon qizi

Toshkent Davlat Transport Universiteti, Toshkent, O’zbekiston.

xolmurodovagulnoza3@gmail.com

Annotatsiya: Ushbu ishning asosiy maqsadi ko’chiruvchi sirtdan hosil qilingan sirtlarning to’la egriligi ikkita alohida o’zgaruvchili funksiyaning ko’paytmasi bo’lgan holda tenglamasini topish. Bu masala izotrop fazoda 1-turdagi ko’chiruvchi sirt regressiya chetidan doimiy masofada joylashgan sirtlarning tenglamasi uchta holda alohida topilgan. Bundan tashqari, bu sirtlar uchun parabolik Monge-Ampere tenglamasini yechish orqali sirt tenglamasi topilgan.

Kalitso’zlar: Izotrop fazo, ko’chiruvchi sirtlar, ko’chiruvchi sirt regressiya chetidan doimiy masofada joylashgan sirtlar, to’la va o’rta egriliklar, parabolik Monge-Ampere tenglamasi.

Поверхность на постоянном расстоянии от края регрессии на поверхности переноса типа 1 в изотропном пространстве. Уравнение Монжа-Ампера.

Холмуродова Гульноза Наримон қизи

Ташкентский Государственный Транспортный Университет,Ташкент, Узбекистан.

xolmurodovagulnoza3@gmail.com

Аннотация: Основная цель данной работы — восстановить поверхности, созданные из поверхностей переноса, поскольку их полная кривизна является произведением двух функций с отдельными переменными. Эта задача решалась для трех случаев в изотропном пространстве и уравнения поверхностей на постоянном расстоянии от края регрессии на поверхность переноса типа 1 были найдены для каждого случая отдельно. Более того, уравнение поверхности было найдено путем решения параболического уравнения Монжа-Ампера для этих поверхностей.

Ключевые слова: изотропное пространство, поверхности переноса, поверхности на постоянном расстоянии от края регрессии на поверхности переноса, полная и средняя кривизны, параболическое уравнение Монжа-Ампера.

Introduction

Initial concepts of isotropic space are given in the works of K.Strubecker.[1] In the work of H.Sachs [2] described groups of the motion in the isotropic space. Currently, the differential geometry of surfaces is studying by several geometers in this space. An isotropic space is a space with a degenerate metric. It differs from Euclidian space.[3] In addition, the total and mean curvatures of the surfaces in isotropic space differ from the Euclidian case. If the surface $z = z(x, y)$ is uniquely projected onto the Oxy plane, then the total curvature is the following form:

$K = z_{xx} z_{yy} - z_{xy}^2$. The right side of this formula is called the Monge-Ampere equation in this space. [4] Hence, some classes of surfaces were found in the case the total curvature is zero or equals to a nonzero constant. In particular, it was well considered for transfer surfaces.[5,6,8] M.E.Aydin classified transfer surfaces with constant total curvature in the isotropic space.[5] Besides that, Sh. Ismoilov solved the surface recovering problem for the case the total curvature is product of two separate variable functions [6]. In addition, in the article [7] surfaces at a constant distance from the edge of regression on a surface was studied in Euclidian space. The equations of the surfaces generated by translation surfaces of type 1 were found for the case the total curvature $K^h = 0$ in the isotropic space in the article of M.K.Karacan and others [8]. In this paper, the equations of the surfaces at a constant distance from the edge of regression on a transfer surface of type 1 are found for the case $K^h = \varphi(x)\psi(y)$. And the parabolic Monge-Ampere equation is solved.

Preliminaries.

Basic concepts of an isotropic space R_3^2

Let there be given an affine space A_3 with the coordinate system $Oe_1e_2e_3$. $X(x_1, y_1, z_1)$ and $Y(x_2, y_2, z_2)$ be vectors in A_3.

Definition 1. If the scalar product of the vectors $X(x_1, y_1, z_1)$ and $Y(x_2, y_2, z_2)$ is defined by the following:

$$\begin{cases} (X, Y)_1 = x_1x_2 + y_1y_2 & \text{if } x_1x_2 + y_1y_2 \neq 0 \\ (X, Y)_2 = z_1z_2 & \text{if } x_1x_2 + y_1y_2 = 0 \end{cases} \qquad (1)$$

then A_3 is called an isotropic space R_3^2. [9]

It is known that the norm of the vector X is defined by the formula $|\vec{X}| = \sqrt{(\vec{X}, \vec{X})}$ and the distance d between points $M(x_1, y_1, z_1)$ and $N(x_2, y_2, z_2)$

$$d=\begin{cases}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} & \text{if } \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\neq 0\\ |z_2-z_1| & \text{if } \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=0\end{cases} \quad (2)$$

The metric defined by this way is called a degenerate metric. Geometry in a plane of an isotropic space is Euclidian, if it is not parallel to the axis Oz. If it is parallel to the Oz, then the geometry on it is the Galilean.[10] The motion of the isotropic space is given by the following formula:

$$\begin{cases}x'=x\cos\alpha-y\sin\alpha+a\\ y'=x\sin\alpha+y\cos\alpha+b\\ z'=h_1x+h_2y+z+c\end{cases}. \quad (3)$$

Consider a regular surface in the R_3^2 and its vector equation is given by the following:

$$r(u,v)=x(u,v)i+y(u,v)j+z(u,v)k. \quad (4)$$

Then the total and mean curvatures of this surface are defined as in the Euclidian case:

$$K=\frac{LN-M^2}{EG-F^2} \qquad 2H=\frac{EN-2FM+GL}{EG-F^2} \quad (5)$$

where, E,F,G and L,M,N are the coefficients of the first and second fundamental forms of the surface, relatively. If the surface Φ is given an explicit form $z=z(x,y)$, which is the one-valued projection onto the Oxy plane, then for the total and mean curvatures of this surface following is valid:

$$K=z_{xx}z_{yy}-z_{xy}^2, \qquad 2H=z_{xx}+z_{yy} \quad (6)$$

Where, the right side of the above formulas Monge-Ampere and Laplacian operators, respectively.[4]

Transfer surfaces in the isotropic space R_3^2.

M.E.Aydin classified types of transfer surfaces in isotropic space.[11] It is known that the transfer surfaces of type 1 are obtained by translating two planar curves. The surface parametrization is given by

$$r(u,v)=\alpha(u)+\beta(v) \quad (7)$$

The transfer surfaces of type 1 is defined by the one-valued projection onto the Oxy plane for the case both $\alpha(u)$, $\beta(v)$ are isotropic planar curves. And vector equation of this surface is

$$r(u,v)=ui+vj+(f(u)+g(v))k$$

where, $\alpha(u)=(u,0,f(u))$ and $\beta(v)=(0,v,g(v))$.[3]

Surfaces at a constant distance from the edge of regression on a transfer surface of type 1 in R_3^2.

Let the surface Φ be admissible surface which is not isotropic tangent planes in isotropic space. n_P be an isotropic unit normal vector at a point P of the surface Φ. Take a unit vector at a point P

$$m_P = d_1 r_u + d_2 r_v + d_3 n_P \qquad (8)$$

where r_u, r_v are tangent vectors at P and $d_1^2+d_2^2=1$. If there is a function h defined by

$$h:\Phi\to\Phi^h,\ h(P)=P+lm(P) \qquad (9)$$

where l is constant, then the surface Φ^h is called the surface at a constant distance from the edge of regression on Φ. Φ and Φ^h are shown by the pair (Φ,Φ^h). If $d_1=d_2=0$, then we have $m(P)=l\cdot n(P)$ and so Φ and Φ^h are parallel surfaces. In this paper, we assume that both d_1 and d_2 are not zero at the same time. Let $r(u,v)$ be a parametrization of the transfer surface Φ. In this case, $\{r_u,r_v\}$ is non-isotropic orthonormal bases of the surface Φ. Let $n(P)$ be a isotropic unit normal vector at a point P and $d_1,d_2,d_3\in\mathbb{R}$ be constant numbers. Then we can write a vector equation of Φ^h is

$$r^h(u,v)=r(u,v)+l\cdot m(u,v)$$

Thus we obtain $r^h(u,v)=r(u,v)+l(d_1r_u+d_2r_v+d_3n(0,0,1))$. If we take $ld_1=\eta$ $ld_2=\mu$ $ld_3=\gamma$ where $\eta^2+\mu^2=l^2$. Thus we get

$$r^h(u,v)=r(u,v)+\eta r_u+\mu r_v+\gamma n(0,0,1). \qquad (10)$$

From this, surfaces at a constant distance from the edge of regression on a transfer surface of Type 1 is given by the following formula:[8]

$$r^h(u,v)=\left(u+\eta, v+\mu, \left(f(u)+\eta f'(u)\right)+\left(g(v)+\mu g'(v)\right)+\gamma\right) \quad (11)$$

Main results.

Let there be given the surfaces at a constant distance from the edge of regression on a transfer surface of Type 1 in the isotropic space. And the parametrization of this is defined the above formula (11). In this part we solve the Monge- Ampere equation for the recovering problem of this type of surfaces in the case the total curvature is the product of two arbitrarily functions.

Theorem 1. If the surface at a constant distance from the edge of regression on a transfer surface of Type 1 is given by the vector equation (11) and the total curvature of this surface is $K=\rho(u)\lambda(v)$, then we get the following:

a)If $d_1 \neq 0$ and $d_2 \neq 0$, then

$$\vec{r}^h(u,v)=\left(u+\eta, v+\mu, \tau\iint\left[\int\rho(u)du\right]du+\frac{1}{\tau}\iint\left[\int\lambda(v)dv\right]dv+\eta\left(C_1u+C_2\right)+\mu\left(\tilde{C}_1v+\tilde{C}_2\right)+\gamma\right)$$

(12) where, $\tau, C_1, C_2, \tilde{C}_1, \tilde{C}_2$ - constant numbers.

b) Either $d_1=0$, $d_2 \neq 0 \Rightarrow$

$$\vec{r}^h(u,v)=\left(u, v+\mu, \tau\iint\left[\int\rho(u)du\right]du+\frac{1}{\tau}\iint\left[\int\lambda(v)dv\right]dv+\left(C_1u+C_2\right)+\mu\left(\tilde{C}_1v+\tilde{C}_2\right)+\gamma\right)$$

or $d_1 \neq 0$, $d_2=0 \Rightarrow$ $\vec{r}^h(u,v)=\left(u+\eta, v, \tau\iint\left[\int\rho(u)du\right]du+\frac{1}{\tau}\iint\left[\int\lambda(v)dv\right]dv+\eta\left(C_1u+C_2\right)+\left(\tilde{C}_1v+\tilde{C}_2\right)+\gamma\right)$

Proof: From the formula (11) the following holds:

$$z^h(u,v)=f(u)+\eta f'(u)+g(v)+\mu g'(v)+\gamma \quad (13)$$

Calculating the coefficients of the first and the second fundamental forms, we obtain the following expressions: $E=1,\ F=0,\ G=1$ $L=f''(u)+\eta f'''(u)$, $M=0$, $N=g''(v)+\mu g'''(v)$ From this, the total and mean curvatures are:

$$K=\left(f''(u)+\eta f'''(u)\right)\left(g''(v)+\mu g'''(v)\right) \qquad 2H=\left(f''(u)+\eta f'''(u)\right)+\left(g''(v)+\mu g'''(v)\right) \quad (14)$$

a) $\left(f''(u)+\eta f'''(u)\right)\left(g''(v)+\mu g'''(v)\right)=\rho(u)\lambda(v) \Rightarrow \dfrac{\left(f''(u)+\eta f'''(u)\right)}{\rho(u)}=\dfrac{\lambda(v)}{\left(g''(v)+\mu g'''(v)\right)}=\tau,\ \tau - const$

The above equations are two third order differential equations with separate variable functions.

a.1) $f'''(u)+\dfrac{1}{\eta}f''(u)=\dfrac{\tau}{\eta}\rho(u)$

If we successively integrate this equation, we get the following:

$$f'(u)+\frac{1}{\eta}f(u)=\int\left[\frac{\tau}{\eta}\int\rho(u)du\right]du+C_1u+C_2$$

From this, we solve the homogeneous part of the equation and then use the method of variation of the constant: $f'(u)+\frac{1}{\eta}f(u)=0 \Rightarrow f_{bir.j}(u)=Ce^{-\frac{u}{\eta}} \Rightarrow f(u)=C(u)e^{-\frac{u}{\eta}}$

$$C'(u)e^{-\frac{u}{\eta}}=\int\left[\frac{\tau}{\eta}\int\rho(u)du\right]du+C_1u+C_2 \Rightarrow C'(u)=e^{\frac{u}{\eta}}\int\left[\frac{\tau}{\eta}\int\rho(u)du\right]du+C_1ue^{\frac{u}{\eta}}+C_2e^{\frac{u}{\eta}}$$

$$C(u)=\int e^{\frac{u}{\eta}}\left[\int\left[\frac{\tau}{\eta}\int\rho(u)du\right]du\right]du+C_1\eta e^{\frac{u}{\eta}}(u-\eta)+C_2\eta e^{\frac{u}{\eta}}+C_3$$

Thus, we get the $f(u)=e^{-\frac{u}{\eta}}\int e^{\frac{u}{\eta}}\left[\int\left[\frac{\tau}{\eta}\int\rho(u)du\right]du\right]du+C_1\eta(u-\eta)+C_2\eta+C_3e^{-\frac{u}{\eta}}$

a.2) The above methods are also valid for solving this equation:

$$g''(v)+\mu g'''(v)=\frac{1}{\tau}\lambda(v) \Rightarrow g'(v)+\frac{1}{\mu}g(v)=\frac{1}{\tau\mu}\int\left[\int\lambda(v)dv\right]dv+\tilde{C}_1v+\tilde{C}_2$$

By the method of variation of the constant we get the $g(v)=\tilde{C}(v)e^{-\frac{v}{\mu}}$

$$\tilde{C}'(v)=\frac{1}{\tau\mu}e^{\frac{v}{\mu}}\int\left[\int\lambda(v)dv\right]dv+\tilde{C}_1ve^{\frac{v}{\mu}}+\tilde{C}_2e^{\frac{v}{\mu}} \qquad \tilde{C}(v)=\int\left[\frac{1}{\tau\mu}e^{\frac{v}{\mu}}\int\left[\int\lambda(v)dv\right]dv\right]dv+\tilde{C}_1\mu e^{\frac{v}{\mu}}(v-\mu)+\tilde{C}_2\mu e^{\frac{v}{\mu}}$$

From this, $g(v)=e^{-\frac{v}{\mu}}\int\left[\frac{1}{\tau\mu}e^{\frac{v}{\mu}}\int\left[\int\lambda(v)dv\right]dv\right]dv+\tilde{C}_1\mu(v-\mu)+\tilde{C}_2\mu$

After putting the corresponding expressions in formula (12), we get the surface equation (12).

b) If $d_1=0$, $d_2\neq 0$ then $\eta=0$ thus the surface total curvature is:

$K=f''(u)(g'(v)+\mu g''(v)) \Rightarrow f''(u)(g'(v)+\mu g''(v))=\rho(u)\lambda(v)$ by applying the above methods in the case a), then we obtain:

$f(u)=\tau\int\left[\int\rho(u)du\right]du+C_1u+C_2$ and $g(v)=e^{-\frac{v}{\mu}}\int\left[\frac{1}{\tau\mu}e^{\frac{v}{\mu}}\int\left[\int\lambda(v)dv\right]dv\right]dv+\tilde{C}_1\mu(v-\mu)+\tilde{C}_2\mu$

Thus, $\vec{r}^{\,b}(u,v)=\left|u,v+\mu,\tau\int\left[\int\rho(u)du\right]du+\frac{1}{\tau}\int\left[\int\lambda(v)dv\right]dv+(C_1u+C_2)+\mu(\tilde{C}_1v+\tilde{C}_2)+\gamma\right|$

In the case, $d_1\neq 0$, $d_2=0 \Rightarrow \mu=0$. Since the solution is symmetric with respect to u and v, the following is valid:

$$\vec{r}^{\,h}(u,v)=\left(u+\eta, v, \tau\int\left[\int\rho(u)du\right]du+\frac{1}{\tau}\int\left[\int\lambda(v)dv\right]dv+\eta(C_1u+C_2)+(\tilde{C}_1v+\tilde{C}_2)+\gamma\right)$$

Theorem is proved.

Theorem 2. If the surfaces at a constant distance from the edge of regression on a transfer surface of Type 1 satisfies the parabolic Monge-Ampere equation

$$z_{xx}z_{yy}-z_{xy}^2=z_x \tag{15}$$

then we get the following for the given surface:

$$z^h(u,v)=C_1(1+\tau\eta)e^{\tau u}+C_2+\frac{v^2}{2\tau}+\tilde{C}_1\mu v+\tilde{C}_2\mu+\gamma \tag{16}$$

where, $C_1, C_2, \tilde{C}_1, \tilde{C}_2, \tau$ - constants.

Proof: Let there be given the regular surface Φ^h and its parametrization is:

$$z^h(u,v)=f(u)+\eta f'(u)+g(v)+\mu g'(v)+\gamma \tag{17}$$

Let this satisfies parabolic Monge-Ampere equation:

$$(f''(u)+\eta f'''(u))(g''(v)+\mu g'''(v))=(f'(u)+\eta f''(u)) \quad \frac{f''(u)+\eta f'''(u)}{f'(u)+\eta f''(u)}=\frac{1}{g''(v)+\mu g'''(v)}=\tau$$, τ - const

We solve these differential equations separately: 1) $\frac{f''+\eta f'''}{f'+\eta f''}=\tau$ and $f'+\eta f''\neq 0$ thus,

$f\neq C_2e^{-\frac{u}{\eta}}+C_1$. $\eta f'''+(1-\tau\eta)f''-\tau f'=0$ this is homogeneous differential equation with constant coefficients. Therefore, we seek the solution in the following form: $f(u)=e^{ku}$

Thus, $e^{ku}\cdot k(\eta k^2+(1-\tau\eta)k-\tau)=0 \Rightarrow k_0=-\frac{1}{\eta}$, $k_1=\tau$, $k_2=0 \Rightarrow f(u)=C_0e^{-\frac{u}{\eta}}+C_1e^{\tau u}+C_2$

$$f(u)+\eta f'(u)=C_1(1+\eta\tau)e^{\tau u}+C_2 \tag{18}$$

2) $\frac{1}{g''+\mu g'''}=\tau \Rightarrow g'+g\cdot\frac{1}{\mu}=\frac{v^2}{2\tau\mu}+\tilde{C}_1v+\tilde{C}_2 \Rightarrow g(v)=\frac{v^2}{2\tau}-v\frac{\mu}{\tau}+\frac{\mu^2}{\tau}+\tilde{C}_1\mu v-\tilde{C}_1\mu^2+\tilde{C}_2\mu$

If we use the variation of constants method, then we get the above expressions:

$$g(v)+\mu g'(v)=\frac{v^2}{2\tau\mu}+\tilde{C}_1\mu v+\tilde{C}_2\mu \tag{19}$$

If we put (18)-(19) into the (17) then we obtain (16).

Theorem is proved.

References:

1. Strubecker K. Differentialgeometrie des isotropin Raumes II.// Math Zeitschrift. 1942. p.743-777.

2. Sachs H. Isotrop Geometri des Raumes.//Vieweg Verlag, Braunchweig.

1990.p.43-50.

3.Artykbaev A, Ismoilov Sh.Sh.. Surface recovering by a given total and mean curvature in isotropic space R_3^2.// Palestine Journal of Mathematics. 2022.p.351-361.

4.Ismoilov Sh.Sh.,Kholmurodova G.N. Equality of surfaces in Euclidian and semi-Euclidian spaces according to geometric characteristics, Monge-Ampere equation.// Bulletin of the Institute of Mathematics. 2023. p.39-46.

5. Aydin M.E. A generalization of translation surfaces with constant curvature in the isotropic space.// Journal of Geometry. 2016. p .603-615.

6.Ismoilov Sh.Sh. Geometry of the Monge-Ampere equation in an isotropic space.// Uzbek Mathematical Journal.2022. p.66-77.

7.Tarakci O., Hacisalioglu H. Surfaces at a constant distance from the edge of regression on a surface.// Applied Mathematics and Computation. 2004. p.81-93.

8. Karacan M.K., Cakmak A, Kiziltug S, Es H.Surfaces generated by translation surfaces of type 1 in I_3^1.// Iranian Journal of Mathematical Sciences and Informatics. 2021.p.123-135.

9. Artykbaev A, Sokolov D.D. Geometry as a whole in space time.// Tashkent. Fan 1991.

10. Artykbaev A,Ismoilov Sh.Sh.Special mean and total curvature of a dual surface in isotropic spaces.// International Electronic Journal of Geometry. 2022.p.1-10.

11. Aydin M.E. Classifications of translation surfaces in isotropic geometry with constant curvature.// Ukrainian Mathematical Journal 2020. p.329-347.