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CONTROL PROBLEM FOR A PSEUDO-PARABOLIC EQUATION WITH NEUMANN CONDITION

Abstract: In this paper, we consider the control problem involving the pseudo-parabolic equation in a bounded one-dimensional domain. Finding an admissible control function is required to achieve a given average temperature in the domain. The existence of such an admissible control was proved using the Laplace transform method.

Keywords: psevdo-parabolic equation; integral equation; control problem; boundary value problem; Laplace transform; admissible control; heat process; mathematical model.

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ЗАДАЧА УПРАВЛЕНИЯ ДЛЯ ПСЕВДОПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С УСЛОВИЕМ НЕЙМАНА

Аннотация: В данной работе рассматривается задача управления псевдопараболическим уравнением в ограниченной одномерной области. Требуется найти приемлемую функцию управления для достижения заданной средней температуры в полевых условиях. Существование такого управления было доказано с помощью метода замены Лапласа.

Ключевые слова: псевдо параболическое уравнение; интегральное уравнение; задача управления; краевая задача; преобразование Лапласа; допустимый контроль; тепловой процесс; математическая модель.

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NEYMAN SHARTLI PSEVDO PARABOLIK TENGLAMA UCHUN BOSHQARUV MASASI

Annotatsiya: Ushbu maqolada biz chegaralangan bir o'lchovli sohada psevdoparabolik tenglamani boshqarish masalasini ko'rib chiqamiz. Sohadagi berilgan o‘rtacha temperaturaga erishish uchun joiz boshqaruv funksiyasini topish talab qilinadi. Laplas almashtirish usuli yordamida bunday boshqaruv funksiyasi mavjudligi isbotlangan.

Kalit so‘zlar: psevdo parabolik tenglama; integral tenglama; boshqaruv masalasi; chegaraviy qiymat masalasi; Laplas almashtirishi; joiz boshqaruv; issiqliq almashinish jarayoni; matematik model.

1. Introduction

Consider the pseudo-parabolic equation in the domain $\Omega=\{(x,t):\ 0<x<l,\ t>0\}$:

$$u_t(x,t)=u_{xx}(x,t)+u_{xxt}(x,t),\quad (x,t)\in\Omega, \tag{1.1}$$

with boundary value conditions

$$u_x(0,t)=\mu(t),\qquad u_x(l,t)=0, \tag{1.2}$$

and initial value condition

$$u(x,0)=0,\quad 0\le x\le l. \tag{1.3}$$

We say that the function $\mu(t)$ is an *admissible control,* if this function is differentiable on the half-line $t\ge 0$ and satisfies the following constraints

$$\mu(0)=0,\quad |\mu(t)|\le 1,\quad t\ge 0.$$

We consider the weight function $\rho(x)\in W_2^2(0,l)$, which satisfy the following conditions

$$\rho'(x)\le 0,\quad \rho''(x)\ge 0,$$

$$\frac{1}{l}\int_0^l \rho(x)dx=1. \tag{1.4}$$

Let

$$\rho(x)=\sum_{k=0}^{\infty}\rho_k\cos\frac{\pi kx}{l},$$

where

$$\rho_k=\frac{2}{l}\int_0^l \rho(x)\cos\frac{\pi kx}{l}dx,\ k=1,2,\ldots.$$

$$\rho_0=\frac{1}{l}\int_0^l \rho(x)dx=1, \tag{1.5}$$

In the present work we consider the following problem:

Control problem. *For a given function $\theta(t)$ problem consist looking for the admissible control $\mu(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem (1.1)-(1.3) exists and satisfies the following equation*

$$\int_0^l \rho(x)u(x,t)dx = \theta(t). \qquad (1.6)$$

From the increasing interest in physics and mathematics, a lot of effects have been devoted to the studies of boundary control problems for the heat equation in recent years. The optimal control problem for the second order parabolic type equations was studied by Friedman and Fattorini [1, 2]. Time-optimal problems with control on the boundary for the second order parabolic equation have been treated by Egorov [3], and he proved a Bang-bang principle in the special case.

The boundary control problem for a heat equation with a piecewise smooth boundary in a n-dimensional domain was studied by Albeverio and Alimov [4, 5] and an estimate for the minimum time required to reach a given average temperature was found. In [6], the boundary control problem with a positive weight function placed under the integral in the n-dimensional domain for the homogeneous heat conduction equation was studied.

The latest results on boundary control problems for the heat conduction equation are studied in works [7-10], [11-13]. These articles are mainly devoted to the problems of finding the boundary control function for the heat transfer equation in one and two-dimensional domain. In [14], an estimate of the minimum time for a given average temperature in a two-dimensional domain was obtained.

Boundary control problems for pseudo-parabolic equations are studied in works [15-17]. In these works, the control problem is studied for the pseudo-parabolic type equation, but the proof of the control function is proved using the Laplace transform method.

A lot of information about optimal control problems is given in detail in the monographs of Friedman, Fursikov and Lions [18-20]. General numerical optimization and optimal boundary control have been studied in a great number of publications such as [21]. The practical approaches to optimal control of the heat equation are described in publications like [22].

In our work, it is proved that the control function exists. In Section 2, the given problem is reduced to the Volterra integral equation of the first kind. Finally, required estimate for the kernel is obtained and proved the existence of the control function using the Laplace transform method in the section 3.

2. Main integral equation

We consider the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < l,$$

with boundary conditions

$$X'(0) = 0, \quad X'(\pi) = 0, \quad 0 \le x \le l.$$

Then we get

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2, \qquad X_k = \cos\frac{\pi k x}{l}, \quad k = 0,1,...$$

Definition 2.1. *By the solution of the problem (1.1)-(1.3) we understand the function* $u(x,t)$ *represent in the form*

$$u(x,t) = \mu(t)\frac{(l-x)^2}{2l} - w(x,t) \tag{2.1}$$

where the function $w(x,t)$ *with the regularity* $w(x,t) \in C^{2,1}_{x,t}(\Omega) \cap C(\overline{\Omega})$, $w_x \in C(\overline{\Omega})$ *is the solution to the problem:*

$$w_t(x,t) - w_{xx}(x,t) - w_{xxt} = \frac{(l-x)^2}{2l}\mu'(t) - \frac{1}{l}$$

with boundary condition

$$w_x(0,t) = 0, \quad w_x(l,t) = 0,$$

and initial condition

$$w(x,0) = 0, \quad 0 \le x \le l.$$

Consequently, we have (see [23, 24])

$$w(x,t) = \frac{l}{6}\mu(t) - \frac{1}{l}\mu(t) - \frac{1}{l}\int_0^t \mu(s)ds + \\ + \frac{2l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(1+\lambda_k)k^2}\times \\ \times\left(\int_0^t \mu'(s)e^{-q_k(t-s)}ds\right)\cos\frac{\pi k x}{l} \tag{2.2}$$

where $q_k = \dfrac{\lambda_k}{1+\lambda_k}$.

According to the solution (2.2) and from (2.1) we have solution of the initial-boundary problem (1.1)-(1.3)

$$u(x,t) = \mu(t)\frac{(l-x)^2}{2l} - \frac{l}{6}\mu(t) + \\ + \frac{1}{l}\mu(t) + \frac{1}{l}\int_0^t \mu(s)ds - \\ - \frac{2l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(1+\lambda_k)k^2}\times \\ \times\left(\int_0^t \mu'(s)e^{-q_k(t-s)}ds\right)\cos\frac{\pi k x}{l} \tag{2.3}$$

Using the solution (2.3) and the condition (1.6), we can write

$$\theta(t) = \int_0^l \rho(x)u(x,t)dx = \mu(t)\int_0^l \rho(x)\frac{(l-x)^2}{2l}dx - \\ - \frac{l}{6}\mu(t)\int_0^l \rho(x)dx + \frac{1}{l}\mu(t)\int_0^l \rho(x)dx + \\ + \frac{1}{l}\int_0^l \rho(x)dx\int_0^t \mu(s)ds - \\ - \frac{2l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(1+\lambda_k)k^2}\times \\ \times\left(\int_0^t \mu'(s)e^{-q_k(t-s)}ds\right)\int_0^l \rho(x)\cos\frac{\pi k x}{l}dx. \tag{2.4}$$

By (1.4), (2.4), and the properties of the function $\mu(t)$, have

$$\theta(t) = \int_0^l \rho(x)\frac{(l-x)^2}{2l}dx - \mu(t)\frac{l^2}{6} + \mu(t) + \\ + \int_0^t \mu(s)ds - \frac{l^2}{\pi^2}\sum_{k=0}^{\infty}\frac{\rho_k}{1+\lambda_k}\int_0^t e^{-q_k(t-s)}\mu'(s)ds = \\ = \int_0^l \rho(x)\frac{(l-x)^2}{2l}dx - \mu(t)\frac{l^2}{6} + \mu(t) +$$

$$+\int_0^t \mu(s)ds - \mu(t)\frac{l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{\rho_k}{(1+\lambda_k)k^2} +$$

$$+\frac{l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{\rho_k \lambda_k}{(1+\lambda_k)^2 k^2}\int_0^t e^{-q_k(t-s)}\mu(s)ds \tag{2.5}$$

Note that

$$\frac{(l-x)^2}{2l} = \frac{l}{6} + \frac{2l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos\frac{\pi k x}{l}.$$

According to the condition (1.4) and Parseval equality, we get

$$\int_0^l \rho(x)\frac{(l-x)^2}{2l}dx = \frac{l^2}{6} + \frac{l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{\rho_k}{k^2}. \tag{2.6}$$

By (2.5) and (2.6), we obtain

$$\theta(t) = \mu(t) + \mu(t)\frac{l^2}{\pi^2}\sum_{k=0}^{\infty}\frac{\rho_k \lambda_k}{(1+\lambda_k)k^2} +$$

$$+\frac{l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{\rho_k \lambda_k}{(1+\lambda_k)^2 k^2}\int_0^t e^{-q_k(t-s)}\mu(s)ds.$$

We set

$$B(t) = 1 + \sum_{k=1}^{\infty}\beta_k e^{-q_k t}, \tag{2.7}$$

and

$$\beta_k = \frac{l^2}{\pi^2}\frac{\rho_k \lambda_k}{(1+\lambda_k)^2 k^2},$$

$$\alpha = 1 + \frac{l^2}{\pi^2}\sum_{k=0}^{\infty}\frac{\rho_k \lambda_k}{(1+\lambda_k)k^2}. \tag{2.8}$$

Then we get the main integral equation

$$\alpha\mu(t) + \int_0^t B(t-s)\mu(s)ds = \theta(t), \qquad t > 0. \tag{2.9}$$

Lemma 2.1 [25]. Let $g(y) \ge 0$ and $g'(y) \le 0$. Then the inequality holds

$$\int_0^{n\pi} g(y)\sin y\,dy \ge 0, \quad y \in [0,\infty], \quad n = 1,2,\ldots$$

Lemma 2.2. *For the coefficients* $\{\rho_k\}_{k\in N}$ *defined by (1.5) the estimate*

$$0 \le \rho_k \le \frac{C}{k^2}, \quad k = 1,2,\ldots$$

is valid.

Proof. From (1.5), we write

$$\rho_k = \frac{2}{l}\int_0^l \rho(x)\cos\frac{k\pi x}{l}dx = \frac{2}{k\pi}\rho(x)\sin\frac{k\pi x}{l}\Big|_{x=0}^{x=l} -$$

$$-\frac{2}{k\pi}\int_0^l \rho'(x)\sin\frac{k\pi x}{l}dx = -\frac{2}{k\pi}\int_0^l \rho'(x)\sin\frac{k\pi x}{l}dx. \tag{2.10}$$

By conditions (4) and Lemma 2.1 we obtain $\rho_k \ge 0$. Then, from (2.10) we can write

$$\rho_k = -\frac{2}{k\pi}\int_0^l \rho'(x)\sin\frac{k\pi x}{l}dx =$$

$$= \frac{2l}{k^2\pi^2}\rho'(x)\cos\frac{k\pi x}{l}\Big|_{x=0}^{x=l} -$$

$$-\frac{2l}{k^2\pi^2}\int_0^l \rho''(x)\cos\frac{k\pi x}{l}dx =$$

$$= \frac{2l}{k^2\pi^2}[\rho'(l)(-1)^k - \rho'(0)] + \frac{o(1)}{k^2},$$

where $\rho'(l)(-1)^k - \rho'(0) \ge 0.$

Then we obtain

$$0 \le \rho_k \le \frac{C}{k^2}.$$

Lemma 2.2 is proved.

Preposition 2.1. *A function* $B(t)$ *defined by (2.7) is continuous on the half-line* $t \ge 0.$

Proof. Indeed, from (2.7) and Lemma 2.2 we obtain

$$1 \le B(t) \le 1 + const\sum_{k=1}^{\infty}\frac{1}{k^6}.$$

Preposition 2.1. is proved.

3. Solution of the Control problem

Denote by $W(M)$ the set of function $\theta \in W_2^1(-\infty,+\infty)$, $\theta(t)=0$ for $t\le 0$ which satisfies the condition

$$\|\theta(t)\|_{W_2^1(R_+)}\le M$$

Theorem 3.1. *There exists* $M>0$ *such that for any function* $\theta\in W(M)$ *the solution* $\mu(t)$ *of the equation (2.9) exists and satisfies condition*

$$|\mu(t)|\le 1.$$

The solve equation *(2.9)*, we use the Laplace transform method. We introduce the notation

$$\tilde{\mu}(p)=\int_0^{\infty} e^{-pt}\mu(t)dt.$$

Using the convolution property from equation (2.9) we obtain

$$\alpha\tilde{\mu}(\rho)+\tilde{B}(p)\tilde{\mu}(p)=\tilde{\theta}(p).$$

Consequently,

$$\tilde{\mu}(p)=\frac{\tilde{\theta}(p)}{\alpha+\tilde{B}(p)},$$

where $p=\delta+i\xi,\quad \delta>0,\ \xi\in R$, and

$$\mu(t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\tilde{\theta}(p)}{\alpha+\tilde{B}(p)}e^{pt}dp=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\theta}(\delta+i\xi)}{\alpha+\tilde{B}(\delta+i\xi)}e^{(\delta+i\xi)t}d\xi. \tag{3.1}$$

It is clear that

$$\int_0^{\infty} e^{-q_k t}e^{-pt}dt=\frac{1}{p+q_k}.$$

We can write

$$\tilde{B}(p)=\int_0^{\infty}B(t)e^{-pt}dt=\frac{1}{p}+\sum_{k=0}^{\infty}\frac{\beta_k}{p+q_k},$$

where $B(t)$ defined by (2.6) and we have

$$\alpha+\tilde{B}(\delta+i\xi)=\alpha+$$

$$+\frac{1}{\delta+i\xi}+\sum_{k=1}^{\infty}\frac{\beta_k}{\delta+i\xi+q_k}=$$

$$\mathrm{Re}(\alpha+\tilde{B}(\delta+i\xi))+\mathrm{Im}(\alpha+\tilde{B}(\delta+i\xi)),$$

where

$$\mathrm{Re}(\alpha+\tilde{B}(\delta+i\xi))=\alpha+$$

$$+\frac{\delta}{\delta^2+\xi^2}+\sum_{k=1}^{\infty}\frac{\beta_k(\delta+q_k)}{(\delta+q_k)^2+\xi^2},$$

$$\mathrm{Im}\,\tilde{B}(\delta+i\xi)=-\frac{\xi}{\delta^2+\xi^2}-$$

$$-\frac{2\xi}{\pi}\sum_{k=1}^{\infty}\frac{1}{(\delta+q_k)^2+\xi^2}.$$

We know that

$$(\delta+q_k)^2+\xi^2\le\left((\delta+q_k)^2+1\right)\left(1+\xi^2\right)$$

,

or

$$\frac{1}{(\delta+q_k)^2+\xi^2}\ge\frac{1}{1+\xi^2}\cdot\frac{1}{(\delta+q_k)^2+1},$$

and

$$\frac{1}{\delta^2+\xi^2}\ge\frac{1}{1+\xi^2}\cdot\frac{1}{\delta^2+1}.$$

Consequently, we get the estimate

$$\left|\operatorname{Re}\tilde{B}(\delta+i\xi)\right|=\alpha+\frac{\delta}{\delta^2+\xi^2}+$$

$$+\sum_{k=1}^{\infty}\frac{\beta_k(\delta+q_k)}{(\delta+q_k)^2+\xi^2}\geq$$

$$\geq\alpha+\frac{1}{1+\xi^2}\left(\frac{\delta}{1+\delta^2}+\sum_{k=1}^{\infty}\frac{\beta_k(\delta+q_k)}{(\delta+q_k)^2+1}\right)=$$

$$=\alpha+\frac{C_{1\delta}}{1+\xi^2}, \qquad (3.2)$$

and

$$\left|\operatorname{Im}\tilde{B}(a+i\xi)\right|=\frac{|\xi|}{\delta^2+\xi^2}+$$

$$+|\xi|\sum_{k=1}^{\infty}\frac{1}{(\delta+q_k)^2+\xi^2}\geq$$

$$\geq\frac{|\xi|}{1+\xi^2}\left(\frac{1}{1+\delta^2}+\sum_{k=1}^{\infty}\frac{1}{(\delta+q_k)^2+1}\right)=,$$

$$=\frac{|\xi|C_{2\delta}}{1+\xi^2} \qquad (3.3)$$

where

$$C_{1\delta}=\frac{\delta}{1+\delta^2}+\sum_{k=1}^{\infty}\frac{\beta_k(\delta+q_k)}{(\delta+q_k)^2+1},$$

$$C_{2\delta}=\frac{1}{1+\delta^2}+\sum_{k=1}^{\infty}\frac{1}{(\delta+q_k)^2+1}.$$

From (3.2) and (3.3), we have

$$\left|\tilde{B}(\delta+i\xi)\right|\geq\alpha+\frac{C_{3\delta}}{\sqrt{1+\xi^2}}\geq\alpha, \qquad (3.4)$$

where $C_{3a}=\min\{C_{1a},C_{2a}\}$.

Then, when $a\to 0$ from (3.1) we obtain the equality

$$\mu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\theta}(i\xi)}{\alpha+\tilde{B}(i\xi)}e^{i\xi t}d\xi. \qquad (3.5)$$

Lemma 3.1 *Suppose that the function $\theta(t)$ belongs to W(M). Then for the imaginary part of the Laplace transform of function $\theta(t)$ the following estimate holds*

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\zeta)\right|d\zeta<+\infty.$$

Proof. Using the formula for integration by parts in (3.1), we can write

$$\tilde{\theta}(\sigma+i\zeta)=\int_0^{\infty}e^{-(\sigma+i\zeta)t}\theta(t)dt=$$

$$=\theta(t)\frac{e^{-(\sigma+i\zeta)t}}{\sigma+i\zeta}\Bigg|_{t=0}^{t=\infty}+\frac{1}{\sigma+i\zeta}\int_0^{\infty}e^{-(\sigma+i\zeta)t}\theta'(t)dt.$$

Then, we have

$$(\sigma+i\zeta)\tilde{\theta}(\sigma+i\zeta)=\int_0^{\infty}e^{-(\sigma+i\zeta)t}\theta'(t)dt,$$

Further for $\sigma\to 0$ we get

$$i\zeta\tilde{\theta}(i\zeta)=\int_0^{\infty}e^{-i\zeta t}\theta'(t)dt.$$

Besides

$$\tilde{\theta}(i\zeta)=\int_0^{\infty}e^{-i\zeta t}\theta(t)dt.$$

Thus, we can write the following inequality

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\zeta)\right|^2(1+\zeta^2)d\zeta\leq C_2\|\theta\|^2_{W_2^1(R_+)},$$

where $C_2=const>0$.

By using elementary identities and inequalities we have

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\zeta)\right|d\zeta=\int_{-\infty}^{+\infty}\frac{\left|\tilde{\theta}(i\zeta)\right|}{1+\zeta^2}d\zeta+$$

$$+|\xi|\sum_{k=1}^{\infty}\frac{1}{(\delta+q_k)^2+\xi^2}\leq$$

$$\leq\left(\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\zeta)\right|^2d\zeta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\frac{1}{\left(1+\zeta^2\right)^2}d\zeta\right)^{\frac{1}{2}}+$$

$$+\left(\int_{-\infty}^{+\infty}\zeta^2\left|\tilde{\theta}(i\zeta)\right|^2d\zeta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\frac{\zeta^2}{\left(1+\zeta^2\right)^2}d\zeta\right)\leq$$

$$\leq C\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\zeta)\right|^2\left(1+\zeta^2\right)d\zeta\leq C_2\|\theta\|^2_{W_2^1(R_+)}.$$

Lemma 3.1 is proved.

Proof of theorem 3.1. According to (3.4), (3.5) and lemma 3.1, we can write

$$|\mu(t)|\leq\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\left|\tilde{\theta}(i\xi)\right|}{\left|\alpha+\tilde{B}(i\xi)\right|}d\xi\leq$$

$$\leq\frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\xi)\right|\sqrt{1+\xi^2}d\xi\leq$$

$$\leq\frac{1}{2\pi\alpha}C\|\theta\|_{W_2^2(R_+)}\leq\frac{CM}{2\pi\alpha}=1,$$

where $M=\frac{2\pi\alpha}{C}$.

Theorem 1 is proved.

Conclusion. In this work, the control problem for the pseudo-parabolic equation in the one-dimensional domain was considered. The mixed problem was reduced to the Volterra integral equation by the method of separation of variables. The existence of the control function was proved using the method of Laplace transform.

REFERENCES

1. A. Friedman, Optimal control for parabolic equations, J. Math. Anal. Appl., Vol. 18, 1967, pp. 479-491.
2. H.O. Fattorini, Time-Optimal control of solutions of operational differential equations, SIAM J. Control, Vol. 2, 1964, pp. 49-65.
3. Yu.V. Egorov, Optimal control in Banach spaces, Dokl. Akad. Nauk SSSR, Vol. 150, 1963, pp. 241-244 [in Russian].
4. S. Albeverio and Sh.A. Alimov, On one time-optimal control problem associated with the heat exchange process. Applied Mathematics and Optimization, Vol. 57, 2008, pp. 58-68.
5. Sh.A. Alimov, On a control problem associated with the heat transfer process. Eurasian Mathematical Journal, Vol. 1, 2010, pp. 17-30.
6. Sh.A. Alimov and F.N. Dekhkonov, On the time-optimal control of the heat exchange process. Uzbek Mathematical Journal, No. 2, 2019, pp. 4-17.

7. F.N. Dekhkonov, On the control problem associated with the heating process, Mathematical notes of NEFU, Vol. 29, 2022, pp. 62-71.
8. F.N. Dekhkonov, Boundary control problem for the heat transfer equation associated with heating process of a rod, Bulletin of the Karaganda University. Mathematics Series, Vol. 110, 2023, 63-71.
9. F.N. Dekhkonov and E.I. Kuchkorov, On the time-optimal control problem associated with the heating process of a thin rod, Lobachevskii Journal of Mathematics, Vol. 44, 2023, pp. 1134-1144.
10. F.N. Dekhkonov, On the time-optimal control problem for a heat equation, Bulletin of the Karaganda University. Mathematics Series, Vol. 111, 2023, pp. 28-38.
11. N. Chen, Y. Wang, and D. Yang, Time–varying bang–bang property of time optimal controls for heat equation and its applications, Syst. Control Lett., Vol. 112, 2018, pp. 18-23.
12. V.A. Il'in and E.I. Moiseev, Optimization of boundary controls of string vibrations, Russian Math. Surveys, Vol. 60, 2005, pp. 1093-1119.
13. Z.K. Fayazova, Boundary control of the heat transfer process in the space, Russian Mathematics (Izvestiya VUZ. Matematika), Vol. 63, 2019, pp. 71-79.
14. F.N. Dekhkonov, On a time-optimal control of thermal processes in a boundary value problem, Lobachevskii Journal of Mathematics, Vol. 43, 2022, pp. 192-198.
15. F.N. Dekhkonov, On a boundary control problem for a pseudo-parabolic equation, Communications in Analysis and Mechanics, Vol. 15, 2023, pp. 289-299.
16. F.N. Dekhkonov, Boundary control problem associated with a pseudo-parabolic equation, Stochastic Modeling and Computational Sciences, vol.3, 2023, pp. 117-128.
17. Z.K. Fayazova, Boundary control for a Psevdo-Parabolic equation, Mathematical notes of NEFU, Vol. 25, 2018, pp. 40-45.
18. A. Friedman, Differential equations of parabolic type, XVI, Englewood Cliffs, New Jersey, 1964.
19. A.V. Fursikov, Optimal control of distributed systems, Theory and applications, Translations of Math. Monographs, vol. 187, Amer. Math. Soc., Providence, 2000.
20. J.L. Lions, Contróle optimal de systèmes gouvernés par des équations auxdérivées partielles. Dunod Gauthier-Villars, Paris, 1968.
21. A. Altmüller and L. Grüne, Distributed and boundary model predictive control for the heat equation. Technical report, University of Bayreuth, Department of Mathematics, 2012.

22. S. Dubljevic and P.D. Christofides, Predictive control of parabolic PDEs with boundary control actuation. Chemical Engineering Science, Vol. 61, 2006, pp. 6239-6248.
23. A.N. Tikhonov and A.A. Samarsky, Equations of Mathematical Physics, Nauka, Moscow, 1966.
24. V.S. Vladimirov Equations of Mathematical Physics, Marcel Dekker, New York, 1971.
25. Alimov, Sh.A. and Dekhqonov, F.N. (2019). On a control problem associated with fast heating of a thin rod. Bulletin of National University of Uzbekistan, 2(1), 1-14.