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DIFFERENTIAL GAMES WITH NON-STATIONARY CONSTRAINTS OF LANGENHOP TYPE ON CONTROLS

Annotation. *In this work, we examined the pursuit-evasion differential game of one pursuer and one evader with non-stationary constraints of Langenhop-type, which is one of certain integral constraints on controls of both players. We used a* Π*-strategy providing to finish the pursuit problem for the pursuer. Also, we succeeded in obtaining the necessary conditions for evasion by using the proposed strategy for the evader.*

Keywords: *game, pursuit, evasion, differential equation, Langenhop type constraint, pursuer, evader, strategy, control function.*

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BOSHQARUVLARI LANGENHOP TIPLI NOSTATSIONAR CHEGARALANISHLI DIFFERENTSIAL O‘YINLAR

*Annotatsiya***.** *Ushbu ishda biz ikkala o‘yinchining boshqaruvlari asosiy integral chegaralanishdan biri bo‘lgan Langenhop tipidagi nostatsionar chegaralanishga ega bo‘lgan bitta quvlovchi va bitta qochuvchili differensial o‘yinini o‘rgandik. Biz quvlovchi uchun quvish masalasini hal qilishni taminlaydigan* Π*-strategiyadan foydalandik. Shuningdek, biz qochish uchun taklif qilingan strategiyadan foydalanib, qochish uchun muhim shartlarni olishga muvaffaq bo‘ldik.*

Kalit so‘zlar: *o‘yin, quvish, qochish, differensial tenglama, Langenhop tipli chegaralanish, quvlovchi, qochuvchi, strategiya,boshqaruv funksiyasi.*

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ДИФФЕРЕНЦИАЛЬНЫЕ ИГРЫ С НЕСТАЦИОНАРНЫМИ ТИПА ЛАНГЕНХОП ОГРАНИЧЕНИЯМИ НА УПРАВЛЕНИЯ

Аннотация. *В данной работе мы рассмотрели дифференциальную игру преследования-убегания одного преследователя и одного убегающего с нестационарными типа Лангенхопа ограничениями, которые являются одним из некоторых интегральных ограничений на управления обоих игроков. Мы использовали стратегию для преследователя, чтобы решить задачу преследования. Также нам удалось получить необходимые условия для убегания, используя предложенную стратегию для убегающего.*

Ключевые слова: *игра, преследование, убегания, дифференциальное уравнение, ограничение Лангенхопа типа, преследователя, убегающего, стратегия, функция управления.*

Introduction

Differential game theory has become a widely accepted approach for studying controlled processes by multiple players in situations involving conflict and uncertainty. This field originating from the works of Isaacs [1] has been further developed and refined by other mathematicians such as Pontryagin [2], Krasovskii and Subbotin [3], Pshenichnyi [4] and Petrosjan [5]. In terms of the application, the study of differential games is of significant interest because they involve players whose control functions satisfy a variety types of constraints. The field of differential games primarily focuses on problems where geometric, integral, or mixed constraints are applied to control functions (see [6-10])). In a recent study by Samatov et al. [6-10], a new form of integral constraint was proposed as Langenhop-type constraint. These works was focused on pursuit-evasion problems involving one pursuer and one evader in the space R^n. In the present work, we studied pursuit-evasion problems with non-stationary constraints of Langehop type on controls. The Pursuer's Π-strategy [6-10] was defined and the sufficient conditions for solving the pursuit problem were presented. In addition, a specific strategy was proposed for the Evader in dealing with the evasion problem.

Materials and Methods

Let two players: a pursuer $\mathbf{P}$ and an evader $\mathbf{E}$ moving in R^n space. Both players of the differential game studied in this paper have systems of the following form:

$$\mathbf{P}: \quad \dot{x} = u, \quad x(0) = x_0, \quad (1) \qquad \mathbf{E}: \quad \dot{y} = v, \quad y(0) = y_0, \quad (2)$$

where $x, y, u, v \in R^n$, $n \geq 1$; x_0 and y_0 are vectors describing the initial states of the pursuer $\mathbf{P}$ and the evader $\mathbf{E}$ respectively. It is accepted that $x_0 \neq y_0$. $u(t)$ and $v(t)$ are vectors representing the control velocity vectors of the pursuer $\mathbf{P}$ and the evader $\mathbf{E}$ respectively.

This work presents three classes of controls for both players. We introduce three classes of controls for the pursuer $\mathbf{P}$ and the evader $\mathbf{E}$:

1) $\mathbf{U}_{LT}$ ($\mathbf{V}_{LT}$) is the set of all admissible controls $u(\cdot)$ ($v(\cdot)$), satisfying Langenhop type constraints (briefly, LT -constraint):

$$|u(t)| \leq \alpha(t)\left(\rho - k\int_0^t |u(s)|\,ds\right), \qquad \text{for almost every } 0 \leq t < \bar{\tau}, \quad (3)$$

$$|v(t)| \leq \alpha(t)\left(\sigma - k\int_0^t |v(s)|\,ds\right), \qquad \text{for almost every } 0 \leq t < \tilde{\tau}, \quad (4)$$

where $\bar{\tau} = \sup\{t : \rho - k\int_0^t |u(s)|\,ds \geq 0\}$, $\tilde{\tau} = \sup\{t : \sigma - k\int_0^t |v(s)|\,ds \geq 0\}$;

2) $\mathbf{U}_G$ ($\mathbf{V}_G$) is the set of all admissible controls $u(\cdot)$ ($v(\cdot)$) satisfying Geometric constraints (briefly, G -constraints):

$$|u(t)| \leq \rho\alpha(t)e^{-kA(t)}, \qquad \text{for almost every } t \geq 0; \quad (5)$$

$$|v(t)| \leq \sigma\alpha(t)e^{-kA(t)}, \qquad \text{for almost every } t \geq 0; \quad (6)$$

3) $\mathbf{U}_I$ ($\mathbf{V}_I$) is the set of all admissible controls $u(\cdot)$ ($v(\cdot)$) satisfying Integral constraints (briefly, I -constraints):

$$\int_0^t |u(s)|\,ds \le \frac{\rho}{k}\left(1 - e^{-kA(t)}\right),\ t \ge 0, \qquad (7) \qquad \int_0^t |v(s)|^2\,ds \le \frac{\sigma}{k}\left(1 - e^{-kA(t)}\right), \quad t \ge 0, \qquad (8)$$

where $\alpha(t)$ is positive scalar function, $A(t) = \int_0^t \alpha(s)ds$ and ρ, k are positive parametric numbers in (3)-(8).

Given the players admissible controls $u(\cdot)$ and $v(\cdot)$, from (1)-(2) the corresponding trajectory $x(t)$ and $y(t)$ of the players at any time are found by

$$x(t) = x_0 + \int_0^t u(s)ds, \qquad y(t) = y_0 + \int_0^t v(s)ds.$$

The aim of the pursuer $\mathbf{P}$ is to capture the evader $\mathbf{E}$, i.e. to reach the state of $x(t) = y(t)$ (the pursuit problem), while the aim of the evader $\mathbf{E}$ aims to maintain the inequality $x(t) \ne y(t)$ (the evasion problem).

Lemma 1. $\mathbf{U}_G \subset \mathbf{U}_{LT} \subset \mathbf{U}_I$ and $\mathbf{V}_G \subset \mathbf{V}_{LT} \subset \mathbf{V}_I$ for almost all $t \ge 0$.

Proof. From (5) , we have

$$a) \quad |u(t)| \le \rho\alpha(t)e^{-kA(t)} = \alpha(t)\left(\rho - k\int_0^t \rho\alpha(s)e^{-kA(s)}ds\right) = \alpha(t)\left(\rho - k\int_0^t |u(s)|\,ds\right).$$

Therefore, we derive $U_G \subset U_{LT}$. $b)$ Let $\omega = -k\int_0^t |u(s)|\,ds$. Then $\dot{\omega}(t) = -k\,|u(t)|$ and $\omega(0) = 0$. From (3) we have $\dot{\omega}(t) \ge -k\alpha(t)\omega(t) - k\rho\alpha(t)$ for almost every $0 \le t \le \tau$. By multiplying both sides of the last inequality by $e^{kA(t)}$, we obtain $d\left(\omega(t)e^{kA(t)}\right) \ge -k\rho\alpha(t)e^{kA(t)}$. We integrate both sides of this inequality from 0 to t, we get $\omega(t)e^{kA(t)} \ge \rho\left(1 - e^{kA(t)}\right) \Rightarrow -k\int_0^t |u(s)|\,ds \ge \rho\left(e^{-kA(t)} - 1\right).$ It follows that $\mathbf{U}_{LT} \subset \mathbf{U}_I$ for almost every $t \ge 0$. This proves the lemma 1.

Lemma 2. If $u(\cdot)$ is belong to one of the introduced classes $\mathbf{U}_G$ or $\mathbf{U}_I$, then $x(t) \in B_\mu[x_0]$ for each $t \ge 0$, where $B_\mu[x_0]$ represents the ball with radius $\mu = \rho / k$ and centered at point x_0.

Proof. Assume that $u(\cdot) \in \mathbf{U}_G$. Then using (1) and (5) we obtain

$$|x(t)-x_0|\le\int_0^t|u(s)|\,ds\le\int_0^t\rho\alpha(s)e^{-kA(s)}ds=\frac{\rho}{k}\left(1-e^{-kA(t)}\right)<\frac{\rho}{k}$$ for every $t\ge 0$.

Or assume that $u(\cdot)\in\mathbf{U}_I$. Then using (1) and (7) we obtain

$$|x(t)-x_0|\le\int_0^t|u(s)|\,ds\le\frac{\rho}{k}\left(1-e^{-kA(t)}\right)<\frac{\rho}{k}$$. This proves the lemma 2.

The following significant result is reached using lemmas 1 and 2.

Corollary 1. If $u(\cdot)\in U_{LT}$, then the pursuer cannot leave the ball $B_\mu[x_0]$

Discussion of results

We define the pursuer's parallel approach strategy and sufficient conditions for the pursuit to be completed.

Definition 1. The function $\mathbf{u}_{LT}(t,v)=v-\lambda_{LT}(t,v)\xi_0$ is called a $\mathbf{\Pi}_{LT}$-strategy of the pursuer $\mathbf{P}$, if and only if $\rho\ge\sigma$, where

$\lambda_{LT}(t,v)=\langle v,\xi_0\rangle+\sqrt{\langle v,\xi_0\rangle^2+2\delta\alpha(t)e^{-kA(t)}|v|+\left(\delta\alpha(t)e^{-kA(t)}\right)^2}$, $\delta=\rho-\sigma$, $\xi_0=z_0/|z_0|$ and $\langle v,\xi_0\rangle$ -the scalar product between two vectors v and ξ_0 in R^n.

We take note that: $|\mathbf{u}_{LT}|=|v(t)|+\delta\alpha(t)e^{-kA(t)},\qquad t\ge 0$

Definition 2. If positive root of the equation

$$\Lambda_{LT}(t)=0 \tag{9}$$

exists with respect to t, then the smallest positive root of the equation (9) is referred to as the guaranteed pursuit time and denoted as $\mathbf{T}_{LT}$, where $\Lambda(t)=1-\dfrac{\delta}{k|z_0|}\left(1-e^{-kA(t)}\right)$.

Theorem 1. If $\delta>k|z_0|$ and (9) has the smallest positive root, then the $\mathbf{\Pi}_{LT}$-strategy ensures that pursuit is completed in the LT-Game with time $[0,\mathbf{T}_{LT}]$.

Proof. Assuming the evader $\mathbf{E}$ chooses any control $v(\cdot)\in V_{LT}$ and the pursuer $\mathbf{P}$ applies the $\mathbf{\Pi}_{LT}$-strategy, the initial value problem can be written using equations (1)-(2): $\dot z=\mathbf{u}_{LT}(t,v(t))-v(t)=\lambda_{LT}(t,v(t))\xi_0,\qquad z(0)=z_0$, where $z=x-y$. As a result, we obtain

$$z(t) = \Lambda_{LT}(t, v(\cdot))\xi_0, \tag{10}$$

where $\Lambda_{LT}(t,v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t \lambda_{LT}(s, v(s))ds$. We observe that the function $\Lambda_{LT}(t,v(\cdot))$ is continuous function that decreases for every $t \geq 0$ and we obtain

$$\Lambda_{LT}(t,v(\cdot)) \leq 1 - \frac{1}{|z_0|}\int_0^t \delta\alpha(s)e^{-kA(t)}ds \leq 1 - \frac{\delta}{k|z_0|}\left(1 - e^{-kA(t)}\right) = \Lambda_{LT}(t)$$

In accordance the conditions of theorem 1 imply that $\Lambda_{LT}(\mathbf{T}_{LT}) = 0$ and there exists time θ such that $\Lambda_{LT}(\theta, v(\cdot)) = 0$. By using (10), we can conclude that $z(\theta) = 0$ or $x(\theta) = y(\theta)$, where $\theta \leq \mathbf{T}_{LT}$.

Now, we see that $\mathbf{\Pi}_{LT}$-strategy satisfies the constraint (3) within the time interval $[0;\theta]$. Assume that the evader applies any control $v(\cdot)$ from the set V_{LT}. By using equation (4) ,the admissibility of the strategy $\mathbf{\Pi}_{LT}$-strategy is obtained as follows: $|\mathbf{u}_{LT}(t)| = |v(t)| + \delta\alpha(t)e^{-kA(t)} \leq \alpha(t)\left(\sigma - k\int_0^t |v(s)|\,ds\right) + \delta\alpha(t)e^{-kA(t)}$. This shows the admissibility of $\mathbf{\Pi}_{LT}$-strategy. This concludes the proof of theorem 1.

Let us now examine the game from the perspective of the evader

Definition 3. The function $\mathbf{v}_{LT}(t) = -\sigma\alpha(t)e^{-kA(t)}\xi_0$ is called the $\mathbf{E}_{LT}$-strategy of the evader.

Theorem 2. If $\sigma \geq \rho$, then the $\mathbf{E}_{LT}$-strategy ensures evasion in the LT-Game with time interval $[0, +\infty)$.

Proof. We first see that the $\mathbf{E}_{LT}$ strategy is admissible. The admissibility of the $\mathbf{E}_{LT}$ strategy is obtained as follows:

$$|\mathbf{v}_{LT}(t)| = \sigma\alpha(t)e^{-kA(t)} \leq \alpha(t)\left(\sigma - k\int_0^t \sigma\alpha(s)e^{-kA(s)}ds\right) \leq \alpha(t)\left(\sigma - k\int_0^t |\mathbf{v}_{LT}(s)|\,ds\right)$$

Now, assume that the pursuer $\mathbf{P}$ chooses any control $u(\cdot) \in \mathbf{U}_{LT}$ and the evader $\mathbf{E}$ applies the $\mathbf{E}_{LT}$-strategy. From (1)-(2) , we obtain

$$|z(t)| = |z_0 - \int_0^t \mathbf{v}_{LT}(s,v(s))ds + \int_0^t u(s)ds| \geq |z_0 - \int_0^t \mathbf{v}_{LT}(s,v(s))ds| - |\int_0^t u(s)ds| \geq$$

$$\geq |z_0 - \int_0^t \mathbf{v}_{LT}(s,v(s))ds| - \int_0^t |u(s)|\,ds = |z_0| + \frac{\sigma}{k}\left(1 - e^{-k\Lambda(s)ds}\right) - \int_0^t |u(s)|\,ds \; .$$

From (6) we obtain $|z(t)| \geq |z_0| + \frac{\sigma - \rho}{k}\left(1 - e^{-k\Lambda(t)}\right) > 0.$ Hence, $z(t) \neq 0$ or $x(t) \neq y(t)$. This concludes the proof of theorem 2.

Conclusion. We have studied pursuit-evasion differential games of one pursuer and one evader with non-stationary constraints of Langehop type on controls. The following are our main contributions:

1) If the lemmas 1 and 2 are satisfied, we have shown that the pursuer(the evader) then the pursuer cannot leave the ball that we found.
2) We have used a method of parallel approach of simple motion differential game with one pursuer and one evader.
3) We have found necessary and sufficient conditions in the evasion game.

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