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MODIFIED PRESTON'S LEMMA

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Annotation: *In this work, modified Preston's lemma is proved.*

Key words: *Cayley tree, Preston's lemma.*

O'zgartirilgan Prestonning lemmmasi

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Annotatsiya: *Bu ishda o'zgartirilgan Prestonning lemmasi isbotlangan.*

Kalit so'zlar: *Keli daraxti, Prestonning lemmasi.*

Модифицированная лемма Престона

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Аннотация: *В данной работе доказана модифицированная лемма Престона.*

Ключевые слова: *Дерево Кэли, лемма Престона.*

Solving phase transition problems for all models of statistical mechanics is reduced to solving multiparameter high-order equations. The fact that there is no exact general method of solving such equations makes it difficult to express its solutions in radicals. Therefore, it is often limited to solving the problem of the presence of positive solutions of such equations. Preston’s lemma is used to find the number of Gibbs measures for some models [1].

Theorem 1. [1] The equation

$$\left(\frac{1+x}{b+x}\right)^k=\alpha\cdot x \quad (1)$$

(with $x\geq 0, k\geq 1$, $a > 0$, $b > 0$) has one solution if either $k=1$ or $b\leq\left(\frac{k+1}{k-1}\right)^2$. If $k>1$ and $b>\left(\frac{k+1}{k-1}\right)^2$ then there exist $\eta_1(b,k),\eta_2(b,k)$with $0<\eta_1(b,k)<\eta_2(b,k)$ such that the equation has three solutions if $\eta_1(b,k)\text{¿}\alpha<\eta_2(b,k)$ and has two solutions if either $\alpha=\eta_1(b,k)$ or $\alpha=\eta_2(b,k)$. In fact $\eta_i(b,k)=\frac{1}{x_i}\cdot\left(\frac{1+x_i}{b+x_i}\right)^k$, where $x_1, x_2$are the solutions of

$$x^2+[2-(b-1)(k-1)]\cdot x+b=0. \quad (2)$$

From some problems of the theory of Gibbs measures [2] we get an equation of the following form:

$$\frac{a\cdot x^2+b\cdot x+c}{d\cdot x^2+e\cdot x+f}=\alpha\cdot x, \quad (3)$$

where $x\geq 0, a, b, c, d, e, f>0$.

Remark 1. If in equation (3*)* $a = d = 0$, $b = c = e = 1$, $f=b$, then equation (3) is a special case of equation (1) in case $k=1$. This equation was studied in [1].

Theorem 2. Let n be the number of positive solutions of equation (3). The following assertions are valid for n:

1. If the conditions

$$\begin{cases} a\cdot f<c\cdot d \\ a\cdot e<b\cdot d \\ b\cdot f<c\cdot e \end{cases}$$

are fulfilled, then $n = 1$.

2. If the conditions

$$\begin{cases} a\cdot f>c\cdot d \\ a\cdot e>b\cdot d \\ b\cdot f>c\cdot e \end{cases} \text{ or } \begin{cases} a\cdot f>c\cdot d \\ a\cdot e>b\cdot d \\ b\cdot f<c\cdot e \end{cases} \text{ or } \begin{cases} a\cdot f<c\cdot d \\ a\cdot e<b\cdot d \\ b\cdot f>c\cdot e \end{cases} \text{ or } \begin{cases} a\cdot f>c\cdot d \\ a\cdot e<b\cdot d \end{cases} \text{ or } \begin{cases} a\cdot f<c\cdot d \\ a\cdot e>b\cdot d \end{cases}$$

are fulfilled, then the following cases are fulfilled:

Case 2.1. When there are such numbers $\eta_1 := \eta_1(a,b,c,d,e,f), \eta_2 := \eta_2(a,b,c,d,e,f)\,(0<\eta_1<\eta_2)$ then

$$n=\begin{cases} 1, if\ \alpha \notin [\eta_1,\eta_2], \\ 2, if\ \alpha \in \{\eta_1,\eta_2\}, \\ 3, if\ \alpha \in (\eta_1,\eta_2) \end{cases}$$

where

$$\eta_i=\frac{1}{x_i}\cdot\frac{a x_i^2+b x_i+c}{d x_i^2+e x_i+f},$$

$x_1, x_2$are the positive solutions of equation

$$ad\,x^4+2\,bd\,x^3+(be+3\,cd-af)\,x^2+2ecx+cf=0\,(4)$$

Case 2.2. If equation (4) has no positive solutions, then *n = 1*.

Proof. Let's denote the left part of the given equation (3) by $F_1(x)$, that is

$$F_1(x)=\frac{a\cdot x^2+b\cdot x+c}{d\cdot x^2+e\cdot x+f}.(5)$$

The first and second order derivatives of function $F_1(x)$ are as follows:

$$F_1'(x)=\frac{(ae-bd)\cdot x^2+2(af-cd)\cdot x+bf-ce}{(d\cdot x^2+e\cdot x+f)^2},(6)$$

$F_1''(x)=¿$

$$2\frac{(b d^2-ade)\cdot x^3+3(c d^2-adf)\cdot x^2+3(cde-bdf)\cdot x+a f^2+c e^2-cdf-bfe}{(d\cdot x^2+e\cdot x+f)^3}$$

$$(7)$$

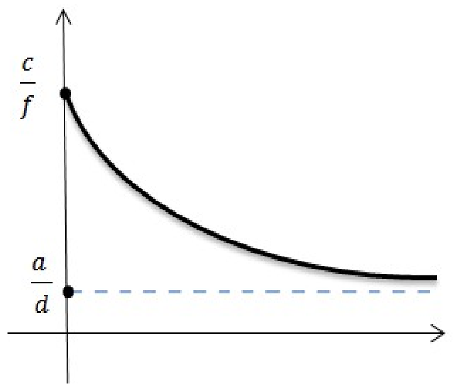
1. If system of inequalities

$$\begin{cases} a\cdot f<c\cdot d \\ a\cdot e<b\cdot d \\ b\cdot f<c\cdot e \end{cases} \quad (8)$$

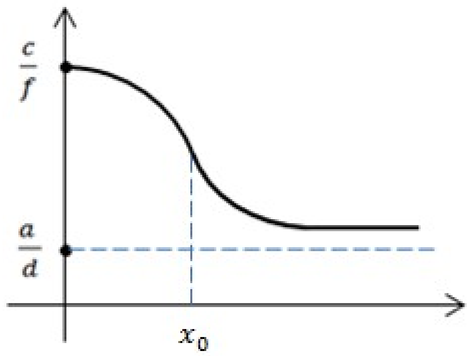
is satisfied, $F_1'(x)<0$, from which function (5) is decreasing in $x \in [0;+\infty¿$. $F_1''(x)=0$ equation has at most 1 positive solution, according to Descartes' theorem, when the system of inequalities (8) is satisfied. Therefore, the function $F_1(x)$has at most 1 positive inflection point.

1.1. If the graph of the function $F_1(x)$ does not have a positive inflection point when the system of inequalities (8) is fulfilled, we can see the graph of the function $F_1(x)$ in general in Graph 1.

1.2. If the graph of the function $F_1(x)$ has 1 positive inflection point when the system of inequalities (8) is fulfilled and we denote it by x_0, then the graph of the function $F_1(x)$ is convex at $[0;x_0)$, concave at $(x_0;+\infty)$ and we can see the graph of the function $F_1(x)$ in general in Graph 2.



Graph 1.



Graph 2.

When this system of inequalities (8) is fulfilled, the graph of the function $F_1(x)$ intersects the straight line $y=\alpha\cdot x$ at only 1 point, so equation (3) has only 1 positive solution when the system of inequalities (8) is fulfilled, that is, $n=1$.

2. If system of inequalities

$$\begin{cases} a\cdot f>c\cdot d \\ a\cdot e>b\cdot d \\ b\cdot f>c\cdot e \end{cases} \qquad (9)$$

is satisfied, $F_1'(x)>0$, from which function (5) is increasing in $x\in[0;+\infty)$.

Solving the equation

$$x\cdot F_1'(x)=F_1(x) \qquad (10)$$

helps us to determine the number of positive solutions of the equation (4). Equation (10) looks like equation (4).

According to Descartes' theorem, equation (4) can have at most 2 positive solutions when the system of inequalities (9) is satisfied.

$F_1''(x)=0$ equation has at most 1 positive solution, according to Descartes' theorem, when the system of inequalities (9) is satisfied. Therefore, the function $F_1(x)$ has at most 1 positive inflection point.

2.1. If the graph of function $F_1(x)$ does not have an inflection point when the system of inequalities (9) is fulfilled, then the straight line $y=\alpha\cdot x$ is not tangent line of the graph of the function $F_1(x)$, and therefore equation (4) does not have a positive solution and we can see the graph of the function $F_1(x)$ in general in Graph 3.

So, when the system of inequalities (9) is fulfilled, the graph of the function $F_1(x)$ intersects with the straight line $y=\alpha\cdot x$, since the equation (4) does not have a positive solution, at only 1 point, that is, *n=1*.

2.2. If equation (4) has 2 such positive solutions x_1, x_2 when the system of inequalities (9) is satisfied, then $\eta_1 := \eta_1(a,b,c,d,e,f), \eta_2 := \eta_2(a,b,c,d,e,f)$ be found and let $0<\eta_1<\eta_2$, then

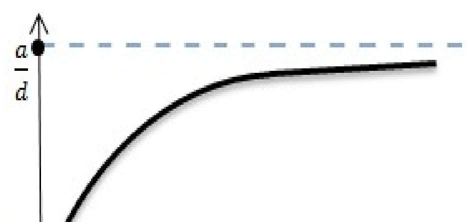
$$n=\begin{cases}1, if\ \alpha \notin [\eta_1,\eta_2],\\ 2, if\ \alpha \in \{\eta_1,\eta_2\},\\ 3, if\ \alpha \in (\eta_1,\eta_2),\end{cases}$$

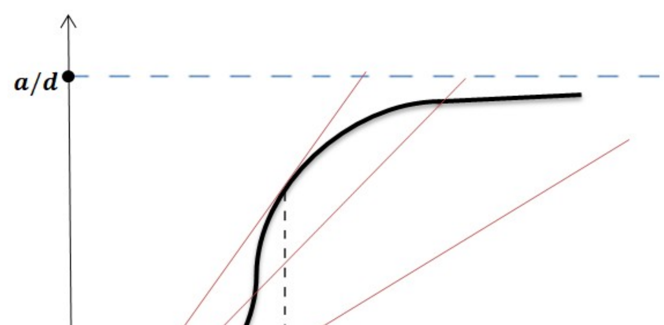
where

$$\eta_i=\frac{1}{x_i}\cdot\frac{a x_i^2+b x_i+c}{d x_i^2+e x_i+f},$$

x_1, x_2 are the positive solutions of equation (4).

If the graph of the function $F_1(x)$ has 1 positive inflection point when the system of inequalities (9) is fulfilled, and we denote it by x_0 then the graph of the function $F_1(x)$ is concave in $[0; x_0)$, convex in $(x_0; +\infty)$ and we can see the graph of the function $F_1(x)$ in general in Graph 4.





Graph 3. Graph 4.

3. If the system of inequalities

$$\begin{cases} a\cdot f>c\cdot d\,, \\ a\cdot e>b\cdot d\,, \\ b\cdot f<c\cdot e \end{cases} \qquad (11)$$

is fulfilled, then the function $F_1(x)$ is decreasing in $\left[0\,;\frac{cd-af+\sqrt{(af-cd)^2-(ae-bd)(bf-ce)}}{ae-bd}\right]$

and increasing in $\left[\frac{cd-af+\sqrt{(af-cd)^2-(ae-bd)(bf-ce)}}{ae-bd};+\infty\right.$¿ .

$F_1^{''}(x)$ = 0 equation has at most 1 positive solution, according to Descartes' theorem, when the system of inequalities (11) is satisfied. Therefore, the function $F_1(x)$ has at most 1 positive inflection point.

Solving the equation (10) helps us to determine the number of positive solutions of the equation (3). Equation (10) looks like equation (4).

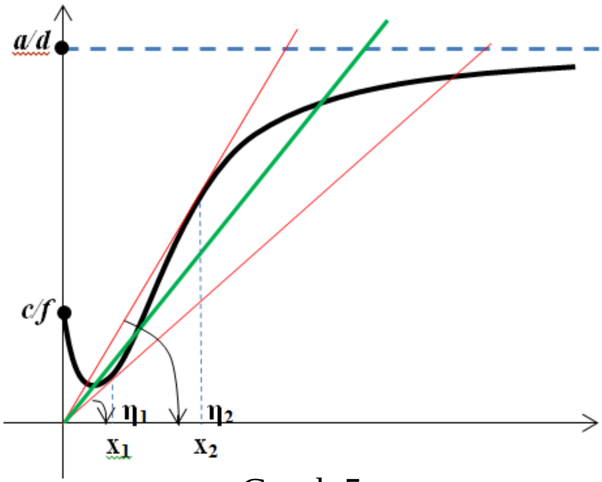
According to Descartes' theorem, equation (4) can have at most 2 positive solutions when the system of inequalities (11) is satisfied.

3.1. If equation (4) has 2 such positive solutions $x_1, x_2$when the system of inequalities (11) is satisfied, then $\eta_1 := \eta_1(a,b,c,d,e,f), \eta_2 := \eta_2(a,b,c,d,e,f)$be found and let $0<\eta_1<\eta_2$, then

$$n=\begin{cases} 1, if\ \alpha \notin [\eta_1,\eta_2], \\ 2, if\ \alpha \in \{\eta_1,\eta_2\}, \\ 3, if\ \alpha \in (\eta_1,\eta_2). \end{cases}$$

If the graph of the function $F_1(x)$has 1 positive inflection point when the system of inequalities (11) is fulfilled, and we denote it by x_0, then the graph of the function $F_1(x)$

is concave in $[0;x_0)$, convex in $(x_0;+\infty)$and we can see the graph of the function $F_1(x)$ in general in Graph 5.



Graph 5

All other cases are proved as above. The proof of the Theorem is complete.

Remark 2. Using the results obtained in Theorem 2, it is possible to solve phase transition problems for some models of statistical mechanics.

LITERATURE

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