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BOUSSINESQ TIPIDAGI DIFFERENSIAL TENGLAMA UCHUN INTEGRAL SHART BILAN BERILGAN MASALA

Annotatsiya. Maqolada $u_{tt}(t)+Au_{tt}(t)+Au(t)=f$ ko‘rinishidagi Boussinesq tipidagi tenglama uchun $u'(+0)=\psi$ va $\int_0^T u(t)dt=\varphi$ integral chegaraviy shartni qanoatlantiruvchi yechimni topish masalasi o‘rganilgan, bu yerda $A:H\to H$ o‘z-o‘ziga qo‘shma, chegaralanmagan, musbat H separabel Hilbert fazosida aniqlangan operator. Masalaning yechimining mavjud va yagonaligi Furye usulidan foydalanib ko‘rsatilgan.

Kalit so‘zlar: Boussinesq tipidagi tenglama, integral chegaraviy shart, Hilbert fazosi, operator, Furye usuli.

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ЗАДАЧА С ИНТЕГРАЛЬНЫМ УСЛОВИЕМ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТИПА БУССИНЕКА

Аннотация. В работе изучается задача нахождения решения, удовлетворяющего интегральным граничным условиям $u'(+0)=\psi$ и $\int_0^T u(t)dt=\varphi$ для уравнения типа Буссинеска вида $u_{tt}(t)+Au_{tt}(t)+Au(t)=f$, где $A:H\to H$ — самосопряженный, неограниченный, положительный H оператор, определенный в сепарабельном гильбертовом пространстве. Существование и единственность решения задачи доказаны с помощью метода Фурье.

Ключевая слова. Уравнение типа Буссинеска, интегральное граничное условие, гильбертово пространство, оператор, метод Фурье.

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PROBLEM WITH AN INTEGRAL CONDITION FOR A DIFFERENTIAL EQUATION OF BOUSSINEQ -TYPE

Annotation. The paper studies the problem of finding a solution satisfying the integral boundary conditions $u'(+0)=\psi$ and $\int_0^T u(t)dt=\varphi$ for the Boussinesq-type equation of the form $u_{tt}(t)+Au_{tt}(t)+Au(t)=f$, where $A:H\to H$ is a self-adjoint, unbounded, positive H separable operator defined in Hilbert space. The existence and uniqueness of the solution of the problem are shown using the Fourier method.

Key words. Boussinesq - type equation, integral boundary condition, Hilbert space, operator, Fourier method.

Kirish

Ushbu maqolada Boussinesq tipidagi tenglama uchun integral chegaraviy shart bilan berilgan masala o‘rganiladi.

Quyidagi masalani qaraylik:

$$\begin{cases} u_{tt}(t)+Au_{tt}(t)+Au(t)=f, \qquad 0<t\le T, \\ \int_0^T u(t)\,dt=\varphi, \\ u'(+0)=\phi, \end{cases} \tag{1}$$

bu yerda φ, ϕ va f - H Hilbert fazosining elementlari. $A:H\to H$ operator $D(A)\subset H$ aniqlanish sohasiga ega o‘z-o‘ziga qo‘shma, musbat, chegaralanmagan ixtiyoriy operator bo‘lsin. Faraz qilaylik, A operator H fazoda to‘la ortonormal $\{\vartheta_k\}$ xos funksiyalar sistemasiga va mos ravishda $\{\lambda_k\}$ musbat xos sonlar to‘plamiga ega bo‘lsin. Shu bilan birga A operator xos sonlari ketma-ketligi noldan boshqa limit nuqtaga ega bo‘lmasin. Xos qiymatlarni qayta nomerlash yordamida ularni kamaymaydigan qilib nomerlab olamiz, ya’ni $0<\lambda_1\le\lambda_2\le\cdots\to+\infty$ kabi yozib olamiz.

Boussinesq xususiy hosilali differensial tenglamalari 1872 yilda Jozef Boussinesq tomonidan kiritilgan ([1] ga qarang) va u turli jarayonlarning modelini ifodalaydi. Boussinesq tenglamalari qatlamda suyuqlik sirtining harakatini (suyuqlik to‘lqin harakatlarini) tavsiflaydi, ya’ni suyuqlikning g‘ovakli idishdan oqib o‘tayotganda erkin sirtining shaklini ifodalaydi. Boussinesq tipidagi tenglamalar uchun boshlang‘ich va chegaraviy masalalar [2] - [10] maqolalarda o‘rganilgan. Ushbu [7] - [10] maqolalarda Boussinesq tipidagi tenglamalar uchun qo‘yilgan masalaning yechimi mavjud va yagonaligini ta’minlovchi shartlar olingan.

Ta’rif. Agar $u(t) \in C((0,T];H)$ absolyut uzluksiz funksiya $u_{tt}(t),\ Au_{tt}(t),\ Au(t) \in C((0,T];H)$ xossalarga ega bo‘lib, (1) masalaning barcha shartlarini qanoatlantirsa, u holda $u(t)$ funksiyaga (1) masalaning yechimi deyiladi.

Teorema. Agar $\varphi, \phi \in D(A)$, $f \in H$ va $\sin \mu_k T \neq 0$ bo‘lsin. U holda (1) masalaning yagona yechimi mavjud va u quyidagicha ko‘rinishga ega:

$$u(t) = \sum_{k=1}^{\infty} \frac{\cos \mu_k t}{\sin \mu_k T} \left(\varphi_k \mu_k - \frac{f_k T \mu_k}{\lambda_k} + \frac{\phi_k \cos \mu_k T}{\mu_k} - \frac{\phi_k}{\mu_k} \right) \vartheta_k + \left(\frac{f_k}{\lambda_k} + \frac{\phi_k}{\mu_k} \sin \mu_k t \right) \vartheta_k,$$

bu yerda $\mu_k = \sqrt{\frac{\lambda_k}{1+\lambda_k}}$.

Biz masalani ikkita qismga ya’ni bir jinsli va bir jinsli bo‘lmagan qismlarga ajratib o‘rganamiz.

Yechimning mavjudligi

Ushbu bo‘limda biz (1) masalani yechish bilan sh ug‘illanamiz. (1) masalani yechish uchun biz quyidagi ikkita masalani yechishimiz kifoya:

$$\begin{cases} V_{tt}(t) + AV_{tt}(t) + AV(t) = f, \quad 0 < t \leq T, \\ V(+0) = 0, \\ V'(+0) = 0. \end{cases} \tag{2}$$

va

$$\begin{cases} W_{tt}(t) + AW_{tt}(t) + AW(t) = 0, \quad 0 < t \leq T, \\ \int_0^T W(t)dt = \psi, \\ W'(+0) = \phi, \end{cases} \tag{3}$$

bu yerda $\psi = \varphi - \int_0^T V(t)dt$ kabi aniqlanadi ($\psi \in H$ berilgan funksiya). $V(t)$ va $W(t)$ funksiyalar yig‘indisi (1) masalaning yechimi va $u(t) = V(t) + W(t)$ ko‘rinishda belgilaymiz.

(2) masalani Furye usulidan foydalanib yechimini quyidagi qator ko‘rinishida qidiramiz

$$V(t) = \sum_{k=1}^{\infty} V_k(t)\vartheta_k.$$

(2) masalani Furye qatoriga yoyib ushbu masalaga kelamiz

$$\begin{cases} V_k''(t) + \mu_k^2 V_k(t) = \dfrac{1}{1+\lambda_k} f_k, \quad 0 < t \le T, \\ V_k(+0) = 0, \\ V_k'(+0) = 0. \end{cases} \tag{4}$$

(4) masalaning yechimi quyidagicha ko‘rinishda bo‘ladi (qarang, [11])

$$V_k(t) = \frac{f_k}{\lambda_k}(1 - \cos \mu_k t). \tag{5}$$

Shunday qilib, (2) masalaning formal yechimi quyidagicha ko‘rinishga ega:

$$V(t) = \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k}(1 - \cos \mu_k t)\vartheta_k.$$

Endi (3) masalani yechish bilan shug‘ullanamiz. Masalaning yechimini ushbu Furye usulidan foydalanib topamiz.

$$W(t) = \sum_{k=1}^{\infty} W_k(t)\vartheta_k. \tag{6}$$

(6) qatorni (3) masalaning birinchi tengligiga qo‘yib, quyidagi oddiy differensial tenglamani hosil qilamiz:

$$W_k''(t) + \lambda_k W_k''(t) + \lambda_k W_k(t) = 0,$$

soddalashtirib qaytadan yozib olamiz

$$W_k''(t) + \mu_k^2 W_k(t) = 0.$$

(3) masalaning shartlaridan foydalanib ushbu masalaga kelamiz:

$$\begin{cases} W_k''(t) + \mu_k^2 W_k(t) = 0, \\ \int_0^T W_k(t)dt = \psi_k, \\ W_k'(+0) = \phi_k. \end{cases} \qquad (7)$$

Ushbu masalani yechish uchun avval birinchi tenglamani yechib quyidagi tenglikni olamiz:

$$W_k(t) = a_k \cos \mu_k t + b_k \sin \mu_k t.$$

$W_k'(+0) = \phi_k$ shartdan foydalanib, $b_k = \frac{\phi_k}{\mu_k}$ ekanligini topamiz. Integral shartdan foydalanib

$$a_k = \frac{1}{\sin \mu_k T}\left(\mu_k \psi_k + \frac{\phi_k \cos \mu_k T}{\mu_k} - \frac{\phi_k}{\mu_k} \right)$$

tenglikni hosil qilamiz.

Demak, (7) masalaning yechimi quyidagi ko’rinishga ega ekan:

$$W_k(t) = \frac{\cos \mu_k t}{\sin \mu_k T}\left(\mu_k \psi_k + \frac{\phi_k \cos \mu_k T}{\mu_k} - \frac{\phi_k}{\mu_k} \right) + \frac{\phi_k}{\mu_k} \sin \mu_k t.$$

Bundan (3) masalaning yechimi

$$W(t) = \sum_{k=1}^{\infty} \left[\frac{\cos \mu_k t}{\sin \mu_k T}\left(\mu_k \psi_k + \frac{\phi_k \cos \mu_k T}{\mu_k} - \frac{\phi_k}{\mu_k} \right) + \frac{\phi_k}{\mu_k} \sin \mu_k t \right]$$

ko’rinishida ekanligini topamiz.

Agar $\psi = \varphi - \int_0^T V(t)dt$ va (5) ni hisobga olsak, u holda quyidagi tenglikka ega bo’lamiz:

$$\psi_k = \varphi_k - \frac{f_k}{\lambda_k}\left(T - \frac{1}{\mu_k} \sin \mu_k T \right).$$

Yuqorida aytkanimizdek, $u(t) = V(t) + W(t)$ ekanligini hisobga olsak, u holda (1) masalaning formal yechimi ushbu

$$u(t) = \sum_{k=1}^{\infty} \frac{\cos \mu_k t}{\sin \mu_k T}\left(\varphi_k \mu_k - \frac{f_k T \mu_k}{\lambda_k} + \frac{\phi_k \cos \mu_k T}{\mu_k} - \frac{\phi_k}{\mu_k} \right) \vartheta_k + \left(\frac{f_k}{\lambda_k} + \frac{\phi_k}{\mu_k} \sin \mu_k t \right) \vartheta_k \qquad (8)$$

tenglik bilan aniqlanishini topamiz.

(8) qator bilan aniqlangan $u(t)$ funksiya haqiqiy yechim ekanligini ko‘rsatish uchun bu funksiyaning ta’rifning barcha shartlarini qanoatlantirishini tekshirib chiqamiz. Buning uchun $Au_{tt}(t)\in C((0,T]:H)$ ekanligini ko‘rsatish yetarli bo‘ladi. Buni ko‘rsatish uchun avval (10) qatorning qismiy yig‘indisini $S_n(t)$ orqali belgilab olamiz:

$$S_n(t)=\sum_{k=1}^{n}\frac{\cos\mu_k t}{\sin\mu_k T}\left(\varphi_k\mu_k-\frac{f_k T\mu_k}{\lambda_k}+\frac{\phi_k\cos\mu_k T}{\mu_k}-\frac{\phi_k}{\mu_k}\right)\vartheta_k+\left(\frac{f_k}{\lambda_k}+\frac{\phi_k}{\mu_k}\sin\mu_k t\right)\vartheta_k \tag{9}$$

(9) tenglikka $\frac{\partial^2}{\partial t^2}$ va A operatorlarni hadma - had qo‘llaymiz. U holda ushbu tenglikni hosil qilamiz

$$A\left(\frac{\partial^2}{\partial t^2}S_n(t)\right)=\sum_{k=1}^{n}\mu_k^2\frac{\cos\mu_k t}{\sin\mu_k T}\left(-\varphi_k\mu_k+\frac{f_k T\mu_k}{\lambda_k}-\frac{\phi_k\cos\mu_k T}{\mu_k}+\frac{\phi_k}{\mu_k}\right)\lambda_k\vartheta_k-\mu_k^2\left(\frac{\phi_k}{\mu_k}\sin\mu_k t\right)\lambda_k\vartheta_k$$

Parseval tengligidan foydalanib quyidagi tenglikka ega bo‘lamiz

$$\left\|A\left(\frac{\partial^2}{\partial t^2}S_n(t)\right)\right\|^2=\sum_{k=1}^{n}\mu_k^4\lambda_k^2\frac{\cos\mu_k t}{\sin\mu_k T}\left|\varphi_k\mu_k+\frac{f_k T\mu_k}{\lambda_k}+\frac{\phi_k\cos\mu_k T}{\mu_k}+\frac{\phi_k}{\mu_k}\right|^2+\mu_k^4\lambda_k^2\left|\frac{\phi_k}{\mu_k}\sin\mu_k t\right|.$$

Agar, barcha $k\geq 1$ uchun $|\mu_k|\leq 1$, $\frac{1}{\mu_k^2}=1+\frac{1}{\lambda_k}$ ekanligini hisobga olsak, quyidagi baho o‘rinli bo‘ladi

$$\left\|A\left(\frac{\partial^2}{\partial t^2}S_n(t)\right)\right\|^2\leq\sum_{k=1}^{n}\lambda_k^2\left|\frac{\varphi_k\cos\mu_k t}{\sin\mu_k T}\right|^2+\sum_{k=1}^{n}\lambda_k^2\left|\frac{f_k T\cos\mu_k t}{\lambda_k\sin\mu_k T}\right|^2+\sum_{k=1}^{n}\lambda_k^2\left|\frac{\phi_k\cos\mu_k T\cos\mu_k t}{\mu_k\sin\mu_k T}\right|^2+$$

$$+\sum_{k=1}^{n}\lambda_k^2\left|\frac{\phi_k\cos\mu_k t}{\mu_k\sin\mu_k T}\right|^2+\sum_{k=1}^{n}\lambda_k^2\left|\frac{\phi_k}{\mu_k}\sin\mu_k t\right|^2$$

$$\left\|A\left(\frac{\partial^2}{\partial t^2}S_n(t)\right)\right\|^2\leq C\left(\sum_{k=1}^{n}\lambda_k^2|\varphi_k|^2+\sum_{k=1}^{n}\lambda_k^2|\phi_k|^2+\sum_{k=1}^{n}|f_k|^2\right).$$

Demak, agar $\varphi,\phi\in D(A)$ va $f\in H$ bo‘lsa $Au_{tt}(t)\in C((0,T];H)$ bo‘lar ekan. Xuddi shunga o‘xshash $u(t),u_{tt}(t),Au_{tt}(t)\in C((0,T];H)$ ekanligini ko‘rsatish mumkin.

Yechimning yagonaligi.

(1) masalaning yechimining yagonaligini isbotlash uchun standart usuldan foydalanamiz, ya’ni quyidagi masala aynan nol yechimga ega ekanligini ko‘rsatish yetarli. Aytaylik ushbu masalaning

$$\begin{cases} u_{tt}(t) + Au_{tt}(t) + Au(t) = 0, \quad 0 < t < T; \\ \int_0^T u(t)dt = 0, \\ u'(+0) = 0 \end{cases} \tag{10}$$

$u(t)$ ixtiyoriy yechim bo‘lsin, u holda $u_k(t) = (u(t), \vartheta_k)$ bo‘ladi. $u(t) = \sum_{k=1}^{\infty} u_k(t)\vartheta_k$ va A operatorning o‘z - o‘ziga qo‘shma ekanligidan (10) ga ko‘ra ushbu tenglikka ega bo‘lamiz

$$\frac{\partial^2 u_k}{\partial t^2} = -(Au_{tt} + Au, \vartheta_k) = -(u_{tt} + u, \lambda_k \vartheta_k) = -\lambda_k \frac{\partial^2 u_k}{\partial t^2} - \lambda_k u_k.$$

Bundan quyidagi masalaga kelamiz

$$\begin{cases} u_k''(t) + \mu_k^2 u_k(t) = 0, \\ \int_0^T u_k(t)dt = 0, \\ u_k'(+0) = 0 \end{cases}$$

Bu masalani yechib barcha k, $k \geq 1$ lar uchun $u_k(t) = 0$ ekanligini topamiz. $\{\vartheta_k\}$ xos funksiyalarning to‘laligidan $u(t) \equiv 0$ ekanligi kelib chiqadi. Teorema to‘la isbotlandi

Foydalanilgan adabiyotlar

1. Boussinesq, Theorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pures Appl., Deux. Ser. 17, 55–108 (1872).

2. J. Banasiak, N. A. Manakova and G. A. Sviridyuk, Positive solutions to Sobolev type equations with relatively p-sectorial operators, / Vestn. Yuzh.-Ural. Univ., Ser. Mat. Model. Program. 13 (2), 17–32 (2020).

3. N. N. Solovyova, S. A. Zagrebina and G. A. Sviridyuk, Sobolev type mathematical models with relatively positive operators in the sequence spaces, Vestn. Yuzh. Ural. Univ., Ser. Mat., Mekh., Fiz. 9 (4), 27–35 (2017).
4. A. G. Sveshnikov, A. B. Alshin, M. O. Korpusov and Yu. D. Pletner, Linear and Nonlinear Sobolev-Type Equations (Fizmatlit, Moscow, 2007) [in Russian].
5. Sh. Alimov and A. Khalmukhamedov, On a non-local problem for a Boussinesq type differential equation / Lobachevskii Journal of Mathematics, Vol. 43, No. 4, pp. 916–923, (2022).
6. T. K. Yuldashev, Mixed differential equation of a Boussinesq type, / Vestn. Volgogr. Univ., Ser.: Mat. Fiz. 2 (33), 13–26 (2016).
7. T. K. Yuldashev, Solvability of a boundary value problem for a differential equation of the Boussinesq type, / Differ. Equat. 54, 1384–1393 (2018).
8. T. K. Yuldashev, Determination of the coeffcient in nonlocal problem for a Boussinesq type integro-differential equation with degenerate kernel, / Vladikavk. Mat. Zh. 21 (2), 67–84 (2019).
9. A. R. Khalmukhamedova and E. I. Kuchkorova, On the Solvability of a Nonlocal Problem for a Boussinesq-Type Differential Equation, , Russian Mathematics, 2023, Vol. 67, No. 10, pp. 54–62.
10. Kh. T. Dekhkonov, Yu. E. Fayziev and R. R. Ashurov, On the non-local problem for a Boussinesq type equations, Lobachevskii Journal of Mathematics, 2024, Vol. 45, No. 3, pp. 1023–1031.
11. A. N. Tikhonov, A. A. Samarskiy, Equations of Mathematical Physics (Moscow State University Publishing House 1977) [in Russian].