Gellerstedt masalasi uchta singulyar koeffitsiyentga ega boʻlgan uch oʻlchovli aralash tipdagi tenglama uchun.

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Annotatsiya. Ushbu ishda uch oʻlchovli aralash tipdagi tenglama uchun Gellerstedt masalasi oʻrganilgan, bunda uchta singulyar koeffitsiyentdan iborat soha yarim silindr va ikki uchburchak shaklidagi toʻgʻri prizma bilan chegaralangan. Spektral tahlil usuli yordamida qoʻyilgan masalaning yagona yechimga ega ekanligi isbotlangan. Oʻrganilgan masalaning yechimi ikki marta Furye-Bessel qatorlari yigʻindisi shaklida qurilgan.

Kalit soʻzlar: Gellerstedt masalasi, singulyar koeffitsiyentlar, Bessel funksiyasi, Gauss gipergeometrik funksiyasi.

Задача Геллерстедта для трехмерного уравнения смешанного типа с тремя сингулярными коэффициентами

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Аннотация. В работе исследована задача Геллерстедта для трехмерного уравнения смешанного типа с тремя сингулярными коэффициентами в области, состоящей из полуцилиндра и две треугольной прямой призмы. Методом спектрального анализа доказана однозначная разрешимость поставленной задачи. Решение исследуемой задачи построено в виде суммы двойного ряда Фурье-Бесселя.

Ключевые слова: задача Геллерстедта, сингулярные коэффициенты, функция Бесселя, гипергеометрическая функция Гаусса.

The Gellerstedt Problem for a Three-Dimensional Mixed-Type Equation with Three Singular Coefficients

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Abstract. This work investigates the Gellerstedt problem for a three-dimensional mixed-type equation with three singular coefficients in a domain composed of a semi-cylinder and two triangular prisms. Using the method of spectral analysis, the existence and uniqueness of the considered problem have been proved.

The solution to the studied problem has been constructed as the sum of a double series.

Keywords: Gellerstedt problem, singular coefficients, Bessel function, Gauss hypergeometric function.

I. Introduction. Problem Statement

Let $\Omega = \{(x, y, z) : (x, y) \in \Delta,\ z = (0, c)\}$, where Δ is a finite simply connected domain of the xOy plane, bounded for $y \geq 0$ by the arc $\bar{\sigma}_0 = \{(x, y) : x^2 + y^2 = 1,\ y \geq 0\}$ and for $y \leq 0$ by the segments $\overline{AB} = \{(x, y) : y = -x - 1,\ x \in [-1, -1/2]\}$, $\overline{OB} = \{(x, y) : y = x,\ x \in [-1/2, 0]\}$, $\overline{OC} = \{(x, y) : y = -x,\ x \in [0, 1/2]\}$, $\overline{CD} = \{(x, y) : y = x - 1, x \in [1/2, 1]\}$, $O = O(0,0)$, $A = A(-1, 0)$, $B = B(-1/2, -1/2)$, $C = C(1/2, -1/2)$, $D = D(1, 0)$.

In the domain Ω, we consider the following equation

$$U_{xx} + \operatorname{sgn} y \cdot U_{yy} + U_{zz} + \frac{2\beta}{x} U_x + \frac{2\beta}{|y|} U_y + \frac{2\gamma}{z} U_z = 0, \tag{1}$$

where $U = U(x, y, z)$, and $\beta, \gamma \in R$ such that $\beta, \gamma \in (0, 1/2)$.

Let us introduce the following notations: $\Delta_0 = \Delta \cap \{y > 0\}$, $\Delta_1 = \Delta \cap \{(x > 0) \cup (y < 0)\}$, $\Delta_2 = \Delta \cap \{(x < 0) \cup (y < 0)\}$, $\Omega_j = \Delta_j \times (0, c)$, $j = \overline{0,2}$, $S_0 = \{(x, y, z) : \sigma_0 \times (0, c)\}$, $S_1 = \{(x, y, z) : OB \times (0, c)\}$, $S_2 = \{(x, y, z) : OC \times (0, c)\}$, $S_3 = \{(x, y, z) : (x, y) \in \Delta, z = 0\}$, $S_4 = \{(x, y, z) : (x, y) \in \Delta, z = c\}$, $\Omega^{\pm} = \Omega \cap \{\pm x > 0, z \in (0, c)\}$, $\Delta^{\pm} = \Delta \cap \{\pm x > 0\}$.

In the domain Ω, Equation (1) belongs to the mixed type, i.e., it is the elliptic type in Ω_0 - and hyperbolic type in Ω_1 and Ω_2, where $x = 0$, $y = 0$ and $z = 0$ are planes of singularity for the equation, and the equation changes its type as it crosses the rectangles $\bar{\Omega} \cap \{y = 0\}$.

For Equation (1) in the domain Ω, we investigate the following problem:

Problem G. Find a function $U(x, y, z)$ satisfying Equation (1) in the domain Ω and the following conditions:

$$U(x, y, z) \in C(\bar{\Omega}) \cap C^{2,2,2}_{x,y,z}(\Omega_0 \cup \Omega_1 \cup \Omega_2); \tag{2}$$

$$U(x, y, z)\big|_{\bar{S}_0} = F(x, y, z); \tag{3}$$

$$U(x, y, z)\big|_{\bar{S}_1} = 0,\ U(x, y, z)\big|_{\bar{S}_2} = 0; \tag{4}$$

$$U(x, y, z)\big|_{\bar{S}_3} = 0,\ U(x, y, z)\big|_{\bar{S}_4} = 0. \tag{5}$$

Also it satisfies the following matching condition:

$$\lim_{y\to -0}(-y)^{2\beta}U_y(x,y,z)=\lim_{y\to +0}y^{2\beta}U_y(x,y,z),\quad x\in(-1,0)\cup(0,1),\ z\in(0,c), \tag{6}$$

where $F(x,y,z)$ is a given function.

The Gellerstedt problem on the plane for various mixed-type equations has been studied in the references [1]-[18]. However, numerous studies dedicated to this problem predominantly address two-dimensional problems. The three-dimensional problem G, when $\beta=\gamma=0$, was investigated in [19]. The Gellerstedt problem for three-dimensional mixed-type equations with singular coefficients remains poorly studied.

In the present work, by the spectral analysis method we will prove the unique solvability of the problem G.

2. Construction of Non-trivial Solutions for the Problem {(1), (2), (4), (5), (6)}

We find non-trivial solutions to the equation (1) satisfying the conditions (2) and (4)-(6). By separating variables by the formula $U(x,y,z)=u(x,y)Z(z)$, from the equation (1) and the conditions (4), (5), and (6), we obtain the following eigenvalue problems:

$$u_{xx}+(\operatorname{sgn} y)u_{yy}+\frac{2\beta}{x}u_x+\frac{2\beta}{|y|}u_y-\lambda u=0,\ \ (x,y)\in\Delta, \tag{7}$$

$$u(x,y)\big|_{\overline{OB}}=0,\ -1/2\le x\le 0,\ \ u(x,y)\big|_{\overline{OC}}=0,\ 0\le x\le 1/2, \tag{8}$$

$$\lim_{y\to -0}(-y)^{2\beta}u_y(x,y)=\lim_{y\to +0}y^{2\beta}u_y(x,y),\quad x\in(-1,0)\cup(0,1), \tag{9}$$

$$Z''(z)+\frac{2\gamma}{z}Z'(z)+\lambda Z(z)=0,\ 0<z<c, \tag{10}$$

$$Z(0)=0,\ \ Z(c)=0. \tag{11}$$

The problem {(10),(11)} has non-trivial solutions of the form (see, for example, [20], [21], [22])

$$Z_m(z)=z^{1/2-\gamma}J_{1/2-\gamma}(\sigma_m z/c),\ m\in N, \tag{12}$$

where $J_l(z)$ - is the first kind Bessel function of order l [23], and σ_m - is the m-th positive root of the equation $J_{1/2-\gamma}(\sqrt{\lambda}c)=0$, $\lambda_m=(\sigma_m/c)^2$, $m\in N$.

According to the theory of Bessel functions [23], the system of eigenfunctions (10) forms an orthogonal and complete system in the space $L_2(0,c)$ with the weight $z^{2\gamma}$.

Now, we consider the problem {(7)-(9)} at $\lambda=\lambda_m$ in the domain Δ. We will seek the solution $u(x,y)$ for the problem {(7)-(9)} in the following form:

$$u(x,y)=u_1(x,y)+u_2(x,y),\ (x,y)\in\bar{\Delta}^+, \tag{13}$$

where the functions $u_1(x,y)$ and $u_2(x,y)$ are defined respectively as:

$$u_1(x,y)=\frac{1}{2}\left[u(x,y)+u(-x,y)\right],\ (x,y)\in\bar{\Delta}^+, \tag{14}$$

$$u_2(x,y)=\frac{1}{2}\left[u(x,y)-u(-x,y)\right],\ (x,y)\in\bar{\Delta}^+. \tag{15}$$

Obviously, if $(x,y)\in\bar{\Delta}^+$, then $(-x,y)\in\bar{\Delta}^-$, where $\bar{\Delta}=\bar{\Delta}^+\cup\bar{\Delta}^-$.

Next, we consider the boundary problem, which can be formulated with respect to the auxiliary function $u_2(x,y)\in C\left(\bar{\Delta}^+\right)\cap C^{2,2,2}_{x,y,z}\left(\Delta_0^+\cup\Delta_1\right)$:

$$u_{2xx}+\operatorname{sgn} y\cdot u_{2yy}+\frac{2\beta}{x}u_{2x}+\frac{2\beta}{|y|}u_{2y}-\lambda_m u_2=0,\ (x,y)\in\Delta^+, \tag{16}$$

$$u_2(0,y)=0,\ y\in[0,1], \tag{17}$$

$$u_2(x,y)\big|_{\overline{OC}}=u_2(x,-x)=0,\ x\in[0,1/2], \tag{18}$$

$$\lim_{y\to-0}(-y)^{2\beta}u_{2y}(x,y)=\lim_{y\to+0}y^{2\beta}u_{2y}(x,y),\quad x\in(0,1), \tag{19}$$

where $\Delta_0^+=\Delta^+\cap\{y>0\}$.

Let us investigate this problem. First, consider this problem in the domain Δ_1 when $\lambda=\lambda_m$, i.e., consider the following problem:

$$u_{2xx}-u_{2yy}+\frac{2\alpha}{x}u_{2x}-\frac{2\beta}{y}u_{2y}-\lambda_m u_2=0,\ (x,y)\in\Delta_1, \tag{20}$$

$$u_2(x,-x)=0,\ x\in[0,1/2]. \tag{21}$$

The solution to this problem is sought in the form

$$u(x,y)=X(\xi)Y(\eta),\ \text{where } \xi=\sqrt{x^2-y^2},\ \eta=x^2/\xi^2. \tag{22}$$

Then, for the functions $X(\xi)$ and $Y(\eta)$, we have the following conditions $X(0)=0$, $\left|\lim\limits_{\eta\to+\infty}Y(\eta)\right|<+\infty$ and equations

$$\xi^2X''(\xi)+(1+4\beta)\xi X'(\xi)-\left[\lambda_m\xi^2+\mu\right]X(\xi)=0,\ \xi>0; \tag{23}$$

$$\eta(1-\eta)Y''(\eta)+\left[1/2+\beta-(1+2\beta)\eta\right]Y'(\eta)+\frac{1}{4}\mu Y(\eta)=0,\ \eta>1, \tag{24}$$

where $\mu\in R-$ is the separation constant.

The solutions to equation (23), satisfying the condition $X(0)=0$, exist for $\mu>0$ and they (up to a constant multiplier) have the form

$$X(\xi)=\xi^{-2\beta}I_\omega(\sigma_m\xi/c),\quad \omega>2\beta,\ m\in N, \tag{25}$$

where $\omega=\sqrt{4\beta^2+\mu}$, and $I_l(x)$– is a Bessel function of imaginary argument of order l [23].

Equation (24) is a Gaussian hypergeometric equation. Its general solution is given by [24]

$$Y(\eta)=c_1\eta^{-\beta-\omega/2}F(\beta+\omega/2,1/2+\omega/2,1+\omega;1/\eta)+ \\ c_2\eta^{-\beta+\omega/2}F(\beta-\omega/2,1-\omega/2,1-\omega;1/\eta), \qquad (26)$$

where c_1 and c_2 are arbitrary constants.

Since $\omega-2\beta>0$, from (26) it follows that to obtain a function bounded as $\eta\to+\infty$, in the formula one must set $c_2=0$, resulting in

$$Y(\eta)=c_1\eta^{-\beta-\omega/2}F(\beta+\omega/2,1/2+\omega/2,1+\omega;1/\eta). \qquad (27)$$

Therefore, continuous and non-trivial solutions in $\overline{\Delta}_1$ for the problem (20), (21), according to (22), (25), and (27), are defined by the equations

$$u_{2m}^{-}(x,y)=c_1x^{-2\beta-\omega}\left(x^2-y^2\right)^{\omega/2}I_\omega\left(\sigma_m\sqrt{x^2-y^2}/c\right)\times \\ \times F\left(\beta+\frac{\omega}{2},\frac{1+\omega}{2},1+\omega;1-\frac{y^2}{x^2}\right),\ c_1\neq 0,\ m\in N. \qquad (28)$$

From here, using well-known formulas [24]

$$F(A,B,C;1)=\Gamma(C)\Gamma(C-A-B)/\left[\Gamma(C-A)\Gamma(C-B)\right],\ C-A-B>0, \qquad (29)$$

$$F(A,B,C;x)=(1-x)^{C-A-B}F(C-A,C-B,C;x), \qquad (30)$$

we find

$$\begin{cases}\tau_m^{-}(x)=\lim\limits_{y\to-0}u_{2m}^{-}(x,y)=c_1k_1(\omega)x^{-2\beta}I_\omega(\sigma_m x/c),\ x\in[0,1];\\ \nu_m^{-}(x)=\lim\limits_{y\to-0}(-y)^{2\beta}\dfrac{\partial}{\partial y}u_{2m}^{-}(x,y)=c_1k_2(\omega)x^{-1}I_\omega(\sigma_m x/c),\ x\in(0,1),\end{cases} \qquad (31)$$

where $\Gamma(z)-$ is Euler's gamma function [24],

$$k_1(\omega)=\frac{\Gamma(1+\omega)\Gamma(1/2-\beta)}{\Gamma(1-\beta+\omega/2)\Gamma\left[(1+\omega)/2\right]},\ k_2(\omega)=\frac{2\Gamma(1+\omega)\Gamma(1/2+\beta)}{\Gamma(\beta+\omega/2)\Gamma\left[(1+\omega)/2\right]}. \qquad (32)$$

Now, we consider the problem {(16)-(19)} when $\lambda=\lambda_m$ in the domain Δ_0^{+}, i.e., consider the following problem:

$$u_{2xx}+u_{2yy}+\frac{2\beta}{x}u_{2x}+\frac{2\beta}{y}u_{2y}-\lambda_m u_2=0,\ (x,y)\in\Delta_0^{+}, \qquad (33)$$

$$u_2(0,y)=0,\ y\in[0,1]. \qquad (34)$$

By separating variables with the formula

$$u_2(x,y) = Q(\rho) S(\varphi), \tag{35}$$

where $\rho = \sqrt{x^2 + y^2}$, $\varphi = arctg(y/x)$, from the equation (33) and the conditions $u_2 \in C(\overline{\Delta}_0^+)$, (34), we have the following problems:

$$\rho^2 Q''(\rho) + (1+4\beta)\rho Q'(\rho) - \left[(\sigma_m \rho / c)^2 + \tilde{\mu}\right] Q(\rho) = 0, \ \rho \in (0,1), \tag{36}$$

$$|Q(0)| < +\infty; \tag{37}$$

$$S''(\varphi) + 4\beta ctg(2\varphi) S'(\varphi) + \tilde{\mu} S(\varphi) = 0, \ \varphi \in (0, \pi/2), \tag{38}$$

$$S(\pi/2) = 0, \tag{39}$$

where $\tilde{\mu} \in R$ is the separation constant.

First, we investigate the problem {(36), (37)}. The general solution to the equation (36) is defined as [23]

$$Q_m(\rho) = c_3 \rho^{-2\beta} I_{\tilde{\omega}}(\sigma_m \rho / c) + c_4 \rho^{-2\beta} K_{\tilde{\omega}}(\sigma_m \rho / c), \ \rho \in [0,1], \tag{40}$$

where $\tilde{\omega} = \sqrt{4\beta^2 + \tilde{\mu}}$, c_3 and c_4 are arbitrary constants, $K_l(x)$ is a MacDonald function of order l [23].

From (40), it follows that the solutions to equation (36), satisfying condition (37), exist for $\tilde{\mu} \geq 0$ and they are defined by

$$Q_m(\rho) = c_3 \rho^{-2\beta} I_{\tilde{\omega}}(\sigma_m \rho / c), \ \tilde{\omega} \geq 2\beta, \ m \in N. \tag{41}$$

Now, let us consider the problem {(38), (39)}. The general solution to equation (38) has the form

$$S(\varphi) = c_5 F\left(\beta + \tilde{\omega}/2, \beta - \tilde{\omega}/2, \beta + 1/2; \sin^2\varphi\right) +$$

$$+ c_6 (\sin\varphi)^{1-2\beta} F\left[(1+\tilde{\omega})/2, (1-\tilde{\omega})/2, (3/2) - \beta; \sin^2\varphi\right], \tag{42}$$

where c_5 and c_6 are arbitrary constants.

By satisfying the function (42) to condition (39) and applying the formulas (29), (30), we obtain $c_6 = k_3(\tilde{\omega}) c_5$, where

$$k_3(\tilde{\omega}) = -\frac{\Gamma(1/2+\beta)\Gamma(1-\beta-\tilde{\omega}/2)\Gamma(1-\beta+\tilde{\omega}/2)}{\Gamma(3/2-\beta)\Gamma((1-\tilde{\omega})/2)\Gamma((1+\tilde{\omega})/2)}. \tag{43}$$

By substituting in (42) $c_6 = k_3(\tilde{\omega}) c_5$ and assuming $c_5 = 1$ (without loss of generality), we have

$$S(\varphi) = F\left(\beta + \tilde{\omega}/2, \ \beta - \tilde{\omega}/2, \beta + 1/2; \sin^2\varphi\right) +$$

$$+ k_3(\tilde{\omega}) (\sin\varphi)^{1-2\beta} F\left[(1+\tilde{\omega})/2, (1-\tilde{\omega})/2, 3/2 - \beta; \sin^2\varphi\right]. \tag{44}$$

Based on (35), (41), and (44), we conclude that continuous and non-trivial solutions in $\overline{\Delta}_0^+$ for the problem {(33), (34)} are given by

$$u_{2m}^{+}(x,y)=c_3\rho^{-2\beta}I_{\tilde{\omega}}(\sigma_m\rho/c)\Big\{F\big(\beta+\tilde{\omega}/2,\ \beta-\tilde{\omega}/2,\beta+1/2;\sin^2\varphi\big)+$$
$$+k_3(\tilde{\omega})(\sin\varphi)^{1-2\beta}F\big[(1+\tilde{\omega})/2,(1-\tilde{\omega})/2,3/2-\beta;\sin^2\varphi\big]\Big\},\ c_3\neq 0,\ m\in N, \qquad (45)$$

where $\rho=\sqrt{x^2+y^2}$, $\varphi=arctg(y/x)$.

From this, by direct calculation, we find

$$\begin{cases}\tau_m^{+}(x)=\lim\limits_{y\to+0}u_{2m}(x,y)=c_3x^{-2\beta}I_{\omega}(\sigma_m x/c),\ x\in[0,1];\\ \nu_m^{+}(x)=\lim\limits_{y\to+0}y^{2\beta}\dfrac{\partial}{\partial y}u_{2m}(x,y)=c_3(1-2\beta)k_3(\tilde{\omega})x^{-1}I_{\omega}(\sigma_m x/c),\ x\in(0,1).\end{cases} \qquad (46)$$

Further, based on the conditions $u(x,y)\in C(\overline{\Delta})$ and (9) the following equalities follow:

$$\begin{cases}\tau_m^{-}(x)=\tau_m^{+}(x),\ x\in[0,1],\\ \nu_m^{-}(x)=\nu_m^{+}(x),\ x\in(0,1).\end{cases} \qquad (47)$$

By substituting (31) and (46) into (47) and assuming $\omega=\tilde{\omega}$, we have a homogeneous system of equations regarding c_1 and c_3:

$$\begin{cases}k_2(\omega)c_1-(1-2\beta)k_3(\omega)c_3=0,\\ k_1(\omega)c_1-c_3=0,\end{cases} \qquad (48)$$

where $k_1(\omega)$, $k_2(\omega)$, $k_3(\omega)$ - are constants, defined by formulas (32), (43).

System (48) must have a nontrivial solution. Therefore, the determinant of this system equals zero. By forming the principal determinant of system (48) and setting it to zero, and then using the formula [24] $\Gamma(z)\Gamma(1-z)=\pi/\sin(z\pi)$, $z\notin Z$, we derive a trigonometric equation in the terms of ω: $\sin\left[\pi(\beta+\omega/2)\right]+\cos(\omega\pi/2)=0$. Extracting the solutions of this equation and considering the condition $\omega>2\beta$, we find

$$\omega_n=2n-\beta-1/2,\ n\in N. \qquad (49)$$

Based on (49), the numbers $\mu_n=\omega_n^2-4\beta^2$, $n\in N$ are the eigenvalues for the problems {(24), $\left|\lim\limits_{\eta\to+\infty}Y(\eta)\right|<+\infty$} and {(38),(39)}.

Considering the proofs above and equations (28), (45), $\omega=\tilde{\omega}=\omega_n$, $c_3=k_1(\omega)c_1$ (as follows from system (48)), we conclude that the functions

$$u_{2nm}(x,y)=\begin{cases} c_3\rho^{-2\beta}I_{\omega_n}(\sigma_m\rho/c)\left\{F\left(\beta+\frac{\omega_n}{2},\beta-\frac{\omega_n}{2},\beta+\frac{1}{2};\sin^2\varphi\right)+\right. \\ \left.+k_3(\omega_n)(\sin\varphi)^{1-2\beta}F\left(\frac{1+\omega_n}{2},\frac{1-\omega_n}{2},\frac{3}{2}-\beta;\sin^2\varphi\right)\right\},(x,y)\in\bar{\Delta}_0^+; \\ \left[c_3/k_1(\omega_n)\right]x^{-2\beta-\omega_n}\left(x^2-y^2\right)^{\omega_n/2}I_{\omega_n}\left(\sigma_m\sqrt{x^2-y^2}/c\right)\times \\ \times F\left(\beta+\frac{\omega_n}{2},\frac{1+\omega_n}{2},1+\omega_n;1-\frac{y^2}{x^2}\right),\ (x,y)\in\bar{\Delta}_1,\ n,m\in N \end{cases} \tag{50}$$

are continuous and nontrivial in $\bar{\Delta}^+$ solutions to the problem {(16)-(19)}.

Then, the functions

$$U_{2nm}(x,y,z)=u_{2nm}(x,y)Z_m(z),\ n,m\in N, \tag{51}$$

where $Z_m(z)$ and $u_{nm}(x,y)$ are functions defined by equations (12) and (50), are continuous and nontrivial in Ω^+ solutions to the equation (1), satisfying the conditions (2), $U(x,y,z)|_{\bar{S}_2\cup\bar{S}_3\cup\bar{S}_4}=0$ and $U(0,y,z)=0$.

Note that in work [25], it was proved that the function (44) for $\tilde{\omega}=\omega_n$ is rewritten as

$$S_n(\varphi)=\frac{2^{1/2-\beta}}{\Gamma(3/2-\beta)}(\cos\varphi)^{1-2\beta}F\left[(1-\omega_n)/2,(1+\omega_n)/2,3/2-\beta;\cos^2\varphi\right]=$$

$$=(\sin 2\varphi)^{1/2-\beta}\mathrm{P}^{\beta-1/2}_{(\omega_n-1)/2}(-\cos 2\varphi), \tag{52}$$

where $\mathrm{P}_a^b(x)-$ is the Legendre function [16].

From Lemma 3 of work [26] based on Theorem 2 of work [27], for $\beta\in(0,1/2)$, it follows that the system of eigenfunctions (52) is complete in the space $L_2(0,\pi/2)$.

Now, we consider the function (14). Based on formula (14), it is straightforward to derive the following boundary problem, which can be considered over the interval $0<x<1$ due to the symmetry of function $u=u(x,y)$ relative to $x=0$:

$$u_1(x,y)\in C\left(\bar{\Delta}^+\right)\cap C^{2,2,2}_{x,y,z}\left(\Delta_0^+\cup\Delta_1\right), \tag{53}$$

$$u_{1xx}+\operatorname{sgn}y\cdot u_{1yy}+\frac{2\beta}{x}u_{1x}+\frac{2\beta}{y}u_{1y}-\lambda_m u_1=0,\ (x,y)\in\Delta^+, \tag{54}$$

$$u_{1x}(x,y)|_{x=0}=0,\ y\in(0,1),\ u_1(x,y)|_{\overline{OC}}=u_1(x,-x)=0,\ x\in[0,1/2], \tag{55}$$

$$\lim_{y\to -0}(-y)^{2\beta}u_{1y}(x,y)=\lim_{y\to +0}y^{2\beta}u_{1y}(x,y),\ \ x\in(0,1). \tag{56}$$

Similarly to the problem {(16)-(19)}, we find continuous and nontrivial solutions in Δ^+ for the problem {(53)-(56)} in the form

$$u_{1nm}(x,y)=\begin{cases} c_3\rho^{-2\beta}I_{\theta_n}(\sigma_m\rho/c)\left\{F\left(\beta+\dfrac{\theta_n}{2},\beta-\dfrac{\theta_n}{2},\beta+\dfrac{1}{2};\sin^2\varphi\right)+\right.\\ \left.+\tilde{k}_3(\theta_n)(\sin\varphi)^{1-2\beta}F\left(\dfrac{1+\theta_n}{2},\dfrac{1-\theta_n}{2},\dfrac{3}{2}-\beta;\sin^2\varphi\right)\right\},(x,y)\in\bar{\Delta}_0;\\ \left[c_3/k_1(\theta_n)\right]x^{-2\beta-\theta_n}\left(x^2-y^2\right)^{\theta_n/2}I_{\theta_n}\left(\sigma_m\sqrt{x^2-y^2}/c\right)\times\\ \times F\left(\beta+\dfrac{\theta_n}{2},\dfrac{1+\theta_n}{2},1+\theta_n;1-\dfrac{y^2}{x^2}\right),\ (x,y)\in\bar{\Delta}_1,\ n,m\in N \end{cases} \tag{57}$$

where $\tilde{k}_3(\theta_n)=-\dfrac{\Gamma(\beta+1/2)\Gamma\left[(1-\theta_n)/2\right]\Gamma\left[(1+\theta_n)/2\right]}{\Gamma(3/2-\beta)\Gamma(\beta+\theta_n/2)\Gamma(\beta-\theta_n/2)}$, $\rho=\sqrt{x^2+y^2}$, $\varphi=arctg(y/x)$, $\theta_n=2n+\beta-3/2,\ n\in N$.

Then, the functions

$$U_{1nm}(x,y,z)=u_{1nm}(x,y)Z_m(z),\ n,m\in N, \tag{58}$$

where $Z_m(z)$ and $u_{1nm}(x,y)$ are functions defined by equations (12) and (57), are continuous and nontrivial in Ω^+ solutions to the equation (1), satisfying the conditions (2), $U(x,y,z)\big|_{\bar{S}_2\cup\bar{S}_3\cup\bar{S}_4}=0$ and $x^{2\beta}U_x(x,y,z)\big|_{x=0}=0$.

3. Reducing Problem G to Problem T (Tricomi) in the Domain Ω^+. Uniqueness of the Solution to Problem T

We seek the solution $U(x,y,z)$ to the Problem G in the following form:

$$U(x,y,z)=U_1(x,y,z)+U_2(x,y,z),\ (x,y,z)\in\bar{\Omega}^+, \tag{59}$$

where the functions $U_1(x,y,z)$ and $U_2(x,y,z)$ are defined respectively as:

$$U_1(x,y,z)=\frac{1}{2}\left[U(x,y,z)+U(-x,y,z)\right],\ (x,y,z)\in\bar{\Omega}^+, \tag{60}$$

$$U_2(x,y,z)=\frac{1}{2}\left[U(x,y,z)-U(-x,y,z)\right],\ (x,y,z)\in\bar{\Omega}^+. \tag{61}$$

Clearly, if $(x,y,z)\in\bar{\Omega}^+$, then $(-x,y,z)\in\bar{\Omega}^-$.

Next, we consider the boundary problem that can be formulated with respect to the auxiliary function $U_2=U_2(x,y,z)$:

$$U_2(x,y,z)\in C\left(\bar{\Omega}^+\right)\cap C^{2,2,2}_{x,y,z}\left(\Omega^+\setminus\{y=0\}\right), \tag{62}$$

$$U_{2xx}+\operatorname{sgn} y\cdot U_{2yy}+U_{2zz}+\frac{2\beta}{x}U_{2x}+\frac{2\beta}{|y|}U_{2y}+\frac{2\gamma}{z}U_{2z}=0,\ (x,y,z)\in\Omega^+, \tag{63}$$

$$U_2(x,y,z)\big|_{\bar{S}_0\cap\{x\geq 0\}}=\frac{1}{2}\left[F(x,y,z)-F(-x,y,z)\right]=\tilde{F}(x,y,z), \tag{64}$$

$$U_2(x,y,z)\big|_{x=0}=0,\ y\in[0,1],\ z\in[0,c], \tag{65}$$

$$U_2(x,y,z)\big|_{\bar{S}_2}=U_2(x,-x,z)=0,\ x\in[0,1/2],\ z\in[0,c], \tag{66}$$

$$U_2(x,y,z)\big|_{\bar{S}_3\cap\{x\geq 0\}}=0,\ U_2(x,y,z)\big|_{\bar{S}_4\cap\{x\geq 0\}}=0, \tag{67}$$

$$\lim_{y\to -0}(-y)^{2\beta}U_{2y}(x,y,z)=\lim_{y\to +0}y^{2\beta}U_{2y}(x,y,z),\ \ x\in(0,1),\ z\in(0,c). \tag{68}$$

Thus, the formulated boundary problem {(62)-(68)} concerning the function $U_2(x,y,z)$ represents a three-dimensional Tricomi problem in Ω^+. We prove the unique solvability of this problem.

Theorem 1. If a solution to Problem T exists and the condition $\lim_{\varphi\to 0}(\sin 2\varphi)^{2\beta}V_\varphi(\rho,\varphi,z)=2^{2\beta}(1-2\beta)k_3(\omega_n)V(\rho,0,z)$ is satisfied, then it is unique, where $U(x,y,z)=V(\rho,\varphi,z)$ - is a solution to Problem T in the domain $\Omega_0\cap\{x>0\}$, and ρ,φ,z - are cylindrical coordinates linked with Cartesian coordinates by $\rho=\sqrt{x^2+y^2}$, $\varphi=arctg(y/x),\ z=z$.

Proof. Let $U(x,y,z)=V(\rho,\varphi,z)$ - be a solution to Problem T in the domain $\Omega_0\cap\{x>0\}$. In these coordinates, the condition (64) is written as

$$V(1,\varphi,z)=f(\varphi,z),\ \ \varphi\in[0,\pi/2],\ z\in[0,c], \tag{69}$$

where $f(\varphi,z)=\tilde{F}(\cos\varphi,\sin\varphi,z)$.

Using $V(\rho,\varphi,z)$ and the eigenfunctions (12), (52), we compose the following function:

$$\vartheta_{nm}(\rho)=d_m\int_0^c\int_0^{\pi/2}V(\rho,\varphi,z)(\sin 2\varphi)^{2\beta}S_n(\varphi)z^{2\gamma}Z_m(z)\,d\varphi dz,\ n,m\in N, \tag{70}$$

where $d_m=2\left[cJ_{3/2-\gamma}(\sigma_m)\right]^{-2},\ m\in N$.

For the function $\vartheta_{nm}(\rho)$, the following equation is valid:

$$B_{2\beta}^{\rho}\vartheta_{nm}^{\varepsilon_1\varepsilon_2}(\rho) = -\frac{d_m}{\rho^2}\int_{\varepsilon_2}^{c-\varepsilon_2}\left\{\left\{(\sin 2\varphi)^{2\beta}\left[V_{\varphi}S_n(\varphi) - VS_n'(\varphi)\right]\right\}\Big|_{\varphi=\varepsilon_1}^{\varphi=\pi/2-\varepsilon_1} -\right.$$

$$\left. -\mu_n\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}V(\rho,\varphi,z)(\sin 2\varphi)^{2\beta}S_n(\varphi)d\varphi\right\}z^{2\gamma}Z_m(z)dz -$$

$$-d_m\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left\{\left\{\left[V_z(\rho,\varphi,z)Z_m(z) - V(\rho,\varphi,z)Z_m'(z)\right]z^{2\gamma}\right\}\Big|_{z=\varepsilon_2}^{z=c-\varepsilon_2} -\right.$$

$$\left. -(\sigma_{\gamma m}/c)^2\int_{\varepsilon_2}^{c-\varepsilon_2}V(\rho,\varphi,z)z^{2\gamma}Z_m(z)dz\right\}(\sin 2\varphi)^{2\beta}S_n(\varphi)d\varphi,$$

where ε_1 and ε_2 are sufficiently small positive numbers.

From here, passing to the limit as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, considering (62), (65), (66), (67), (11), and (39), as well as the conditions of Theorem 1 and the notation (70), we obtain the equality

$$\vartheta_{nm}''(\rho) + \frac{4\beta+1}{\rho}\vartheta_{nm}'(\rho) - \left(\lambda_m + \frac{\mu_n}{\rho^2}\right)\vartheta_{nm}(\rho) = 0, \ \rho \in (0,1).$$

Thus, the function $\vartheta_{nm}(\rho)$ satisfies to the equation (36) with $\tilde{\mu} = \mu_n$. Furthermore, due to the boundary condition (69), from (70) it follows that function $\vartheta_{nm}(\rho)$ satisfies the following boundary condition:

$$\vartheta_{nm}(1) = f_{nm}, \tag{71}$$

where

$$f_{nm} = d_m\int_0^c\int_0^{\pi/2} f(\varphi,z)(\sin 2\varphi)^{2\beta}S_n(\varphi)z^{2\gamma}Z_m(z)d\varphi dz. \tag{72}$$

Therefore, function $\vartheta_{nm}(\rho)$, having the form (70), satisfies the equation (36) with $\tilde{\mu} = \mu_n$ and the conditions (37), (71). Hence, by obeying the general solution

(40) of equation (36) to these conditions, we find the coefficients c_3 and c_4 as $c_3 = f_{nm} / I_{\omega_n}(\sigma_m / c)$, $c_4 = 0$.

Substituting these values into (40), we find $\vartheta_{nm}(\rho)$ in the form:

$$\vartheta_{nm}(\rho) = \rho^{-2\beta} I_{\omega_n}(\sigma_m \rho / c) f_{nm} \Big/ I_{\omega_n}(\sigma_m / c), \ \rho \in [0,1]. \tag{73}$$

To prove Theorem 1, it is sufficient to prove that the homogeneous Problem T has only trivial solutions. Let $\tilde{F}(x,y,z) \equiv 0$, i.e. $f(\varphi,z) \equiv 0$. Then $f_{nm} = 0$ for all $n,m \in N$. Based on this, from (73) and (70), it follows that

$$\int_0^c \int_0^{\pi/2} V(\rho,\varphi,z)(\sin 2\varphi)^{2\beta} S_n(\varphi) z^{2\gamma} Z_m(z) d\varphi dz = 0, \ \rho \in [0,1].$$

From here, due to the completeness of the system of eigenfunctions (12) in the space $L_2(0,c)$ with weight $z^{2\gamma}$ and $V(\rho,\varphi,z) = U(x,y,z) \in C(\overline{\Omega}_0)$, it follows

$$\int_0^{\pi/2} V(\rho,\varphi,z)(\sin 2\varphi)^{2\beta} S_n(\varphi) d\varphi = 0, \ \rho \in [0,1], \ z \in [0,c].$$

Considering the completeness of the system of eigenfunctions (52) in the space $L_2(0,\pi/2)$ and $V(\rho,\varphi,z) = U(x,y,z) \in C(\overline{\Omega})$, from the last equality, it follows that $V(\rho,\varphi,z) = U(x,y,z) \equiv 0$ in $\overline{\Omega}_0 \cap \{x \geq 0\}$.

Using this equality and $U(x,y,z) = V(\rho,\varphi,z)$, it is easy to verify that $U(x,+0,z) \equiv 0, \ \lim\limits_{y \to +0} y^{2\beta} U_y(x,y,z) \equiv 0, \ x \in [0,1], \ z \in [0,c]$.

Then, due to the matching conditions (68), the following equalities are valid:

$$U(x,-0,z) \equiv 0, \ \lim_{y \to -0} (-y)^{2\beta} U_y(x,y,z) \equiv 0, \ x \in [0,1], \ z \in [0,c]. \tag{74}$$

From the results of work [28], it follows that the solution to the equation

$$U_{xx} - U_{yy} + U_{zz} + \frac{2\beta}{x} U_x - \frac{2\beta}{y} U_y + \frac{2\gamma}{z} U_z = 0, \ (x,y,z) \in \Omega_1,$$

satisfying the conditions (74), is identically zero, i.e., $U(x,y,z)\equiv 0,\ (x,y,z)\in\bar{\Omega}_1$. Theorem 1 has been proved.

4. Constructing the Solution to Problem T

Substituting the values $c_{3nm}=d_m f_{nm}\big/ I_{\omega_n}(\sigma_m/c)$ into the equality (50), and then the obtained function into (51), we find the particular solution to Problem T in the form

$$U_{2nm}(x,y,z)=\begin{cases} U^{+}_{2nm}(x,y,z),\ (x,y,z)\in\bar{\Omega}_0\cap\{x\geq 0\},\ n,m\in N,\\ U^{-}_{2nm}(x,y,z),\ (x,y,z)\in\bar{\Omega}_1,\ n,m\in N,\end{cases}$$

where

$$U^{+}_{nm}(x,y,z)=Z_m(z)\vartheta_{nm}(\rho)S_n(\varphi),\ \ \rho=\sqrt{x^2+y^2},\ \ \varphi=arctg(y/x), \tag{75}$$

$$U^{-}_{nm}(x,y,z)=Z_m(z)X_{nm}(\xi)Y_n(\eta)/k_1(\omega_n),\ \xi=\sqrt{x^2-y^2},\ \eta=x^2/\xi^2, \tag{76}$$

$$X_{nm}(\xi)=\xi^{-2\beta}I_{\omega_n}(\sigma_m\xi/c)f_{nm}/I_{\omega_n}(\sigma_m/c), \tag{77}$$

$$Y_n(\eta)=\eta^{-\beta-\omega_n/2}F\big(\beta+\omega_n/2,(1+\omega_n)/2,1+\omega_n;1/\eta\big), \tag{78}$$

$Z_m(z)$, $\vartheta_{nm}(\rho)$, $S_n(\varphi)$ and f_{nm} are defined by equations (12), (73), (52), and (72), respectively.

Theorem 2. If the function $f(\varphi,z)$ satisfies the following conditions:

I. $f(\varphi,z)\in C^{4,5}_{\varphi,z}(\bar{\Pi})$, where $\Pi=\{(\varphi,z):\varphi\in(0,\pi/2),z\in(0,c)\}$;

II. $\left.\dfrac{\partial^j}{\partial\varphi^j}f(\varphi,z)\right|_{\varphi=0}=0$, $\left.\dfrac{\partial^j}{\partial\varphi^j}f(\varphi,z)\right|_{\varphi=\pi/2}=0$, $j=\overline{0,3}$;

III. $\left.\dfrac{\partial^j}{\partial z^j}f(\varphi,z)\right|_{z=0}=0$, $\left.\dfrac{\partial^j}{\partial z^j}f(\varphi,z)\right|_{z=c}=0$, $j=\overline{0,4}$.

Then the solution to Problem T exists and is defined by the formula

$$U_2(x,y,z)=\begin{cases}\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}U^{+}_{2nm}(x,y,z),\ (x,y,z)\in\bar{\Omega}_0\cap\{x\geq 0\},\\ \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}U^{-}_{2nm}(x,y,z),\ (x,y,z)\in\bar{\Omega}_1,\end{cases} \tag{79}$$

where $U^{+}_{2nm}(x,y,z)$, $U^{-}_{2nm}(x,y,z)$ are the functions defined by formulas (75) and (76), respectively.

Theorem 2 can be proved similarly as in works [29], [30], [31].

5. Reducing Problem G to Problem TN (Tricomi-Neumann)

Based on formula (60), it is straightforward to derive the following boundary problem, which can be considered over the interval $0 < x < 1$ due to the symmetry of function $u = u(x, y, z)$ with respect to the plane $x = 0$:

$$U_1(x,y,z) \in C\left(\bar{\Omega}^+\right) \cap C^1\left(\Omega^+\right) \cap C^{2,2,2}_{x,y,z}\left(\Omega^+ \setminus \{y=0\}\right), \tag{80}$$

$$U_{1xx} + \operatorname{sgn} y \cdot U_{1yy} + U_{1zz} + \frac{2\gamma}{z} U_{1z} = 0, \ (x,y,z) \in \Omega^+ \setminus \{y=0\}, \tag{81}$$

$$U_1(x,y,z)\big|_{\bar{S}_0 \cap \{x \geq 0\}} = \frac{F(x,y,z) + F(-x,y,z)}{2} = \Psi(\varphi, z), \ \varphi \in [0, \pi/2], \ z \in [0,c], \tag{82}$$

$$U_{1x}(x,y,z)\big|_{x=0} = 0, \ y \in [0,1], \ z \in [0,c], \tag{83}$$

$$U_1(x,y,z)\big|_{\bar{S}_2} = U_1(x,-x,z) = 0, \ x \in [0,1/2], \ z \in [0,c], \tag{84}$$

$$U_1(x,y,z)\big|_{\bar{S}_3 \cap \{x \geq 0\}} = 0, \ U_1(x,y,z)\big|_{\bar{S}_4 \cap \{x \geq 0\}} = 0, \tag{85}$$

where $\Psi(\varphi, z) = \left[F(\cos\varphi, \sin\varphi, z) + F(-\cos\varphi, \sin\varphi, z)\right]/2$.

Thus, the formulated boundary problem (80)-(85) concerning the function $U_1(x,y,z)$ represents a three-dimensional Tricomi-Neumann problem in Ω^+.

Similarly to Theorem 2, the following theorem can be proved:

Theorem 3. Let function $\Psi(\varphi, z)$ satisfy the conditions of Theorem 2. Then the solution to the TN problem in Ω^+ exists and is defined by the formula

$$U_1(x,y,z) = \begin{cases} \sum\limits_{n=1}^{\infty} \sum\limits_{m=1}^{\infty} Z_m(z) \vartheta_{nm}(\rho) S_n(\varphi), \ \rho \in [0,1], \varphi \in [0, \pi/2], \ z \in [0,c], \\ \sum\limits_{n=1}^{\infty} \sum\limits_{m=1}^{\infty} Z_m(z) X_{nm}(\xi) Y_n(\eta) / k_1(\omega_n), \ \xi \in [0,1], \eta \in [1, \infty), \ z \in [0,c], \end{cases} \tag{86}$$

where $Z_m(z) = z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z / c)$, σ_m - are zeros of the function $J_{1/2-\gamma}(x)$,

$$S_n(\varphi) = (\sin 2\varphi)^{1/2-\beta} \mathrm{P}^{1/2-\beta}_{(\theta_n - 1)/2}(-\cos 2\varphi), \ \varphi = arctg(y/x), \tag{87}$$

$$\vartheta_{nm}(\rho) = \frac{\rho^{-2\beta} I_{\theta_n}(\sigma_m \rho / c) \psi_{nm}}{(\theta_n - 2\beta) I_{\theta_n}(\sigma_m / c) + (\sigma_m / c) I_{\theta_n + 1}(\sigma_m / c)}, \ \rho = \sqrt{x^2 + y^2},$$

$$X_{nm}(\xi) = \frac{\xi^{-2\beta} I_{\theta_n}(\sigma_m \xi / c) \psi_{nm}}{(\theta_n - 2\beta) I_{\theta_n}(\sigma_m / c) + (\sigma_m / c) I_{\theta_n + 1}(\sigma_m / c)}, \ \xi = \sqrt{x^2 - y^2},$$

$$Y_n(\eta) = \eta^{-\beta-\theta_n/2} F\left(\beta+\theta_n/2, (1+\theta_n)/2, 1+\theta_n; 1/\eta\right),\ \eta = x^2/\xi^2,$$

$$\psi_{nm} = d_m \int_0^c \int_0^{\pi/2} \Psi(\varphi, z)(\sin 2\varphi)^{2\beta} S_n(\varphi) z^{2\gamma} Z_m(z)\, d\varphi dz,\ d_m = 2\left[cJ_{3/2-\gamma}(\sigma_m)\right]^{-2},$$

$m \in N$, $\theta_n = 2n + \beta - 3/2,\ n \in N$.

The proof of the uniqueness of the solution to the TN problem relies on the completeness of the system of eigenfunctions (12) in the space $L_2(0,c)$ with weight $z^{2\gamma}$ and the completeness of the system of eigenfunctions (87) in the space $L_2(0,\pi/2)$.

6. Uniqueness and Existence of the Solution to Problem G

Theorem 4. If there exists a solution to the problem G, then it is unique.

Proof. Assume that $F(x,y,z)=0$. Then in the Tricomi and Tricomi-Neumann problems, the specified functions will be equal to zero, i.e., $f(\varphi,z)=0$ and $\Psi(\varphi,z)=0$. It has been proven that the homogeneous Tricomi and Tricomi-Neumann problems have only trivial solutions, i.e., $U_1(x,y,z)\equiv 0$, $(x,y,z)\in\bar{\Omega}^+$ and $U_2(x,y,z)\equiv 0$, $(x,y,z)\in\bar{\Omega}^+$. Substituting these equalities into formula (59), we have $U(x,y,z)\equiv 0, (x,y,z)\in\bar{\Omega}$. Theorem 4 has been proved.

It is known that in Ω^+ solutions to the Tricomi and Tricomi-Neumann problems are determined using functional series (79) and (86) respectively. Substituting them into formula (59), we obtain the solution to Problem G in $\bar{\Omega}$. This completes the investigation of Problem G.

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