${}^{2}R_5$ DA IKKI O‘LCHOVLI TO‘LA SIRTLAR

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Annotatsiya. *Ko‘p o‘lchovli fazolarda birdan katta o‘lchamga ega bo‘lgan sirtlarning geometriyasini o‘rganish geometriyadagi eng qiyin masalalardandir. Ushbu maqolada ikki o‘lchovli sirtlar besh o‘lchovli ikki indeksli psevdoyevklid fazosida qaralgan. Hech qanday to‘rt o‘lchovli gipertekisliklarda yotmaydigan, bir xil urinma tekisliklarga ega va ichki geometriyasi ham bir xil bo‘lgan to‘la sirtlarning mavjudligi bo‘yicha shartlar aniqlangan va teoremalar isbotlangan.*

Kalit so‘zlar: *Besh o‘lchovli psevdoyevklid fazosi, ajraluvchi metrika, haqiqiy, mavhum va nol radiusli sfera, izotrop konus*.

ПОЛНЫЕ ДВУМЕРНЫЕ ПОВЕРХНОСТИ В ${}^{2}R_5$

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Аннотация. *Изучение геометрии поверхностей, имеющих коразмерность больше единицы в многомерных пространствах, является одной из самых сложных задач геометрии. В статье рассматриваются двумерные поверхности в пятимерном псевдоевклидовом пространстве индекса два. Выявляются условия и доказываются теоремы о существовании поверхности, не лежащей в четырехмерной гиперплоскости и имеющей касательные плоскости с одной внутренней геометрией.*

Ключевые слова: *Пятимерное псевдоевклидово пространство, вырожденная метрика, сфера действительного, мнимого и нулевого радиуса, изотропный конус*.

COMPLETE TWO-DIMENSIONAL SURFACES IN ${}^{2}R_5$

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Abstract. *The study of the geometry of surfaces having a codimension greater than one in multidimensional spaces is one of the most difficult problems in geometry. Two-dimensional surfaces in a five-dimensional pseudo-Euclidean space of index two are considered in the article. Conditions are revealed and theorems are proved on the existence of a surface that does not lie in a four-dimensional hyperplane and has tangent planes with one internal geometry.*

Key words: *five-dimensional pseudo-Euclidean space, degenerate metric, sphere of real, imaginary, and zero radius, isotropic cone*.

1 Introduction

Two-dimensional surfaces in five-dimensional space were studied by Yu.A. Aminov and Ya.S. Nasedkin [1-3]. The general theory of m- dimensional surfaces in n- dimensional spaces is presented in the monograph of B.A. Rosenfeld [8]. The addition of the scale dimension to the Minkowski four-dimensional space means the need to use the five-dimensional space of time [11]. D.D. Sokolov and his followers [13] studied the theory of surfaces in a three-dimensional pseudo-Euclidean space and the solution to some problems of geometry “in the whole”.

Recently, the theory of pseudo-Riemannian manifolds has been widely developed in the studies of Bang-Yong Chen [9-10]. N.V. Sherobakina [12] studied the theory of two-dimensional surfaces in semi-Euclidean spaces by the method of external forms.

The study of the geometry of multidimensional surfaces is closely related to the physical interaction of matter. Often, the space-time dimension and the geometrization of physical interactions are conveniently expressed in eight-dimensional space [14-15].

When a multidimensional space is Euclidean, then the geometry of its subspaces is also Euclidean. Moreover, the reduction in dimension does not change the geometric methods used in the study. Experience shows that in the case of pseudo-Euclidean spaces, the geometry of subspaces does not always coincide with the geometry of the enveloping space [4-7].

Therefore, the study of submanifolds of pseudo-Euclidean spaces has an individual character. From this point of view, the study of two-dimensional surfaces in a five-dimensional pseudo-Euclidean space of index two has its own characteristics and requires an individual approach to their study.

2 Main results

The geometry of curves and surfaces in $n-$ dimensional pseudo-Euclidean space is studied in the monograph by B.A. Rosenfeld. But the geometry of surfaces with codimension, that is, $m-$ surfaces when $1<m<n-1$, is not considered in this monograph. In the Euclidean space R_5, the surface F^2, where the codimension is $n-m=3$, was studied. In particular, the problem of the belonging of the surface F^2 to a sphere and a four-dimensional plane was considered.

In the five-dimensional space ${}^2R_5$, the three-dimensional Euclidean space R_3 is its subspace. Therefore, all the diversity of two-dimensional surfaces F^2 in the Euclidean three-dimensional space also exists in ${}^2R_5$. Moreover, in the four-dimensional subspaces of ${}^2R_5$ there also exist two-dimensional surfaces of various types. Therefore, the study of two-dimensional surfaces in the four-dimensional subspaces of ${}^2R_5$ is not of interest. Therefore, the question arises: is there a two-dimensional surface that is not contained in the four-dimensional subspaces of ${}^2R_5$?

Definition. A two-dimensional surface $F^2 \subset {}^2R_5$ is called whole if it is spatial, that is, does not belong to a hyperplane.

We divide the entire two-dimensional surface in ${}^2R_5$ into three more groups. These groups differ from each other in that the geometry on the tangent plane will be of the same type: Euclidean, Minkowski plane, or Galilean plane.

Problem: Is there a complete two-dimensional surface of space ${}^2R_5$ whose tangent plane geometry is of only one type?

The problem is solved in the case when the surface F^2

$$\vec{r}(u,\ v)=\sum_{i=1}^{5} x_i(u,\ v)\vec{e}_i,\ x_i(u,\ v)\in C^k(D) \qquad (1)$$

is defined by the vector equation

$$r(u,v)=x_1(u)e_1+x_2(u)e_2+z(u,v)e_3+y_1(v)e_4+y_2(v)e_5 \qquad (2)$$

where $x_1(u)$, $x_2(u)$, $z(u,v)$, $y_1(v)$, $y_2(v)$ of $C^2(D)$ are functions.

We study the surface in some neighborhood of point $M(u_0,\ v_0)$. Let point $M\in F^2$ have coordinates $(u_0,\ v_0)$.

Example. Consider a two-dimensional surface

$$r(u,\ v)=\rho(u)+v\xi_5,$$

where $\rho(u)$ is the curve in ${}^2R_5$, $\rho(u)\in C^5$.

For this surface not to belong to one four-dimensional subspace, it suffices to require that $k_4\neq 0$. Indeed, two-dimensional surface $r(u,\ v)$ is a linear surface. The curve defined by equation $\rho(u)$, is a directing line. Condition $k_4\neq 0$ means that its fourth-order curvature is nonzero. Therefore, it does not belong to the fourth-order hyperplane. Since the directing line does not belong to a 4D plane, the surface itself does not belong to any 4D hyperplane either. If we assume that parameter v belongs to the segment, then the surface under study can be considered as the surface of a strip.

In the study by Yu. A. Aminov [1]-[3], the existence of two-dimensional surface F_2 lying on four-dimensional sphere S_4 of space R_5 was proved.

The method of superimposed space ${}^2R_5$ can be viewed as Euclidean space ${}^2R_5$. If the set of surfaces F_2 (two-dimensional spaces lying on sphere S_4) is denoted by $F_2(S_4)$, then the following theorem is valid.

Theorem 1. There is no two-dimensional surface from $F_2(S_4)$ where the geometry on the tangent plane is of the same type.

Proof. In the condition of membership of surface F_2 in space R_5 to sphere $S^4 \subset R_5$ is given. The sphere of space ${}^2R_5$, that is, the locus of points equidistant from a given point, is considered in three forms, when the radius of the sphere is real, imaginary, or isotropic.

The sphere in ${}^2R_5$ has the following equation

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = \begin{cases} 1 & \textit{sphere of real radius;} \\ -1 & \textit{sphere of imaginary radius;} \\ 0 & \textit{sphere of zero radius.} \end{cases}$$

If we use the superimposed space method, that is, the coordinate system in ${}^2R_5$ is taken as the coordinate in R_5, then the sphere in R_5 is given by the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1$.

Is it easy to establish unique correspondence to the sphere of spaces ${}^2R_5$ and R_5. The unique correspondence is violated only at points $S_4 \subset R_5$ that are the points of the isotropic sphere. Using this unique correspondence, we prove that each surface from class $F_2(S_4)$ has a tangent plane parallel to the tangent planes, the real sphere and a sphere of imaginary radius. Therefore, surface $F_2 \subset F_2(S_4)$ cannot be a complete surface in ${}^2R_5$. Since two-dimensional surfaces in ${}^2R_5$ are considered, it is necessary to state some facts related to the geometry of two-dimensional planes of space ${}^2R_5$.

The two-dimensional plane in ${}^{2}R_5$ can be viewed as a linear combination of two non-collinear vectors attached to a given point. However, these vectors cannot always be chosen as the basis vectors of a given plane.

If all vectors parallel to the plane have a real norm, then the geometry on the plane is Euclidean. Among the vectors parallel to the plane, there can simultaneously exist vectors with real norm and imaginary norm. In addition, there are two directions corresponding to the isotropic direction. This is the geometry on Minkowski plane.

If among the vectors parallel to the two-dimensional plane, only one vector is isotropic, then the geometry is Galilean. Therefore, in ${}^{2}R_5$, there are three types of two-dimensional planes, the geometry of which is Euclidean, Minkowski or Galilean.

Let us define the necessary and sufficient conditions for the functions that define that the surface belongs to one or another class. The class of surfaces is defined in accordance with the internal geometry of its tangent plane. The proof of the Theorem is complete.

The problem is solved in the case when surface F_2 is given by the vector equation (2). We study the surface in some neighborhood of point $M\left(u_0,\ v_0\right)$. Let point $M \in F_2$ have coordinates $\left(u_0,\ v_0\right)$.

Theorem 2. If the following condition is satisfied at point $M \in F_2$

$$0 < y_{1v}^2 + y_{2v}^2 < z_v^2 < \left(y_{1v}^2 + y_{2v}^2\right)\left[\frac{z_u^2}{x_{1u}^2 + x_{2u}^2} + 1\right],$$

then the geometry on the tangent plane is Euclidean.

Proof. Since the surface under consideration is a vector one, the equation of which is given by equation (2), then $\left|r_u\right| = x_{1u}^2 + x_{2u}^2 + z_u^2 \geq 0$ always holds.

By the hypothesis of the theorem, $z_v^2 > y_{1v}^2 + y_{2v}^2$, therefore, $r_v^2 = z_v^2 - y_{1v}^2 - y_{2v}^2 > 0$. Hence, both tangent vectors are real. In order for the geometry on the plane to be Euclidean, the following condition should be met:

$$|r_u|^2 \cdot |r_v|^2 - (r_u \cdot r_v)^2 > 0.$$

Introducing to this inequality r_u, r_v, according to formula $|\vec{c}| = \sqrt{\alpha^2 |\vec{a}|^2 + 2\alpha\beta(\vec{a} \cdot \vec{b}) + \beta^2 |\vec{b}|^2}$, we obtain

$$z_v^2 < (y_{1v}^2 + y_{2v}^2)\left[\frac{z_u^2}{x_{1u}^2 + x_{2u}^2} + 1\right].$$

The proof of the Theorem is complete.

Theorem 3. If condition

$$1)\ \ z_v^2 > y_{1v}^2 + y_{2v}^2$$

$$2)\ \ z_v^2 > (y_{1v}^2 + y_{2v}^2)\left[\frac{z_u^2}{x_{1u}^2 + x_{2u}^2} + 1\right]$$

is satisfied at point $M \in F_2$, then the geometry on the tangent plane is Minkowski.

Proof. 1) For surfaces F^2 given by vector equation (2), always $r_u^2 > 0$. The fulfillment of the conditions of the theorem means that $r_v^2 < 0$. Therefore, vectors r_u and r_v have real imaginary norm. Obviously, they are not collinear. They can be selected as a basis on the tangent plane. With this basis, the geometry on the tangent plane is Minkowski. The theorem was proven.

2) According to the given equation surface (2), $r_u^2 > 0$. It follows from the conditions of the theorem that $r_v^2 > 0$. Therefore, both tangent vectors to the coordinate lines have real norms. For these two vectors to be basis vectors of the

tangent plane, by the condition of Theorem 2, the inverse inequalities to the inequality indicated in the theorem must hold.

This condition is not satisfied, which means that the geometry on the tangent is not Euclidian. It follows from the inequality that among the vectors lying on the tangent plane, there are vectors that have imaginary norm. The geometry on the tangent plane is Minkowski. The proof of the Theorem is complete.

The case considered in this theorem is special because vectors $r_u,\ r_v$ are not basis vectors on the tangent plane.

Example 1. Let us give an example: the entire two-dimensional surface $F^2 \subset {}^2R_5$, all tangent planes are Minkowski

$$r = (u,\ v) = u^3 e_1 + 3u^3 e_2 + \left(4u^3 - 5v^3\right) e_3 - 2v^3 e_4 - 3v^3 e_5$$

$$r_u = \left\{3u^2,\ 9u^2,\ 12u^2,\ 0,\ 0\right\} \Rightarrow r_u^2 > 0,$$

$$r_v = \left\{0,\ 0,\ -15v^2,\ -6v^2,\ -9v^2\right\} \Rightarrow r_v^2 > 0 \cdot$$

$$z_v^2 < \left(y_{1v}^2 + y_{2v}^2\right)\left[\frac{z_u^2}{x_{1u}^2 + x_{2u}^2} + 1\right] \Rightarrow 225v^2 < \left(36v^2 + 81v^2\right)\left[\frac{144u^2}{9u^2 + 81u^2} + 1\right]$$

Example 2. Entire two-dimensional surface $F^2 \subset {}^2R_5$, all tangent planes Euclidean

$$r = (u,\ v) = 4u^3 e_1 + 9u^3 e_2 - \left(6u^3 + 5v^3\right) e_3 + v^3 e_4 - 2v^3 e_5$$

$$r_u = \left\{12u^2,\ 27u^2,\ -18u^2,\ 0,\ 0\right\} \Rightarrow r_u^2 > 0,$$

$$r_v = \left\{0,\ 0,\ -15v^2,\ 3v^2,\ -6v^2\right\} \Rightarrow r_v^2 > 0 \cdot$$

$$z_v^2 > \left(y_{1v}^2 + y_{2v}^2\right)\left[\frac{z_u^2}{x_{1u}^2 + x_{2u}^2} + 1\right] \Rightarrow 225v^2 > \left(9v^2 + 36v^2\right)\left[\frac{324u^2}{144u^2 + 529u^2} + 1\right].$$

3 Conclusions

It was established that the internal geometry of submanifolds in a pseudo-Euclidean space did not always coincide with the geometry of the space itself. It turned out that in pseudo-Euclidean space, there are subspaces that have Euclidean metric, or a degenerate metric. This singularity of pseudo-Euclidean space metric was studied for two-dimensional planes and the following was proved. It is rather difficult to study two-dimensional surfaces in the general case. Therefore, a class of two-dimensional surfaces was chosen, expressed by the vector equation (2), which meets some requirements of physics. The class of two-dimensional surfaces defined by an equation of the special form (2) was studied. The possibility of studying the surfaces by the little-known method of three-dimensional Euclidean or Minkowski spaces was shown.

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