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ON THE TIME-OPTIMAL CONTROL PROBLEM FOR A FOURTH-ORDER PARABOLIC EQUATION WITH FRACTIONAL CAPUTO DERIVATIVE

Abstract: In the present work, our focus is on a time-optimal boundary control problem for thin-film evolution governed by a fourth-order time-fractional parabolic equation with Caputo derivative. The model accounts for surface-tension effects and anomalous diffusion, and by applying spectral decomposition we reduce the problem to a Volterra integral equation involving eigenfunctions and Mittag-Leffler functions. We construct admissible controls that drive the system to a prescribed average thickness in minimal time and provide explicit estimates depending on model parameters, thereby extending the classical biharmonic control framework to the subdiffusive regime.

Keywords: Time-fractional Caputo derivative, fourth-order parabolic equation, thin-film growth, initial-boundary value problem, Volterra integral equation, admissible control, time-optimal control, spectral analysis; Mittag–Leffler function.

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О ПРОБЛЕМЕ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ВРЕМЕНЕМ ДЛЯ ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА С ДРОБНОЙ ПРОИЗВОДНОЙ КАПУТО

Аннотация: В настоящей работе мы рассматриваем задачу оптимального по быстродействию граничного управления для эволюции тонкой плёнки, описываемую параболическим уравнением четвёртого порядка дробного по времени с производной Капуто. Модель учитывает эффекты поверхностного натяжения и аномальную диффузию, и, применяя спектральное разложение, мы сводим задачу к интегральному уравнению Вольтерры, содержащему собственные функции и функции Миттаг-Леффлера. Мы строим допустимые управления, которые приводят систему к заданной средней толщине за минимальное время и дают явные оценки, зависящие от параметров модели, тем самым расширяя классическую модель бигармонического управления на субдиффузионный режим.

Ключевые слова: Дробная по времени производная Капуто, параболическое уравнение четвёртого порядка, рост тонкой плёнки, начально-краевая задача, интегральное уравнение Вольтерры, допустимое управление, оптимальное по быстродействию управление, спектральный анализ, функция Миттаг-Леффлера.

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KAPUTO KASR TARTIBLI HOSILASI QATNASHGAN TO‘RTINCHI TARTIBLI PARABOLIK TENGLAMA UCHUN OPTIMAL VAQT BOSHQARISH MASASI HAQIDA

Annotatsiya: Hozirgi ishimizda biz Kaputo kasrli hosilasi bilan ifodalangan to‘rtinchi tartibli vaqt bo‘yicha kasrli parabolik tenglama bilan boshqariladigan yupqa qatlam o‘sishiga oid chegara boshqarishning vaqt bo‘yicha optimal masalasini ko’rib chiqamiz. Model sirt taranglashi effektlari va anormal diffuziyani hisobga oladi va spektral yoyish usulini qo‘llash orqali biz masalani xos funksiyalar va Mittag-Leffler funksiyalarini o‘z ichiga olgan Volterra integral tenglamasiga qisqartiramiz. Biz tizimni ma’lum o‘rtacha qalinlikka minimal vaqtda olib boradigan qabul qilinadigan boshqaruvlarni quramiz va model parametrlariga bog‘liq holda aniq baholarni beramiz, shu bilan an’anaviy biegarmonik boshqaruv tuzilmasini subdiffuziv rejimga kengaytiramiz.

Kalit so‘zlar: Vaqt bo‘yicha Kaputo kasrli hosilasi, to‘rtinchi tartibli parabolik tenglama, yupqa qatlam o‘sishi, boshlang‘ich-chegaraviy masala, Volterra integral tenglamasi, qabul qilinadigan boshqaruv, vaqt bo‘yicha optimal boshqaruv, spektral tahlil, Mittag-Leffler funksiyasi.

1 Introduction

In the present work, we study the time-fractional fourth-order parabolic equation

$$D_t^{\alpha} u(x,t) + \Delta^2 u(x,t) = 0, \quad x \in \Omega, \quad 0 < t \le T, \qquad 11()$$

under the boundary conditions

$$\frac{\partial u(x,t)}{\partial \nu} + h_1(x)u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in [0,T],$$ 22()

$$\frac{\partial \Delta u(x,t)}{\partial \nu} + h_2(x)\Delta u(x,t) = -a(x)\mu(t), \quad x \in \partial\Omega, \quad t \in [0,T],$$ 33()

and the initial condition

$$u(x,0) = 0, \quad x \in \Omega,$$ 44()

where $\alpha \in (0,1)$ is the order of the Caputo time-fractional derivative, $\frac{\partial}{\partial \nu}$ denotes differentiation in the outward normal direction, and $\mu(t)$ is the boundary control function.

Here $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ is the ball of radius $R > 0$ in $\mathbb{R}^N$. Its boundary $\partial\Omega$ contains a measurable subset $S \subset \partial\Omega$ with positive $(N-1)$-dimensional surface measure $\operatorname{mes}_{N-1}(S)$, distinct from the N-dimensional Lebesgue measure $|\Omega|$ of the domain. The coefficient functions $h_i(x)$ ($i = 1,2$) and $a(x)$ are piecewise smooth, non-negative, and not identically zero on $\partial\Omega$, and satisfy

$$h_i(x) = 0 \quad \text{for } x \in S, \qquad a(x) = 0 \quad \text{for } x \in \partial\Omega \setminus S,$$

which reflects their respective physical roles on the boundary.

The time-fractional derivative D_t^α is understood in the Caputo sense [1], defined by

$$D_t^\alpha u(\,;t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(\,;s)}{\partial s} ds, \tag{6}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Let $M > 0$ be a fixed constant. A measurable function $\mu : [0,T] \to \mathbb{R}$ is called an admissible control if it satisfies the condition $|\mu(t)| \le M$ almost everywhere on $[0,T]$.

Thin-film growth is a key process in numerous scientific and engineering fields, including semiconductor fabrication and advanced materials science. A deep understanding of its evolution is essential to improve the structural quality, stability, and functional properties of the resulting films. Mathematical models serve as an effective framework for describing the complex mechanisms governing film formation and for constructing predictive tools that can inform and optimize experimental strategies [2]. In recent years, fractional calculus has emerged as a valuable approach for modeling systems with inherent nonlocality and memory effects, providing new insights into anomalous diffusion phenomena in thin-film dynamics.

From a physical perspective, equation 1 models the evolution of a thin-film interface or other biharmonic-type diffusion processes. In the fractional-order case, the dynamics are governed by anomalous subdiffusion in time, arising from long-range memory effects inherent in the medium. The Caputo fractional derivative D_t^α with $\alpha \in (0,1)$ accounts for history-dependent relaxation, where the current state depends on the entire past evolution. Such models are particularly relevant for nanoscale thin-film growth in environments where particle mobility is hindered by trapping phenomena, viscoelastic substrate effects, or other nonlocal temporal interactions.

In the considered setting, the function $u(x,t)$ represents the height of the growth interface at spatial point $x \in \Omega$ and time $t \geq 0$. The operator $\Delta^2 u$ describes surface diffusion driven by the minimization of the system's chemical potential, which is typical in epitaxial growth processes [3]. The fractional time derivative modifies the relaxation rate of interface perturbations, slowing down the smoothing process compared to the classical case $\alpha = 1$. This paper focuses on the time-optimal boundary control problem for the fractional fourth-order parabolic equation 1–4, aiming to determine the minimal time required to achieve a prescribed average film thickness.

It is well documented that initial investigations into optimal time control problems for parabolic partial differential equations (PDEs) were addressed in [4]. In [5], [6], the control problem for a linear parabolic PDE defined on a one-dimensional domain with Robin boundary conditions were studied.

The boundary control problem for a second-order parabolic type equation with a piecewise smooth boundary in an n- dimensional domain was studied in [7] and an estimate for the minimum time required to reach a given average temperature was found. In that work, the authors derived an estimate for the minimal time required to reach a prescribed average temperature using boundary controls of bounded magnitude. Furthermore, thermocontrol models based on second-order parabolic equations were developed in [8].

The minimal time control problem for a fourth-order parabolic equation in a square domain was further investigated in [9], providing estimates for the optimal time. The present paper will focus primarily on the time-fractional models. In [13], a time-optimal boundary control problem for a one-dimensional diffusion equation with a fractional time derivative is studied. The control problem is reduced to a moment problem, and the dependence of the minimal time on the fractional order is analyzed. In [10], the inverse problem of determining the right-hand side of the subdiffusion equation with the fractional Caputo derivative was considered.

In [11], an inverse problem of determining the temperature distribution and a space-dependent heat source from initial and final temperature data was considered in the one-dimensional case, and existence and uniqueness of the solution were proved. In [12], an inverse problem of determining the order of the Caputo fractional derivative in a subdiffusion equation was studied, where existence and uniqueness results were obtained using spectral methods. We note that, in the classical case $\alpha=1$, the problem of determining the minimal time to reach a prescribed average thickness of a thin film was studied in [14].

2 Problem statement

In this Section, we formulate the boundary control problem for the time-fractional fourth-order parabolic equation involving the Caputo derivative.

We introduce the weight function $\rho : \overline{\Omega} \to \mathbb{R}$, which satisfies

$$\rho(x) > 0 \quad \text{for all } x \in \overline{\Omega}, \quad \text{and} \quad \int_\Omega \rho(x)dx = 1. \tag{7}$$

Consider the next time-optimal control problem:

Time-Optimal Problem. *For a given constant* $\theta > 0$ *find the minimal time* $T_{\min} \geq 0$, *so that for* $t \in [0,T]$ *the solution* $u(x,t)$ *of the initial-boundary value problem 1-4 with some admissible control* $\mu(t)$ *exists and for some* $T_1 > T_{\min}$ *with* $T_1 \leq T$ *satisfies the equation*

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)dx = \theta, \quad t \in [T_{\min}, T_1].$$

55()

Let σ denote the surface measure on S induced by the Lebesgue measure. In particular,

$$\int_S d\sigma(x) = mes_{n-1} S.$$

For $0 < \alpha < 1$ and any complex number β, the two-parameter Mittag-Leffler function with complex argument z is defined by (see [16])

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Additionally, we set

$$\Phi_k := \frac{1}{\Lambda_k} \int_{\partial\Omega} w_k(x) a(x) d\sigma(x),$$

66()

where $\Lambda_k = \|w_k(x)\|_{C(\bar{\Omega})} > 0$.

Consider the equation:

$$tE_{\alpha,2}(-\lambda_1 t^{\alpha}) = \frac{\theta|\Omega|}{M\Phi_1}, \quad 0<\alpha<1, \quad t>0.$$

77()

We recall two known properties of the classical Mittag-Leffler function. First, for all $t \geq 0$, it satisfies the decay estimate

$$|E_{\alpha,\beta}(-t)| \leq \frac{C}{1+t},$$

where the constant $C>0$ is independent of t and β. Moreover, for $\alpha \in (0,1)$, the function $E_{\alpha,1}(-t)$ is strictly decreasing on $t \geq 0$ and satisfies

$$0 < E_{\alpha,1}(-t) < 1,$$

with $E_{\alpha,1}(0)=1$ (see [17]). Denote

$$\psi(t) = \frac{M\Phi_1}{|\Omega|} t E_{\alpha,2}(-\lambda_1 t^{\alpha}), \quad t \in (0,T].$$

88()

The function $\psi(t)$ in 8 is continuous and strictly increasing on $(0,T]$, and satisfies $\lim_{t\to 0}\psi(t)=0$ due to the vanishing factor t. Since $E_{\alpha,2}(-\lambda_1 t^{\alpha})$ is positive and decreasing on $t>0$, the product $t \cdot E_{\alpha,2}(-\lambda_1 t^{\alpha})$ attains its maximum at $t=T$. This ensures that $\psi(t)$ reaches its maximum on $(0,T]$ at $\psi(T)$, which is essential for guaranteeing the solvability of the equation $\psi(t)=\theta$.

Theorem 2.1 *Let* $0<\theta<\psi(T)$*, and let* $T^*>0$ *be the solution of equation 7. Then a solution* $T_{\min}$ *of the control problem exists, and the estimate* $T_{\min} \leq T^*$ *is valid.*

3 Reduction to a Volterra integral equation

In this Section, we will consider the reduction of the optimal time control problem to the Volterra integral equation. We introduce the next Green's function

$$G(x,y,t) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k t^{\alpha}) w_k(x) w_k(y), \quad x,y \in \Omega, \quad t>0.$$

It is a classical fact that this function satisfies the next initial-boundary value problem

$$D_t^{\alpha} G(x,y,t) + \Delta^2 G(x,y,t) = 0, \quad x, y \in \Omega, \quad t \in (0,T],$$

with boundary conditions

$$\frac{\partial G(x,y,t)}{\partial \nu} + h_1(x) G(x,y,t) = 0, \quad x, y \in \partial\Omega, \quad t \in [0,T],$$

and

$$\frac{\partial \Delta G(x,y,t)}{\partial \nu} + h_2(x) \Delta G(x,y,t) = 0, \quad x, y \in \partial\Omega, \quad t \in [0,T],$$

and the initial condition

$$G(x,y,0) = \delta(x - y),$$

where $\delta(x - y)$ denotes the Dirac delta function. It is known that the Green's function $G(x,y,t)$ is non-negative (see[15, 18]) .

We determine the solution to the problem 1-4 using the Fourier method as follows:

$$u(x,t) = \int_0^t \mu(\tau) d\tau \int_S G(x,y,t-\tau) a(y) d\sigma(y), \quad x \in \Omega, \quad y \in \Omega, \quad t \in (0,T].$$

Define the auxiliary function

$$W(x,t) := \int_\Omega G(x,y,t) dy, \quad x \in \Omega, \quad t \in (0,T]. \tag{99()}$$

It is clear that the function $W(x,t)$ defined in 9 satisfies the next initial-boundary value problem:

$$D_t^{\alpha} W(x,t) + \Delta^2 W(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T],$$

with boundary conditions

$$\frac{\partial W(x,t)}{\partial \nu} + h_1(x) W(x,t) = 0, \quad x \in \partial\Omega, \quad t \in [0,T],$$

and

$$\frac{\partial \Delta W(x,t)}{\partial \nu} + h_2(x) \Delta W(x,t) = 0, \quad x \in \partial\Omega, \quad t \in [0,T],$$

and the initial condition

$$W(x,0)=1, \quad x\in\Omega.$$

Lemma 3.1 *There exists a constant* $\Lambda_1>0$ *so that the next estimate holds:*

$$W(x,t)\geq\frac{1}{\Lambda_1}E_{\alpha,1}(-\lambda_1 t^{\alpha})w_1(x), \quad x\in\Omega, \quad t\in(0,T]. \tag{1010()}$$

Proof. Define the auxiliary function

$$L(x,t):=W(x,t)-\frac{1}{\Lambda_1}E_{\alpha,1}(-\lambda_1 t^{\alpha})w_1(x).$$

Then $L(x,t)$ satisfies the same equation and boundary condition as $W(x,t)$, with initial data

$$L(x,0)=1-\frac{1}{\Lambda_1}w_1(x)\geq 0.$$

Using the Green's function representation

$$L(x,t)=\int_{\Omega}L(y,0)G(x,y,t)dy,$$

and the facts that $G(x,y,t)\geq 0$ and $L(x,0)\geq 0$, we conclude that $L(x,t)\geq 0$. Thus, the estimate 10 holds.

We define the control kernel $K(t)$ as:

$$K(t):=\int_{\partial\Omega}W(y,t)a(y)d\sigma(y), \tag{1111()}$$

where $W(y,t)$ is given in 9.

Using the integral condition 5 and the solution to the problem 1-4, we derive the next Volterra integral equation:

$$\frac{1}{|\Omega|}\int_0^t K(t-\tau)\mu(\tau)d\tau=f(t), \quad t\in(0,T],$$

where $f(t)=\theta$ for $t\in[T_{\min},T_1]$.

4 Estimate of Minimal Time

Here, we derive an estimate for the minimal time needed to reach a given prescribed average thickness of the thin film within the domain.

Lemma 4.1 *The following estimate holds:*

$$K(t) \geq \Phi_1 E_{\alpha,1}(-\lambda_1 t^{\alpha}), \quad t \in (0,T], \tag{1212()}$$

where Φ_1 *is defined as in 6 for the case* $k = 1$.

We introduce the function

$$H(t) := \frac{1}{|\Omega|}\int_0^t K(t-\tau)d\tau = \frac{1}{|\Omega|}\int_0^t K(\tau)d\tau, \quad t \in (0,T], \tag{1313()}$$

where the second equality follows by the change of variable $\eta = t - \tau$.

Physically, the function $H(t)$ represents the *average thickness* of the thin film over the spatial domain Ω up to time t. It captures the cumulative effect of boundary heat control distributed through $K(\tau)$ on the film's growth. Since $K(\tau)$ encodes the influence of boundary control and spatial weighting, integrating it over time and normalizing by the domain volume $|\Omega|$ provides a meaningful macroscopic measure of how the film thickness evolves on average. This average thickness serves as an important indicator in applications where uniformity and controlled growth rates are critical for the film's structural and functional properties.

It is straightforward to verify that $H(0) = 0$, reflecting the initial absence of the film, and that its time derivative satisfies

$$H'(t) = \frac{1}{|\Omega|}K(t) \geq 0,$$

indicating that the average thickness is a non-decreasing function of time due to the positive influence of the boundary control.

In particular, the value $H(T)$ at the final time $t = T$ corresponds to the total accumulated average thickness over the entire time interval $[0,T]$, given by

$$H(T) = \frac{1}{|\Omega|}\int_0^T K(\tau)d\tau, \quad 0 < T < +\infty.$$

Thus, for any $t \in [0,T]$, the accumulated average thickness $H(t)$ does not exceed the total growth $H(T)$ achieved by time T, emphasizing the cumulative and monotone nature of the thin film formation process.

Lemma 4.2 *Let* $0 < \theta < MH(T)$. *Then there exist a time* $T_{\min} \in (0,T]$ *and a real-valued measurable function* $\mu(t)$ *so that* $|\mu(t)| \leq M$ *, and the next equality holds:*

$$\frac{1}{|\Omega|}\int_0^{T_{\min}} K(T_{\min} - \tau)\mu(\tau)d\tau = \theta. \tag{14}$$

1414()

Proof. This result follows from the monotonicity and continuity of the function $H(t)$. Indeed, if we choose $\mu(t) = M$, then for all $t \in (0,T]$ we have

$$\frac{1}{|\Omega|}\int_0^t K(t-\tau)\mu(\tau)d\tau = \frac{M}{|\Omega|}\int_0^t K(t-\tau)d\tau = MH(t).$$

Due to the fact that $H(t)$ is continuous and strictly increasing, there exists a unique $T_{\min} \in (0,T]$ so that

$$MH(T_{\min}) = \theta.$$

1515()

The central aim of the present work is to obtain a rigorous estimate for the minimal time $T_{\min}$ required to achieve the desired control objective.

Lemma 4.3 *Let*

$$0 < \theta < \psi(T).$$

1616()

Then there exists a value $T_{\min} \in (0,T]$ *so that* $T_{\min} \leq T^*$ *and the equality 14 is satisfied, where* T^* *is the unique solution of equation 7.*

Proof. To derive the desired estimate, we make use of Lemma 4.1. From the inequality established therein, we have

$$H(t) = \frac{1}{|\Omega|}\int_0^t K(\tau)d\tau \geq \frac{\Phi_1}{|\Omega|}\int_0^t E_{\alpha,1}(-\lambda_1\tau^{\alpha})d\tau = \frac{\Phi_1}{|\Omega|} tE_{\alpha,2}(-\lambda_1 t^{\alpha}).$$

1717()

Now, consider the auxiliary equation used to define T^*:

$$T^* E_{\alpha,2}(-\lambda_1 T^{*\alpha}) = \frac{\theta|\Omega|}{M\Phi_1}.$$

1818()

It is known that equation 18 admits a unique solution $T^* > 0$ because we have the condition $\theta < \psi(T)$, where the function $\psi(t)$ is defined by 8.

According to 17 and (21) we derive

$$0 < \frac{\theta}{M} \leq H(T^*).$$

Then it is clear that $T_{\min}$ ($0 < T_{\min} < T^*$) exists, which is a solution to the equation (18).

Lemma 4.4 *Let* $T_{\min} > 0$ *satisfy the equality 15 and the condition 16. Then there exists* $T_1 > T_{\min}$ *and a measurable real-valued function* $\mu(t)$ *so that* $|\mu(t)| \leq M$*, and the equality 5 holds.*

The proof of Theorem 2.1 follows now easily from Lemmas 4.3 and 4.4.

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