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GENERATSIYA FUNKSIYALARI VA ULARNING QO‘LLANILISHI

Annotatsiya. Mazkur maqolada matematik analiz va kombinatorikaning muhim vositalaridan biri bo‘lgan generatsiya funksiyalari haqida so‘z yuritiladi. Generatsiya funksiyasi ketma-ketliklarning umumiy hadini ifodalash, ularni turli algebraik va kombinatorik muammolarda qo‘llash imkonini beruvchi samarali usul hisoblanadi. Ushbu maqolada avvalo generatsiya funksiyasining ta’rifi va uning asosiy xossalari keltiriladi. So‘ngra oddiy va eksponensial generatsiya funksiyalariga oid misollar ko‘rib chiqilib, ular yordamida ayrim ketma-ketliklarning umumiy haddi topiladi.

Kalit so‘zlar. Ketma-ketlik, funksiya, generatsiya funksiyasi, qator, differensial tenglama.

GENERATING FUNCTIONS AND THEIR APPLICATIONS

Abstract. In this article, we discuss generating functions, which are among the important tools of mathematical analysis and combinatorics. A generating function is an effective method that allows us to express the general term of sequences and to apply them to various algebraic and combinatorial problems. In this paper, we first present the definition of generating functions and their main properties. Then, examples of ordinary and exponential generating functions are considered, and with their help, the general terms of certain sequences are derived.

Keywords: Sequence, Function, Generating function, Series, Differential equation.

ПОРОЖДАЮЩИЕ ФУНКЦИИ И ИХ ПРИМЕНЕНИЕ

Аннотация: В данной статье рассматриваются порождающие функции, которые являются одним из важных инструментов математического анализа и комбинаторики. Порождающая функция представляет собой эффективный метод, позволяющий выразить общий член последовательностей и применять их при решении различных алгебраических и комбинаторных задач. В работе сначала приводится определение порождающих функций и их основные свойства. Затем рассматриваются примеры обыкновенных и

экспоненциальных порождающих функций, с помощью которых выводятся общие члены некоторых последовательностей.

Ключевые слова: последовательность, функция, порождающая функция, ряд, дифференциальное уравнение.

Kirish. Matematikaning turli bo‘limlarida ketma-ketliklar bilan ishlash katta ahamiyatga ega. Xususan, kombinatorika va algebraik hisoblashlarda berilgan ketma-ketlikning umumiy hadini topish muhim masalalardan biridir. Bu jarayonda samarali usullardan biri — *generatsiya funksiyalari* hisoblanadi [1-3]. Generatsiya funksiyalari yordamida nafaqat ketma-ketliklarning umumiy hadlarini topish, balki turli kombinator muammolarni yechish, differensial tenglamalarni soddalashtirish, ehtimollar nazariyasi va sonlar nazariyasidagi masalalarni hal qilish mumkin.

Generatsiya funksiyasi oddiy qator ko‘rinishidagi ifodadan iborat bo‘lib, u ketma-ketlikning barcha elementlarini o‘zida mujassamlashtiradi. Shu tufayli, matematikada ko‘plab murakkab natijalarni ixcham va aniq tarzda ifodalash imkonini beradi [4,5].

Mazkur maqolada biz generatsiya funksiyalarining asosiy tushunchalari va ularning ayrim misollardagi qo‘llanishlarini ko‘rib chiqamiz. Xususan, Fibonachchi sonlari uchun generatsiya funksiyasini hosil qilish, xalqaro olimpiada masalalarida generatsiya funksiyasining qo‘llanilishi ham batafsil yoritiladi.

Asosiy qism. Aytaylik, bizga a_n ketma-ketlik berilgan bo‘lsin. Quyidagi

$$f(x)=\sum_{n=0}^{\infty} a_n x^n$$

yoki

$$(a_0, a_1, a_2, a_3, \ldots) \leftrightarrow f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots$$

bo‘lgan $f(x)$ funksiyaga a_n ketma-ketlikning generatsiya funksiyasi deyiladi.

Quyidagi

$$g(x)=\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

funksiyaga esa a_n ketma-ketlikning eksponensial generatsiya funksiyasi deyiladi.

Generatsiya funksiyadagi yig‘indi cheksiz yoki chekli bo‘lishi mumkin. Cheksiz yoki chekli ekanligi a_n ketma-ketlikning hadlari soniga bog‘liq.

Dastlab quyidagi misollarni ko‘raylik.

1-masala. Quyidagi $1, 2, 3, \cdots, n+1, \ldots$ ketma-ketlikning generatsiya funksiyasini toping.

Yechim: Ko‘rish qiyin emaski, $a_n = n+1$. $|x|<1$ uchun:

$$f(x)=\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(n+1)x^n = 1+2x+3x^2+4x^3+\ldots+(n+1)x^n+\ldots=$$

$$=\left(x+x^2+x^3+\ldots\right)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}.$$

Demak, berilgan $a_n = n+1$ ketma-ketlikning generatsiya funksiyasi

$$f(x)=\frac{1}{(1-x)^2}.$$

2-masala. $(1,1,1,\ldots)$ ketma-ketlikning generatsiya funksiyasi $f(x)=\frac{1}{1-x}$ bo‘lishini isbotlang.

Yechim: Biz $f(x)(1-x)=1$ ekanligini ko‘rsatishimiz kerak.

$$\begin{array}{r} \_\; f(x)=1+x+x^2+\ldots \\ xf(x)=\quad x+x^2+\ldots \\ \text{- -} \\ (1-x)f(x)=1+0\cdot x+0\cdot x^2+\ldots \end{array}$$

yoki

$$f(x)=\sum_{n=0}^{\infty} x^n = 1+x+x^2+\ldots+x^n+\ldots=\frac{1}{1-x},\quad |x|<1$$

isbotlandi.

3-masala. $f(x)=\dfrac{1}{x^2-4x+3}$ funksiya qaysi ketmat-ketlikning generatsiya fuksiyasi bo‘ladi?

Yechim: Bu misolni ishlashda quyidagi tenglikdan foydalanamiz:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Dastlab $f(x)$ funksiyani soddalashtirib olamiz:

$$f(x)=\frac{1}{x^2-4x+3}=\frac{1}{(1-x)(3-x)}=\frac{1}{2}\left(\frac{1}{1-x}-\frac{1}{3-x}\right)=$$

$$=\frac{1}{2}\cdot\frac{1}{1-x}-\frac{1}{6}\cdot\frac{1}{1-\frac{x}{3}}=\frac{1}{2}\sum_{n=0}^{\infty}x^n-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{2}-\frac{1}{2\cdot 3^{n+1}}\right)x^n \qquad (1)$$

Boshqa tomondan $f(x)$ funksiya a_n ketma-ketlikning generatsiya funksiyasi, ya’ni quyidagi funksiyaga teng:

$$f(x)=\sum_{n=0}^{\infty}a_n x^n \qquad (2)$$

(1) va (2) tengliklardan

$$a_n=\frac{1}{2}\left(1-\frac{1}{3^{n+1}}\right)$$

ekanini topamiz.

4-masala. Bizga quyidagi ketma-ketlik berilgan bo‘lsin. $F_0=0,\ F_1=1$ va $n\geq 2$ uchun $F_n=F_{n-1}+F_{n-2}$ bo‘lsa, F_n - Fibonachchi sonlarining umumiy hadini toping.

Yechim: Aytaylik Fibonachchi sonlarining generatsiya funksiyasi $f(x)$ bo‘lsin.

$$f(x)=\sum_{n=0}^{\infty}F_n x^n=F_0+F_1x+\sum_{n=2}^{\infty}F_n x^n=x+\sum_{n=2}^{\infty}(F_{n-1}+F_{n-2})x^n=$$

$$=x+\sum_{n=2}^{\infty}F_{n-1}x^n+\sum_{n=2}^{\infty}F_{n-2}x^n=x+\sum_{n=1}^{\infty}F_n x^{n+1}+\sum_{n=0}^{\infty}F_n x^{n+2}=$$

$$=x+x\sum_{n=0}^{\infty}F_nx^n+x^2\sum_{n=0}^{\infty}F_nx^n=x+xf(x)+x^2f(x)=f(x)\Rightarrow\quad f(x)=\frac{x}{1-x-x^2}$$

Ushbu generatsiya funksiyani qatorga yoyamiz:

$$f(x)=\frac{x}{1-x-x^2}=\frac{-x}{\left(x-\frac{\sqrt{5}-1}{2}\right)\left(x+\frac{\sqrt{5}+1}{2}\right)}$$

$\frac{\sqrt{5}-1}{2}=\alpha,\ \frac{\sqrt{5}+1}{2}=\beta$ belgilash kiritamiz:

$$f(x)=\frac{-x}{(x-\alpha)(x-\beta)}=\frac{1}{\sqrt{5}}\left(\frac{\alpha}{\alpha-x}-\frac{\beta}{\beta-x}\right)=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{x}{\alpha}}-\frac{1}{1-\left(-\frac{x}{\beta}\right)}\right)=$$

$$=\frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\left(\frac{x}{\alpha}\right)^n-\sum_{n=0}^{\infty}\left(-\frac{x}{\beta}\right)^n\right)=\frac{1}{\sqrt{5}}\left\{\sum_{n=0}^{\infty}\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^n}-\frac{1}{\left(-\frac{\sqrt{5}+1}{2}\right)^n}\right\}x^n=$$

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)x^n\Rightarrow F_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)\qquad(3)$$

Demak Fibonachchi sonlar ketma-ketliklarning umumiy hadi (3) ga teng bo‘ladi.

5-masala (OMUS-2020, Individual Competition, Problem 4).

a_n sonlar ketma-ketligi uchun $a_0=1, a_1=1$ va $n\geq 2$ uchun $a_n=a_{n-1}+2(n-1)a_{n-2}$ munosabat o‘rinli bo‘lsa, quyidagi yig‘indini hisoblang:

$$\sum_{n=0}^{\infty}\frac{a_n}{n!}$$

Yechim: a_n ketma-ketlikning generatsiya funksiyasini qidiramiz, ya’ni quyidagi ko‘rinishda:

$$f(x)=\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$$

Agar biz $f(x)$ ni topa olsak, yuqoridagi yig‘indini qiymatini ham topa olamiz.

$$f(x)=\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n=a_0+a_1x+\sum_{n=2}^{\infty}\left(\frac{a_{n-1}+2(n-1)a_{n-2}}{n!}\right)x^n=$$

$$=1+x+\sum_{n=2}^{\infty}\frac{a_{n-1}}{n!}x^n+2\sum_{n=2}^{\infty}\frac{(n-1)a_{n-2}}{n!}x^n=1+x+\sum_{n=1}^{\infty}\frac{a_n}{(n+1)!}x^{n+1}+2\sum_{n=0}^{\infty}\frac{(n+1)a_{n-2}}{(n+2)!}x^{n+2}$$

Har ikki tomondan *x* bo‘yicha 1-tartibli hosila olamiz.

$$f'(x)=1+\sum_{n=1}^{\infty}\frac{a_n}{n!}x^n+2x\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n=1+f(x)-1+2x\cdot f(x)=f'(x)$$

$$f'(x)=(1+2x)f(x)\Rightarrow\frac{f'(x)}{f(x)}dx=(1+2x)dx\Rightarrow\ln f(x)=x^2+x+C$$

Bizga ma’lumki $f(0)=1$ bundan $C=0\Rightarrow f(x)=e^{x^2+x}$

$$e^{x^2+x}=\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n\overset{x=1}{\Rightarrow}e^2=\sum_{n=0}^{\infty}\frac{a_n}{n!}$$

Javob: e^2.

6-masala. (IMC 2019, Problem 4) a_n sonlar ketma-ketligi uchun $a_0=1,\ a_1=2$ va $\forall n\in\mathrm{N}$ lar uchun $(n+3)a_{n+1}=(3n+4)a_n+(4n+2)a_{n-1}$ tenglik o‘rinli bo‘lsa, ushbu ketma-ketlikning barcha hadlari butun sonlardan iborat ekanligini isbotlang va a_n ni umumiy ko‘rinishini toping.

Yechim: Dastlab ketma-ketlikning umimiy hadini ko‘rinishini aniqlab olamiz:

$$(n+3)a_{n+1}=(3n+4)a_n+(4n+2)a_{n-1}\quad\Rightarrow a_{n+1}=\frac{3n+4}{n+3}a_n+\frac{4n+2}{n+3}a_{n-1}$$

$$a_{n+1}=3a_n+4a_{n-1}-\frac{5}{n+3}a_n-\frac{10}{n+3}a_{n-1}$$

Ketma-ketligimizni uchun $f(x)=\sum_{n=0}^{\infty}a_nx^n$ geratsiya funksiyani ko‘ramiz.

$$f(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+\sum_{n=2}^{\infty}a_nx^n=1+2x+\sum_{n=1}^{\infty}a_{n+1}x^{n+1}=$$

$$=1+2x+\sum_{n=1}^{\infty}\left(3a_n+4a_{n-1}-\frac{5}{n+3}a_n-\frac{10}{n+3}a_{n-1}\right)x^{n+1}=$$

$$=1+2x+3x\sum_{n=1}^{\infty}a_nx^n+4x^2\sum_{n=1}^{\infty}a_{n-1}x^{n-1}-5\sum_{n=1}^{\infty}\frac{a_n}{n+3}x^{n+1}-10\sum_{n=1}^{\infty}\frac{a_{n-1}}{n+3}x^{n+1}$$

$$\sum_{n=1}^{\infty}a_nx^n=f(x)-1\ ,\ \sum_{n=1}^{\infty}a_{n-1}x^{n-1}=f(x)$$

$$f(x)=1+2x+3x\left(f(x)-1\right)+4x^2f(x)-5\sum_{n=1}^{\infty}\frac{a_n}{n+3}x^{n+1}-10\sum_{n=1}^{\infty}\frac{a_{n-1}}{n+3}x^{n+1}$$

Yuqoridagi tenglikning har ikki tomonini x^2 ga ko‘paytiramiz.

$$x^2f(x)=x^2+2x^3+3x^3\left(f(x)-1\right)+4x^4f(x)-5\sum_{n=1}^{\infty}\frac{a_n}{n+3}x^{n+3}-10\sum_{n=1}^{\infty}\frac{a_{n-1}}{n+3}x^{n+3}$$

$$x^2f(x)=x^2-x^3+3x^3f(x)+4x^4f(x)-5\sum_{n=1}^{\infty}\frac{a_n}{n+3}x^{n+3}-10\sum_{n=1}^{\infty}\frac{a_{n-1}}{n+3}x^{n+3} \tag{4}$$

Quyidagi ikki tenglik o‘rinli

$$\left(\sum_{n=1}^{\infty}\frac{a_n}{n+3}x^{n+3}\right)'=\sum_{n=1}^{\infty}a_nx^{n+2}=x^2\left(f(x)-1\right)$$

$$\left(\sum_{n=1}^{\infty}\frac{a_{n-1}}{n+3}x^{n+3}\right)'=\sum_{n=1}^{\infty}a_{n-1}x^{n+2}=x^3f(x) \tag{5}$$

(4) tenglikdan har ikki tomonidan x ga nisbatan hosila olamiz, (5) dan foydalanib soddalashtiramiz va quyidagi oddiy differensial tenglamaga kelamiz.

$$\left(4x^4+3x^3-x^2\right)f'(x)+\left(6x^3+4x^2-2x\right)f(x)+2x^2+2x=0$$

$$\left(4x^3+3x^2-x\right)f'(x)+\left(6x^2+4x-2\right)f(x)+2x+2=0$$

$$f'(x)+\frac{2(1-3x)}{x(1-4x)}f(x)-\frac{2}{x(1-4x)}=0 \tag{6}$$

(6) differensial tenglamaning integrallovchi ko‘paytuvchisi $\mu(x)=\dfrac{x^2}{\sqrt{1-4x}}$ ga teng. (6) ning har ikki tomonini $\mu(x)$ ga ko‘paytiramiz.

$$\frac{x^2}{\sqrt{1-4x}}f'(x)+\frac{x^2}{\sqrt{1-4x}}\cdot\frac{2(1-3x)}{x(1-4x)}f(x)-\frac{x^2}{\sqrt{1-4x}}\cdot\frac{2}{x(1-4x)}=0$$

$$\left(\frac{x^2}{\sqrt{1-4x}}\cdot f(x)\right)' = \frac{2x}{(1-4x)^{\frac{3}{2}}} \Rightarrow f(x) = \frac{1-2x-\sqrt{1-4x}}{2x^2}$$

Demak, $\dfrac{1-2x-\sqrt{1-4x}}{2x^2} = \sum\limits_{n=0}^{\infty} a_n x^n$

$y=\sqrt{1-4x}$ funksiyani Makloren qatoriga yoysak, quyidagi ko‘rinishga keladi.

$$\sqrt{1-4x} = 1-2x-\sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\cdot 4^n\cdot x^n = 1-2x-\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n+4)!!}\cdot 4^{n+2}\cdot x^{n+2}$$

$$\frac{1-2x-\sqrt{1-4x}}{2x^2} = \sum_{n=0}^{\infty} a_n x^n \Rightarrow \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n+4)!!}\cdot 4^{n+2}\cdot x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{(2n+1)!!}{(2n+4)!!}\cdot 2^{2n+3} = \frac{(2n+1)!!}{(n+2)!}\cdot 2^{n+1} = \frac{(2n+2)!}{(n+1)!\cdot(n+2)!}$$

Yuqoridagi ketma-ketlikning umumiy hadi quyidaginga tengligi kelib chiqdi.

$$a_n = \frac{(2n+2)!}{(n+1)!\cdot(n+2)!}$$

Endi a_n ketma-ketlikning barcha hadlari butun sonlardan iboratligini ko‘rsatamiz.

$$a_n = \frac{(2n+2)!}{(n+1)!\cdot(n+2)!} = \frac{(n+3)(n+4)\ldots(2n+2)}{(n+1)!}$$

n ta ketma-ket kelgan butun sonlar ko‘paytmasi $n!$ga karrali. Shunga ko‘ra

$$(n+3)(n+4)\ldots(2n+2) \,\vdots\, n!$$

Ikkinchi tomondan $(n+3)(n+4)\ldots(2n+2) \,\vdots\, (n+1)!$. Yuqoridagi ikki tenglikka ko‘ra

$$(n+3)(n+4)\ldots(2n+2) \,\vdots\, (n+1)! \Rightarrow \frac{(n+3)(n+4)\ldots(2n+2)}{(n+1)!} \in N \Rightarrow a_n \in N$$

Demak a_n ketma-ketlikning barcha hadlari butun sonlardan iborat va uning umumiy hadi quyidaginga teng.

$$a_n = \frac{(2n+2)!}{(n+1)!\cdot(n+2)!}$$

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