Tukhtabaev Akbarkhuja Mamajonovich

Senior lecturer (PhD) at the department of Mathematics, Namangan state university

Email: akbarxoja.toxtaboyev@mail.ru

FREE ENERGY OF TRANSLATION-INVARIANT *P*-ADIC GENERALIZED GIBBS MEASURES FOR THE ISING MODEL ON THE CAYLEY TREE

Abstract. *We study the free energy of the p-adic Ising model on the Cayley tree of order k. Moreover, we prove its constancy for translation-invariant generalized Gibbs measures and present the explicit formula.*

Key words*: p-adic numbers, Ising model, p-adic generalized Gibbs measure, translation-invariant, free energy.*

To‘xtabayev Akbarxo‘ja Mamajonovich

Namangan davlat universiteti, Matematika kafedrasi katta o‘qituvchisi PhD

KELI DARAXTIDAGI IZING MODELI UCHUN TRANSLYATSION-INVARIANT *P*-ADIK UMUMIYLASHGAN GIBBS O‘LCHOVLARINING OZOD ENERGIYASI

Annotatsiya. *Ushbu ishda k-tartibli Keli daraxtida Izing modelining translyatsion-invariant p-adik umumlashgan Gibbs o‘lchovlari uchun ozod energiyasini o‘rgandik. Shuningdek, bu ozod energiyani o‘zgarmas ekanligini isbotladik va uning uchun formula keltirib chiqardik.*

Kalit so’zlar: *p-adik sonlar, Izing modeli, p-adik umumlashgan Gibbs o‘lchovlari, translyatsion-invariant, ozod energiya.*

Тухтабаев Акбархужа Мамажонович

Старший преподаватель (PhD) кафедры Математики Наманганского государственного университета

СВОБОДНАЯ ЭНЕРГИЯ ТРАНСЛЯЦИОННО-ИНВАРИАНТНЫХ *P*-АДИЧЕСКИХ ОБОБЩЕННЫХ МЕР ГИББСА ДЛЯ МОДЕЛИ ИЗИНГА НА ДЕРЕВЕ КЭЛИ

Аннотация. *Мы изучаем свободную энергию p-адической модели Изинга на дереве Кэли порядка k. Кроме того, мы доказываем ее постоянство (инвариантность) для трансляционно-инвариантных обобщенных мер Гиббса и представляем явную формулу для нее.*

Ключевые слова: *p-адические числа, модель Изинга, p-адическая обобщенная мера Гиббса, трансляционно-инвариантный, свободная энергия.*

Introduction

The *p*-adic Ising model was first studied on the Cayley tree in work [1] as a special case of the 2-state lambda model. In [1], the existence of a unique *p*-adic Gibbs measure was proved for the lambda model (and specifically, the Ising model).

Subsequently, with the aim of extending the class of measures, the concept of *p*-adic generalized Gibbs measures was introduced in work [2]. This work also investigated translation-invariant and periodic generalized Gibbs measures on the Cayley tree of order two. Later, these studies were continued for the Cayley tree of order three in [3] and [4].

Weakly periodic Gibbs measures for the Ising model were studied in [5-6]. Chaotic behavior of Ising-Potts mapping were investigated in [7]. Translation-invariant Gibbs measures on the Cayley tree of order k were studied in [8].

In the real case, the free energy for the Ising model was studied in [9]. It was proven that the free energy for translation-invariant measures is a constant. In the present work, we calculated the free energy for the translation-invariant *p*-adic generalized Gibbs measures on the Cayley tree of order *k* which were defined in [8].

Preliminaries

***p*-adic numbers and *p*-adic measure.** Let p be any prime number. For any nonzero integer a, let $\mathrm{ord}_p a$ be the highest power of p which divides a, i.e., the greatest m such that

$$a \equiv 0 \pmod{p^m}$$
.

If $a = 0$, we agree to write $\operatorname{ord}_p a = \infty$.

Now, for any rational number $x = \frac{a}{b}$, define $\operatorname{ord}_p x = \operatorname{ord}_p a - \operatorname{ord}_p b$. It is easy to check that $\operatorname{ord}_p x$ is well-defined.

Further define a map $|\cdot|_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = \begin{cases} p^{-\operatorname{ord}_p x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(1)

The map defined by (1) is a norm. Furthermore, this norm is a non-Archimedean norm, i.e.

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}$$

for all $x, y \in \mathbb{Q}$.

The norm defined by (1) is called p*-adic norm*.

The strong triangle inequality property follows that

1) if $|x|_p \neq |y|_p$ then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$;

2) if $|x|_p = |y|_p$ then $|x \pm y|_p \leq \max\{|x|_p, |y|_p\}$.

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$ (see [10]).

Ostrowski proved that every non-trivial norm on $\mathbb{Q}$ is equivalent to either p-adic norm for some p, or absolute value. It means that the completion of the field of rational numbers $\mathbb{Q}$ with non-trivial norm is either the field of real numbers $\mathbb{R}$ or one of the fields of p-adic numbers $\mathbb{Q}_p$.

Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}\left(x_0 + x_1 p + x_2 p^2 + \ldots\right), \tag{2}$$

where $x_j, \gamma(x) \in \mathbb{Z}, x_0 \neq 0, \ x_j \in \{0,1,\ldots,p-1\}, j = 1,2,\ldots$.

In this case $|x|_p = p^{-\gamma(x)}$ (see [10]).

Let (X,B) be a measurable space, where B is an algebra of subsets X. A function $\mu : B \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, A_2, \ldots, A_n \in B$ such that $A_i \cap A_j = \emptyset, \ i \neq j$, the following holds:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

A p-adic measure μ is called bounded if $\sup\{|\mu(A)|_p : A \in B\} < \infty$. It is said that p-adic measure is *probabilistic* if $\mu(X) = 1$.

Cayley tree. Let $\Gamma^k = (V,L)$, be a infinite Cayley tree of order $k \geq 1$ with the root $x^0 \in V$ (whose each vertex has exactly $k+1$ edges) (see [11]). Here V is the set of vertices and L is the set of edges. The vertices x and y are called nearest neighbors and they are denoted by $l = \langle x, y\rangle$ if there exists an edge l connecting them. A collection of the pairs $\langle x, x_1\rangle$, $\langle x_1, x_2\rangle$,…, $\langle x_{d-1}, y\rangle$ is called a path from the point x to the point y. The distance $d(x,y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y. We set

$$W_n = \{ x \in V \mid d(x,x^0) = n \}, \ \ V_n = \{ x \in V \mid d(x,x^0) \leq n \}, \ L_n = \{ l = < x, y > \in L \mid x, y \in V_n \}.$$

The set of direct successors of x is defined by

$$S(x) = \{ y \in W_{n+1} : d(x,y) = 1, \ x \in W_n \}.$$

***p*-adic Ising model.** We consider p-adic Ising model on the Cayley tree Γ^k. Let $\mathbb{Q}_p$ be a field of p-adic numbers and $\Phi = \{-1,1\}$. A configuration σ on V is defined by the function $x \in V \to \sigma(x) \in \Phi$. Similarly, one can define the configuration σ_n and $\sigma^{(n)}$ on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) is denoted by

$\Omega = \Phi^{V}$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). For given configurations $\sigma_{n-1} \in \Omega^{V_n}$ and $\varphi^{(n)} \in \Omega^{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1} \vee \omega_n)(x) = \begin{cases} \sigma_{n-1}(x), & if \;\; x \in V_{n-1}, \\ \varphi^{(n)}(x), & if \;\; x \in W_n. \end{cases}$$

The Hamiltonian for p-adic Ising model is defined by

$$H_n(\sigma_n) = N \sum_{\langle x,y \rangle \in L_n} \sigma(x)\sigma(y), \tag{3}$$

where N is positive integer.

Let $h : x \in V \to h_x \in \mathbb{Q}_p$ be function. We define p-adic probability generalized Gibbs measure μ_h^n on Ω_{V_n} by

$$\mu_h^n(\sigma_n) = Z_n^{-1} \rho^{H_n(\sigma_n)} \prod_{x \in W_n} h_{\sigma(x),x}, \tag{4}$$

where $\rho \in Q_p \setminus \{-1, 0, 1\}$, Z_n^{-1} is corresponding normalizing factor. The compatibility conditions for $\mu_h^n(\sigma_n)$, $n \geq 1$ are given by the equality

$$\sum_{\sigma^n \in \Omega_{W_n}} \mu_h^n(\sigma_{n-1} \cup \sigma^n) = \mu_h^{n-1}(\sigma_{n-1}). \tag{5}$$

In this case, by the p-adic analogue of Kolmogorov theorem there exists a unique measure μ_h on the set Ω such that $\mu_h\left(\left\{ \sigma|_{V_n} \equiv \sigma_n \right\}\right) = \mu_h^{(n)}(\sigma_n)$ for all n (see [12]).

Main results

Theorem 1.[8] The measures $\mu_h^n(\sigma_n)$ defined by (4) satisfy the compatibility condition (5), if and only if for any $x \in V$ the following equation holds:

$$h_x = \prod_{y \in S(x)} \frac{\theta h_y + 1}{h_y + \theta}, \tag{6}$$

where $\theta = \rho^{2N}$.

Let $h_x = h, \; \forall x \in V$ then the quantities h_x is called translation-invariant. The corresponding measure μ_h is called translation-invariant measure.

Using above we can write (6) for the translation-invariant h_x the following form

$$h = \left(\frac{\theta h + 1}{h + \theta} \right)^k . \qquad (7)$$

Translation-invariant *p*-adic generalized Gibbs measures for Ising model on the Cayley tree were studied in [2-4, 8].

The free energy is well studied in the real case (see e.g. [9]). In this paper, we are going to analyze in *p*-adic case.

The free energy defined as

$$F(h,\theta) = \lim_{n \to \infty} \frac{1}{|V_n|} \ln |Z_n^{(h)}|_p . \qquad (8)$$

Theorem 2. Let $p \geq 3$, $F(h,\theta)$ be free energy of the translation-invariant *p*-adic generalized Gibbs measures for Ising model on the Cayley tree of order k. Then the free energy $F(h,\theta)$ is a constant. Moreover,

$$F(h,\theta) = \begin{cases} \ln|\theta - 1|_p, & \text{if } |\theta|_p = 1,\ h \neq 1; \\ 0, & \text{if } |\theta|_p = 1,\ h = 1; \\ -\frac{1}{2}\ln|\theta|_p, & \text{if } |\theta|_p < 1; \\ \frac{1}{2}\ln|\theta|_p, & \text{if } |\theta|_p > 1,\ |h|_p = |\theta|_p^{-k}; \\ \frac{2k-1}{2}\ln|\theta|_p, & \text{if } |\theta|_p > 1,\ |h|_p = |\theta|_p^{k}. \end{cases} \qquad (9)$$

Proof. Case 1. Let $|\theta|_p = |\rho^{2N}|_p = 1$.

Case 1.1. Let $h(\neq 1)$ be a translation-invariant *p*-adic generalized Gibbs for the Ising model on the Cayley tree of order *k* and $p \geq 3.$

Then using (17) in [8], we rewrite (8) as follows

$$F(h,\theta) = \lim_{n \to \infty} \frac{k-1}{k^n(k+1)-2} \ln|Z_n^{(h)}|_p =$$
$$= \lim_{n \to \infty} \frac{k-1}{k^n(k+1)-2} \cdot \frac{k^n(k+1)}{k-1} \ln|\theta^2 - 1|_p + \lim_{n \to \infty} \frac{k-1}{k^n(k+1)-2} \cdot \ln|h^2(\theta h^2 + 1) + h^2 + \theta|_p$$
$$= \ln|\theta - 1|_p .$$

Case 1.2. Let $h=1$ and $p\geq 3.$ Then again using (17) in [8], we have $F(h,\theta)=0$. Therefore, we can conclude that

$$F(h)=\begin{cases}\ln|\theta-1|_p, & \text{if } h\neq 1,\\ 0, & \text{if } h=1.\end{cases}$$

Case 2. Let $|\theta|_p=|\rho^{2N}|_p\neq 1$. In [8], the formula of (17) is given as

$$Z_{n,\rho}^{(h)}=\left(\rho^{-N}h+\rho^{N}\right)^{\frac{k(k^n-1)}{k-1}}(h+1),\quad \forall n\in\mathbb{N}.$$

(8) yields that

$$F(h,\theta)=\lim_{n\to\infty}\frac{k-1}{k^{n+1}-1}\cdot\left(\frac{k^{n+1}-k}{k-1}\ln|\rho^{-N}h+\rho^{N}|_p+\ln|h+1|_p\right),$$

or

$$F(h,\theta)=\ln|\rho^{-N}h+\rho^{N}|_p\,.$$

Case 2.1. Let $|\rho|_p<1$ i.e. $|\theta|_p<1$. According to Lemma 4 in [8] $|h|_p=1$. (8) follows that

$$F(h,\theta)=-\frac{1}{2}\ln|\theta|_p\,.$$

Case 2.2. Let $|\rho|_p>1$ i.e. $|\theta|_p>1$. Lemma 4 in [8] and (8) follow that

$$F(h,\theta)=\begin{cases}\frac{1}{2}\ln|\theta|_p, & \text{if } |h|_p=|\theta|_p^{-k},\\ \frac{2k-1}{2}\ln|\theta|_p, & \text{if } |h|_p=|\theta|_p^{k}.\end{cases}$$

Therefore,

$$F(h,\theta)=\begin{cases}\ln|\theta-1|_p, & \text{if } |\theta|_p=1,\ h\neq 1;\\ 0, & \text{if } |\theta|_p=1,\ h=1;\\ -\frac{1}{2}\ln|\theta|_p, & \text{if } |\theta|_p<1;\\ \frac{1}{2}\ln|\theta|_p, & \text{if } |\theta|_p>1,\ |h|_p=|\theta|_p^{-k};\\ \frac{2k-1}{2}\ln|\theta|_p, & \text{if } |\theta|_p>1,\ |h|_p=|\theta|_p^{k}.\end{cases}$$

The theorem has proven.

Remark 1. In real case, in [9] the authors proved that the free energy of translation-invariant Gibbs measures for the Ising model is a constant. Theorem 2 shows that this result also holds for the p-adic case.

Conclusion

Since the free energies of translation-invariant Gibbs measures for the Ising model in both cases (the real and p-adic) be constant functions.

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