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TEKISLIKDA LAPLAS TENGLAMASI UCHUN KOSHI MASALASINING KARLEMAN FUNKSIYASI

Annotasiya. *Ushbu maqola differensial tenglamalar va matematik fiziki ixstisosligi bo‘yicha tadqiqot olib boruvchi ilmiy izlanuvchilar uchun mo‘ljallangan. Maqolada tekislikdagi Laplas tenglamasi uchun Koshi masalasi qaralgan.*

Kalit so‘zlar: *Laplas tenglamasi, Koshi masalasi, nokorrekt masalalar, Karleman funksiyasi, regulyarizasiya, davom ettirish formulalalari.*

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ФУНКЦИЯ КАРЛЕМАНА ЗАДАЧИ КОШИ ДЛЯ УРАВНЕНИЯ ЛАПЛАСА НА ПЛОСКОСТИ

Аннотация*: Статья предназначена для исследователей, проводящих исследования в области дифференциальных уравнений и математической физики. В статье изучается задача Коши для уравнения Лапласа на плоскости.*

Ключевые слова: *уравнение Лапласа, Задача Коши, некорректные задачи, функция Карлемана, регуляризация, формулы продолжения.*

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THE CARLEMAN FUNCTION OF THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION ON THE PLANE

Abstract. *This article is intended for researchers conducting research in the field of differential equations and mathematical physics. The article studies the Cauchy problem for the Laplace equation in the plane.*

Keywords: *Laplace's equation, Cauchy problem, ill-posed problems, Carleman function, regularization, continuation formulas.*

Kirish

G soha chegarasi ∂G ning bir qismi S silliq chiziqda berilgan qiymatlariga koʻra, Laplas tenglamasi uchun Koshi masalasi yechimini davom ettirishi masalasini qaraymiz. Qaralayotgan masala, matematik fizikaning yechim boshlang'ich shartlarga uzluksiz ravishda bog'liq boʻlmagan masalalari qatoriga kiradi. Masalada yechim mavjud va biror yopiq sohada uzluksiz differensiallanuvchi, hamda Koshi shartlari aniq berilgan deb olamiz.

Adabiyotlar tahlili

Nokorrekt masalalarni yechimlar sinfini kompaktga qadar toraytirilsa, bu masala turgʻun boʻlishiga doir birinchi natijalar A.N.Tixonov ishlarida keltirilgan. Laplas tenglamasi uchun Koshi masalasining Karleman funksiyasi tushunchasi M.M.Lavrent'yev tomonidan kiritilgan.[2] Tekislikdagi yoʻlak koʻrinishidagi cheksiz sohada Laplas tenglamasi uchun Koshi masalasining regulyarizatsiyasi V.K.Ivanov tomonidan o‘rganilgan.[3] Ko‘p o‘lchamli fazoda soha chegarasining bir qismi konus bo‘lganda Laplas va Gelmgols tenglamalari uchun Koshi masalasining yechimi Sh.Yarmuxamedov tomonidan Karleman funksiyasini qurish asosida olingan.[4]

Tadqiqot metodologiyasi

1. Masalaning qo‘yilishi. R^2 tekislikda yarim birlik aylana yoyi hamda S silliq chiziq bilan chegaralangan D sohani qaraymiz. Bunda koordinata boshi sohadan tashqarida joylashgan deb olamiz.

$x=(x_1,x_2)\in D$ va $y=(y_1,y_2)\in \bar{D}$ bo‘lsin. Qaralayotgan D sohada

$$\frac{\partial^2 U}{\partial x_1^2}+\frac{\partial^2 U}{\partial x_2^2}=0 \tag{1}$$

Laplas tenglamasi yechimi $U(x)=U(x_1,x_2)\in C^2(D)\cap C^1(\bar{D})$ garmonik funksiyaning Koshi berilganlari S da berilgan:

$$U(x)|_S=f_0(x),\ \left.\frac{\partial U(x)}{\partial n}\right|_S=f_1(x) \tag{2}$$

bu yerda $f_0(x)$ va $f_1(x)$ - berilgan funksiyalar. (2) shartlarga ko‘ra $U(x)$ funksiyani D sohada topish masalasi qaraladi.

2. Karleman funksiyasi qurish sxemasi. (1) – (2) masala yechimini topishda Karleman funksiyasi metodidan foydalanamiz. [2] ga ko‘ra Laplas tenglamasining Karleman funksiyasi ta’rifini keltiramiz:

Ta’rif. S to‘plamning D sohaga nisbatan Karleman funksiyasi deb ikkita x va y nuqtalarga hamda σ musbat sonli parametrga bog‘liq bo‘lgan, quyidagi ikki shartlarni qanoatlantiruvchi $G(x,y,\sigma)$ funksiyaga aytiladi:

$$1)\, G(x,y,\sigma)=\ln\frac{1}{|x-y|}+g(x,y,\sigma), \tag{3}$$

bu yerda $g(x,y,\sigma)$ funksiya y o‘zgaruvchi bo‘yicha D sohada garmonik funksiya;

$$2)\ \int_{\partial D/S}\left(\left|G(x,y,\sigma)\right|+\left|\frac{\partial G(x,y,\sigma)}{\partial n}\right|\right)ds\le\alpha(\sigma); \tag{4}$$

bunda $\alpha(\sigma)$ funksiya $\sigma\to\infty$ da nolga intiladi.

Bu ta’rifda $g(x,y,\sigma)$ funksiya Karleman funksiyasining regulyar qismi deyiladi.

Yuqorida berilgan ta’rifdagi Karleman funksiyasining regulyar qismini

$$g(x,y,N)=-\ln\frac{1}{\rho_1}-\sum_{n=1}^{N}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)\cos n\theta, \tag{5}$$

ko‘rinishda olamiz, bu yerda Karleman funksiyasining parametri σ sifatida N natural son olindi va $\theta=\varphi-\psi;\quad |x|=\rho;\ |y|=\rho_1;\ \varphi=\operatorname{arctg}\frac{x_2}{x_1};\quad \psi=\operatorname{arctg}\frac{y_2}{y_1}.$

Teorema. $G(x,y,N)=\ln\frac{1}{r}+g(x,y,N)$ funksiya qaralayotgan D sohada S yoyi uchun Laplas tenglamasining Karleman funksiyasi bo‘ladi.

Teoremani isbotlashdan oldin quyidagi lemmani isbotlaymiz.

Lemma. (5) tenglik bilan aniqlangan $g(x,y,N)$ funksiya D sohada y o‘zgaruvchi bo‘yicha garmonik funksiya bo‘ladi.

Isbot. $g(x,y,N)$ funksiyani y o‘zgaruvchi bo‘yicha Laplas tenglamasini qanoatlantirishini ko‘rsatamiz ya’ni:

$$\Delta_y g(x,y,N)=-\Delta_y\ln\frac{1}{\rho_1}-\sum_{n=1}^{N}\frac{1}{n}\rho^n\Delta_y\left(\frac{\cos n\theta}{\rho_1^n}\right).$$

$\ln\frac{1}{\rho_1}$ – Laplas tenglamasining fundamental yechimligidan $\Delta_y\ln\frac{1}{\rho_1}=0$.

Qatordagi laplasian ichidagi funksiyani $g^*=\frac{\cos n\theta}{\rho_1^n}$ belgilaymiz.

Δg^* hisoblash uchun Laplas tenglamasini qutb koordinatalar ko‘rinishidan

$$\Delta g^*(\rho_1,\theta)=\rho_1^2 g^*_{\rho_1\rho_1}+\rho_1 g^*_{\rho_1}+g^*_{\theta\theta}$$

foydalanib, tegishli hosilalarni hisoblaymiz:

$$g^*_{\rho_1}=n\rho_1^{-n-1}\cos n\theta; g^*_{\rho_1\rho_1}=n(n+1)\rho_1^{-n-2}\cos n\theta;$$

$$g^*_{\theta}=-n\rho_1^{-n}\sin n\theta;\ g^*_{\theta\theta}=-n^2\rho_1^{-n}\cos n\theta.$$

$$\Delta g^*(\rho_1,\theta)=(n^2+n)\rho_1^{-n}\cos n\theta-n\rho_1^{-n}\cos n\theta- -n^2\rho_1^{-n}\cos n\theta=0.$$

$$\Delta_y g(x,y,N)=0.$$

(5) funksiyaning y o‘zgaruvchi bo‘yicha garmonikligi isbotlandi.

Isbotlangan lemmadan $G(x,y,N)$ funksiya Laplas tenglamasining Karleman funksiyasing 1) shartini qanoatlantirishi kelib chiqadi. Endi $G(x,y,N)$ funksiya ta’rifni 2) shartini qanoatlantirishini ko‘rsatish uchun

$$\int\limits_{\partial D/S}\left(\left|G(x,y,N)\right|+\left|\frac{\partial G(x,y,N)}{\partial n}\right|\right)ds_y,\ x\in D$$

integralni baholaymiz.

Bu yerda
$$G(x,y,N)=\ln\frac{1}{r}-\ln\frac{1}{\rho_1}-\sum_{n=1}^{N}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)\cos n\theta,\ r=|x-y|$$

$G(x,y,N)$ funksiyaning qator ko‘rinishidagi ifodasini topish uchun $|x-y|$ ifodada $x=\rho e^{i\varphi}$, $y=\rho_1 e^{i\psi}$, $\theta=\varphi-\psi$ va $\lambda=\dfrac{\rho}{\rho_1}$ almashtirishlar olib

$$r=\rho_1\sqrt{1+\lambda^2-2\lambda\cos\theta}$$

ko‘rinishda yozib olamiz. Ushbu

$$\ln\frac{1}{r}=\ln\frac{1}{\rho_1}-\ln\sqrt{1+\lambda^2-2\lambda\cos\theta} \tag{6}$$

tenglikdagi $\ln\sqrt{1+\lambda^2-2\lambda\cos\theta}$ ifodani kompleks o‘zgaruvchili $\ln(1-z)$ funksiya $z=\lambda e^{i\theta}$ almashtirishdan so‘ng haqiqiy qismidan quyidagi

$$\ln\sqrt{1+\lambda^2-2\lambda\cos\theta}=-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\cos n\theta \tag{7}$$

tenglikni hosil qilamiz. (6) va (7) dan

$$\ln\frac{1}{r}=\ln\frac{1}{\rho_1}+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)^n\cos n\theta \tag{8}$$

tenglikni olamiz. (8) tenglikdan

$$\sum_{n=N+1}^{\infty}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)^n\cos n\theta=\ln\frac{1}{r}-\ln\frac{1}{\rho_1}-\sum_{n=1}^{N}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)^n\cos n\theta \tag{9}$$

ga kelamiz.

Karleman $G(x,y,N)$ funksiyasing regulyar qismi sifatida

$$g(x,y,N)=-\ln\frac{1}{\rho_1}-\sum_{n=1}^{N}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)^n\cos n\theta$$

funksiyani olamiz. Demak $G(x,y,N)$ funksiyani

$$G(x,y,N)=\sum_{n=N+1}^{\infty}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)^n\cos n\theta$$

ko‘rinishda yozish mumkin.

Oxirgi tenglikning o‘ng tomonidagi qatorni $x\in D,\ y\in\partial D/S$ uchun baholaymiz:

$$\left|G(x,y,N)\right|_{\partial D/S}=\left|\sum_{n=N+1}^{\infty}\frac{1}{n}\left(\frac{\rho}{\rho_1}\right)^n\cos n\theta\right|_{\rho_1=1}\le$$

$$\le\sum_{n=N+1}^{\infty}\frac{\rho^n}{n}=\rho^{N+1}\left(1+\rho+\rho^2+\cdots\right)=\frac{\rho^{N+1}}{1-\rho}. \tag{10}$$

Xuddi shunga o‘xshash G funksiyani $y\in\partial D/S$ nuqtadagi normal bo‘yicha hosilasini baholaymiz:

$$\left|\frac{\partial G}{\partial n}\right|_{\partial D/S}=\left|\frac{\partial G}{\partial\rho_1}\right|_{\rho_1=1}=\left|\sum_{n=N+1}^{\infty}-\frac{\rho^n}{\rho_1^{n+1}}\cos n\theta\right|_{\rho_1=1}=$$

$$=\sum_{n=N+1}^{\infty}\left|-\rho^n\cos n\theta\right|\le\sum_{n=N+1}^{\infty}\rho^n=\frac{\rho^{N+1}}{1-\rho}. \tag{11}$$

Shunday qilib,

$$\int_{\partial D/S}\left(\left|G(x,y,N)\right|+\left|\frac{\partial G(x,y,N)}{\partial n}\right|\right)ds_y \le \frac{4\pi\rho^{N+1}}{1-\rho}. \tag{12}$$

Natijalar va ularning tahlili

(12) tengsizlikning o‘ng tomonidagi ifoda $0<\rho<1$ bo‘lgani uchun $N\to\infty$ da $2\pi\cdot\dfrac{\rho^{N+1}}{1-\rho}$ nolga intiladi.

Grin formulasiga ko‘ra (1), (2) masala yechimi uchun

$$U(x)=\frac{1}{2\pi}\int_{\partial D}\left[\ln\frac{1}{r}\cdot\frac{\partial U(y)}{\partial n}-U(y)\cdot\frac{\partial \ln\frac{1}{r}}{\partial n}\right]dS \quad x\in D \tag{13}$$

integral tasvir o‘rinli.

$g(x,y,N)$ funksiya y o‘zgaruvchi bo‘yicha D sohada garmonikligidan $x\in D$ uchun Grin formulasiga ko‘ra

$$\int_{\partial D}\left[g(x,y,N)\cdot\frac{\partial U(y)}{\partial n_y}-U(y)\cdot\frac{\partial g(x,y,N)}{\partial n_y}\right]dS_y =$$

$$=\int_D\left[g(x,y,N)\cdot_y U(y)-U(y)\cdot_y g(x,y,N)\right]dy=0 \tag{14}$$

tenglik o‘rinli. Masala shartidagi berilganlarni hisobga olib (13) va (14) tengliklardan

$$U(x)=\int_{\partial D/S}\left[G(x,y,N)\cdot\frac{\partial U(y)}{\partial n}-U(y)\cdot\frac{\partial G(x,y,N)}{\partial n}\right]dS_y+$$

$$+\int_S\left[G(x,y,N)\cdot f_1(y)-f_0(y)\cdot\frac{\partial G(x,y,N)}{\partial n}\right]dS_y$$

tenglikni hosil qilamiz. Oxirgi tenglikda $U(x)=U(x_1,x_2)\in C^2(D)\cap C^1(\bar{D})$ hamda (12) tengsizlikni hisobga olgan holda $N\to\infty$ limitga o‘tib

$$U(x)=\lim_{N\to\infty}\frac{1}{2\pi}\int_S\left[G(x,y,N)\cdot f_1(y)-f_1(y)\cdot\frac{\partial G(x,y,N)}{\partial n}\right]dS_y$$

tenglikni olamiz. Teorema isbotlandi.

Xulosa

$x=(x_1,x_2)\in D$ va $y=(y_1,y_2)\in \bar{D}$ bo‘lsin. Qaralayotgan D sohada $\frac{\partial^2 U}{\partial x_1^2}+\frac{\partial^2 U}{\partial x_2^2}=0$ Laplas tenglamasi yechimi $U(x)=U(x_1,x_2)\in C^2(D)\cap C^1(\bar{D})$ garmonik funksiyaning Koshi berilganlari $U(x)|_S=f_0(x),\ \left.\frac{\partial U(x)}{\partial n}\right|_S=f_1(x)$ bu shartlarga ko‘ra $U(x)$ funksiya uchun D sohada $U(x)=\lim\limits_{N\to\infty}\frac{1}{2\pi}\int\limits_S\left[G(x,y,N)\cdot f_1(y)-f_1(y)\cdot\frac{\partial G(x,y,N)}{\partial n}\right]dS_y$ integral tasvir o‘rinli.

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