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The algebra of invariants of the symmetry group of the tetrahedron.

Abstract. This article discusses the invariant algebras of certain finite matrix groups. In the course of the work, the invariant algebras for the symmetry groups of some regular polyhedra were determined. The article makes active use of Gröbner basis theory and the theory of invariants of finite groups.

Keywords: Gröbner basis, Hilbert series, finite group, invariant.

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Tetraedr simmetriya gruppasining invariantlar algebrasi

Annotatsiya. Ushbu maqolada ayrim chekli matritsali gruppalarning invariantlar algebralariga bag‘ishlangan. Ish jarayonida ayrim muntazam ko‘pyoqliklarning simmetriya gruppalari uchun invariantlar algebralari aniqlandi. Maqolani yozish jarayonida Grobner bazislari nazariyasi hamda chekli gruppalar invariantlar nazariyasidan faol foydalanildi.

Kalit so‘zlar: Grobner bazisi, Gilbert qatori, cheklangan guruh, invariant.

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Аннотация. В данной статье рассматриваются алгебры инвариантов некоторых конечных матричных групп. В процессе работы были определены алгебры инвариантов для групп симметрии некоторых правильных многогранников. При написании статьи активно использовались теория базисов Грёбнера и теория инвариантов конечных групп.

Ключевые слова: базис Грёбнера, ряд Гильберта, конечная группа, инвариант.

Introduction.

The classical theory of invariants originated in England in the middle of the 19th century and later, in the second half of the century, became one of the central branches of mathematics. The emergence of invariant theory is connected with three fundamental areas of science: number theory, geometry, and algebra. The initial stage of its development is associated with the work of English mathematicians such as Cayley, Sylvester (who introduced almost all the terminology used in invariant theory, including the term “invariant” itself), and Salmon. At the same time, the works of Jacobi and Hesse in Germany and Hermite in France also played an important role. Later, German mathematicians began to take the leading role in the further development of invariant theory. Among them, Aronhold, Clebsch, Gordan, and especially D. Hilbert stand out for their outstanding contributions.

The tetrahedral symmetry group is one of the important mathematical concepts that connects the boundary points between geometry and algebra. This group represents the symmetric motions of a tetrahedron - that is, all the operations of rotation and translation that map the tetrahedron onto itself. The algebraic description of these symmetries is carried out through the theory of invariants. The main goal of invariant theory is to find polynomials and functions that remain unchanged (invariant) under the action of a given group.

Preliminaries

Definition. Let $G \subset G(n,k)$ be a finite matrix group. A polynomial $f(x) \in k[x_1, x_2, \ldots, x_n]$ is called invariant with respect to the group G (or simply, an invariant of G) if $f(x) = f(A \cdot x)$ for all $A \in G$. The set of all polynomials invariant with respect to G is denoted by $k[x_1, x_2, \ldots, x_n]^G$.

Theorem. Let $G \subset G(n,k)$ be a finite matrix group, then:

1. The set $k[x_1, x_2, \ldots, x_n]^G$ is closed under addition and multiplication and also contains all constants.
2. A polynomial $f \in k[x_1, x_2, \ldots, x_n]$ is invariant with respect to G if and only if its homogeneous components are invariant.

Lemma. Let $G \subset G(n,k)$ be a finite matrix group and let matrices $A_1, A_2, \ldots, A_m \in G$ be such that any matrix $A \in G$ can be represented in the form

$$A = B_1 B_2 \ldots B_t,$$

where $B_i \in A_1, A_2, \ldots, A_m$ for any iii (in other words, $A_1, A_2, \ldots, A_m$ generate G). Then a polynomial $f \in k[x_1, x_2, \ldots, x_n]$ is invariant with respect to G if and only if

$$f(x) = f(A_1 \cdot x) = f(A_2 \cdot x) = \ldots = f(A_m \cdot x).$$

Theorem. Every finite matrix group $\Gamma \subset GL(C^n)$ has n algebraically independent invariants over C that is, the ring $C[x]^\Gamma$ has transcendence degree n.

Recall that the transcendence degree of the algebra $C[x]^{\Gamma}$ is defined as the maximal number of algebraically independent polynomials contained in $C[x]^{\Gamma}$.

Theorem (Hilbert's Finiteness Theorem). The ring of invariants $C[x]^{\Gamma}$ of a finite matrix group $\Gamma \subset GL(C^n)$ is finitely generated.

Theorem (Noether's Degree Bound Theorem). The ring of invariants $C[x]^{\Gamma}$ of a finite matrix group Γ has no more than $\binom{n+|\Gamma|}{n}$ invariants, whose degrees are bounded by the order of the group $|\Gamma|$.

Definition. An algebra A is called *graded* if, as a vector space, it decomposes into a direct sum $A = \oplus_{i=0}^{\infty} A_i$ such that $A_i \cdot A_j \subset A_{i+j}$. The algebra of invariants of a finite group $C[x]^{\Gamma}$ is a graded algebra.

Definition. The *Hilbert series* of an algebra A is the series of the form

$$\sum_{d=0}^{\infty} z^i \cdot \dim A_i .$$

The Hilbert series for the graded algebra $C[x]^{\Gamma}$ is denoted by $\Phi_{\Gamma}(z)$.

Theorem (Molien's Theorem). The Hilbert series of the algebra of invariants $C[x]^{\Gamma}$ is given by

$$\Phi_{\Gamma}(z) = \frac{1}{|\Gamma|} \sum_{\pi \in \Gamma} \frac{1}{\det(id - z\pi)} .$$

Lemma. Let $\Gamma \subset GL(C^n)$ be a finite matrix group. Then the dimension of the subspace of invariants

$$V^{\Gamma} = \{ v \in C^n : \pi v = v \ \ for\, all\ \pi \in \Gamma \}$$

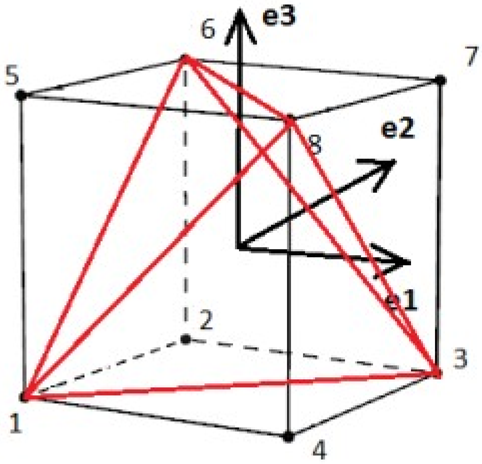
is equal to $\frac{1}{|\Gamma|} \sum_{\pi \in \Gamma} trace(\pi)$.

Lemma. Let $p_1, p_2, \ldots, p_m$ be algebraically independent elements of $C[x]$ having homogeneous degrees $d_1, d_2, \ldots, d_m$. Then the Hilbert series of the subring $R := C[p_1, p_2, \ldots, p_m]$ is equal to

$$H(R,z) = \sum_{n=0}^{\infty} (\dim_c R_d) z^d = \frac{1}{(1-z^{d_1})(1-z^{d_2})\ldots(1-z^{d_m})}.$$

Main part

Let us consider a tetrahedron inscribed in a cube. The rotation group of this tetrahedron is denoted by C_{Th}.



The elements of this group are represented by the following matrices:

$$1)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad 2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad 3)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$4)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad 5)\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad 6)\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$7)\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad 8)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \qquad 9)\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$10)\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad 11)\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad 12)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There are a total of 12 matrices, which represent all the rotational symmetries of the tetrahedron.

The Hilbert series $\Phi_{C_{Th}}(z)$ of the polynomial algebra $C[x,y,z]^{C_{Th}}$, invariant under the tetrahedral group, is calculated using Molien’s theorem as follows:

$$\Phi_{C_{Th}}(z) = \frac{1}{12}\left(\frac{1}{-(-1+z)^3} + 8\frac{1}{1-z^3} + 3\frac{1}{-(1+z)^2(-1+z)} \right)$$

Expanding this series, we obtain the following:

$$\Phi_{C_{Th}}(z) = 1 + z^2 + z^3 + 2z^4 + z^5 + 4z^6 + 2z^7 + 5z^8 + 4z^9 + 7z^{10} + \ldots$$

The fact that the coefficient of z^2 in the series is 1 indicates the existence of a second-degree invariant. We can write this invariant as follows:

$$g_1 = x^2 + y^2 + z^2$$

We construct the Hilbert series for $C[g_1]$.

$$\frac{1}{(1-z^2)} = 1 + z^2 + z^4 + z^6 + z^8 + \ldots$$

However, this series does not contain a term of degree z^3. A third-degree invariant is also needed.

The third-degree invariant is:

$$g_2 = xyz$$

We find the relation between g_1 and g_2. In our case, there is no such relation. Therefore, we construct the Hilbert series for the algebra $C[g_1, g_2]$:

$$\frac{1}{(1-z^2)(1-z^3)} = 1 + z^2 + z^3 + z^4 + z^5 + 2z^6 + 2z^7 + 2z^8 + \ldots$$

In this case as well, an invariant for z^4 is missing. A fourth-degree invariant for the tetrahedral group is found as follows:

$$g_3 = x^4 + y^4 + z^4$$

It is also invariant under the symmetries of the tetrahedron. Now, let us consider the algebra $C[g_1, g_2, g_3]$. Its Hilbert series is:

$$\frac{1}{(1-z^2)(1-z^3)(1-z^4)} = 1 + z^2 + z^3 + 2z^4 + z^5 + 3z^6 + 2z^7 + 4z^8 + \ldots$$

We can see that a sixth-degree invariant is missing. Therefore, we find it. The sixth-degree invariant is:

$$g_4 = x^4y^2 + y^4z^2 + x^2z^4$$

This invariant can be expressed in terms of g_1, g_2, g_3. Using a Gröbner basis, their relation is determined as follows:

$$g_1^6 - 3g_3g_1^4 - 16g_2^2g_1^3 + 3g_3^2g_1^2 - 4g_4g_1^3 + 24g_1g_2^2g_3 + 72g_2^4 - g_3^3 + 4g_4g_3g_1 + 24g_4g_2^2 + 8g_4^2 = 0$$

Thus, the sixth-degree invariant g_4 can be expressed in terms of the previous invariants.

We construct the Hilbert series for the algebra $C[g_1, g_2, g_3, g_4]$:

$$\frac{1}{(1-z^2)(1-z^3)(1-z^4)}+\frac{z^6}{(1-z^2)(1-z^3)(1-z^4)}=1+z^2+2z^4+z^5+4z^6+2z^7+5z^8+4z^9+7z^{10}+10z^{12}+\dots$$

The algebra $C[g_1,g_2,g_3,g_4]$ is a subalgebra of the algebra $C[x,y,z]^{C_{Th}}$.

We have shown that the Hilbert series of the algebra $C[g_1,g_2,g_3,g_4]$ is equal to the Hilbert series of the algebra of invariants of $C[x,y,z]^{C_{Th}}$

$$C[x,y,z]^{C_{Th}}=C\left[x^2+y^2+z^2, xyz, x^4+y^4+z^4, x^4y^2+y^4z^2+x^2z^4\right].$$

Conclusions

The algebra of invariants of the tetrahedral symmetry group occupies an important place in the theory of mathematical invariants. This algebra reflects the symmetry properties of the tetrahedron and studies the expressions that remain unchanged (invariant) under the action of the group on polynomials in multidimensional space. Research shows that the algebra of invariants for the tetrahedral group consists of a finite number of fundamental invariants, through which the entire algebraic structure can be reconstructed. This, in turn, makes it possible to determine the symmetries of geometric objects.

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