Irgashev Baxrom Yusupxanovich[1,2],
Sobitov Ravshan Abdirayimxonovich[3]
[1]**University of Business and Science**
[2]**V.I.Romanovskiy nomidagi Matematika instituti**
[3]**Namangan davlat universiteti**

Riman-Liuvill ma’nosidagi hosila qatnashgan buzilishga ega oltinchi tartibli tenglama uchun aralash masala

Annotatsiya. Ushbu maqolada Riman–Liuvill ma’nosidagi kasr tartibli hosila qatnashgan buzilishga ega oltinchi tartibli differensial tenglama uchun aralash masalaning yechilishi o‘rganiladi. Avvalo, tenglamaning butun va kasr tartibli hosilalar ishtirokidagi xususiyatlari tahlil qilinadi hamda qo‘yilgan masalaning to‘g‘ri qo‘yilganligi shartlari ko‘rib chiqiladi. Chegaraviy va boshlang‘ich shartlar ostida aralash masalaning mavjudligi va yagonaligi uchun zarur bo‘lgan shartlar isbotlanadi. Bundan tashqari, yechimning silliqlik darajasi, buzilishga ega nuqtadagi xolati va uning analitik ko‘rinishlari haqida natijalar keltiriladi. Keltirilgan usulning samaradorligi namunaviy misol orqali tasdiqlanadi.

Kalit so‘zlar. Riman–Liuvill hosilasi; kasr tartibli differensial tenglama; oltinchi tartibli tenglama; buzilish nuqtasi; aralash masala; mavjudlik va yagonalik; chegara shartlari; analitik yechim.

Иргашев Бахром Юсупханович[1,2],
Собитов Равшан Абдираимханович[3]
[1]**University of Business and Science**
[2] **Математический институт имени В.И. Романовского**
[3] **Наманганский государственный университет**

Смешанная задача для уравнения шестого порядка с особенностью, содержащего производную в смысле Римана–Лиувилля

Аннотация. В данной статье исследуется решение смешанной задачи для дифференциального уравнения шестого порядка с особенностью, содержащего производную дробного порядка в смысле Римана–Лиувилля. Прежде всего анализируются свойства уравнения, включающего как целые, так и дробные производные, а также рассматриваются условия корректной постановки задачи. Доказаны необходимые условия существования и единственности решения смешанной задачи при заданных граничных и начальных условиях. Кроме того, приведены результаты о степени гладкости решения, его поведении в точке

особености и его аналитических представлениях. Эффективность предложенного метода подтверждается на примере.

Ключевые слова: производная Римана–Лиувилля; дифференциальное уравнение дробного порядка; уравнение шестого порядка; точка особености; смешанная задача; существование и единственность; граничные условия; аналитическое решение.

Irgashev Bahrom Yusupkhanovich[1,2],
Sobitov Ravshan Abdirayimkhanovich[3]
[1] University of Business and Science
[2] V.I. Romanovskiy Institute of Mathematics
[3] Namangan State University

A Mixed Problem for a Sixth-Order Equation with a Singularity Involving the Riemann–Liouville Derivative

Abstract. This article investigates the solution of a mixed problem for a sixth-order differential equation with a singularity involving a fractional derivative in the sense of Riemann–Liouville. First, the properties of the equation that includes both integer-order and fractional-order derivatives are analyzed, and the conditions ensuring the well-posedness of the problem are examined. The necessary conditions for the existence and uniqueness of the solution under given boundary and initial conditions are proven. In addition, results are provided concerning the smoothness of the solution, its behavior at the singular point, and its analytical representations. The effectiveness of the proposed method is demonstrated through an illustrative example.

Keywords: Riemann–Liouville derivative; fractional differential equation; sixth-order equation; singular point; mixed problem; existence and uniqueness; boundary conditions; analytical solution.

Kirish. So‘ngi o‘n yilliklarda anomal diffuziya, dispersiya jarayonlari, ko‘p qatlamli elastik muhitlar tebranishi, mikro- va nanofizik jarayonlarning matematik modellarida kasr tartibli hosilalar ishtirok etuvchi differensial tenglamalar alohida ahamiyat kasb etmoqda. Mazkur jarayonlar klassik butun tartibli hosilalar orqali yetarli aniqlikda tasvirlab bo‘lmaydi, chunki bunday modellarda xotira va uzoq ta’sir effektlari muhim rol o‘ynaydi. Aynan shuning uchun Riman–Liuvill ma’nosidagi kasr hosila real fizik-mexanik jarayonlarning tarixga bog‘langan dinamikasini aniq ifodalovchi mos operator sifatida keng qo‘llaniladi.

Boshqa tomondan, buzilishga ega yuqori tartibli differensial tenglamalar xususan, yuqori tartibli tenglamalar klassik to‘lqin, nurash, tebranish, filtratsiya va kapillyar-dispersiya jarayonlarini chuqurroq aks ettiruvchi modellardir. Ko‘pincha muhitning qattiq-yumshoq o‘tish zonalari, ko‘p pog‘onali strukturalar va elastik-plastik qatlamlar kabi murakkab fizik effektlar buziluvchi (degeneratsiyalanuvchi) koeffitsientlar bilan ifodalanadi. Bunday tenglamalar uchun aralash masalani qo‘yish esa boshlang‘ich va chegaraviy shartlarning moslashuvini ta’minlash va yechimning mavjudligi, yagona bo‘lishi hamda barqarorligini kafolatlash nuqtai nazaridan juda dolzarbdir[1-2].

Mazkur yo‘nalish bo‘yicha dastlabki va asosiy tadqiqotlar A. M. Naxushev, A. A. Kylbas, S. G. Samko, A.V. Pskhu kabi olimlarning ishlari bilan bog‘liq bo‘lib[3-7], ular kasr tartibli integrallar va Riman–Liuvill hosilasining asosiy xossalari, ular uchun to‘g‘ri qo‘yilgan masalalar nazariyasi, integral operatorlarning spektral xususiyatlarini chuqur tahlil qilganlar.

O‘zbekistonlik olimlar orasida Karimov E., Ruzhansky M., Toshtemirov B tomonidan yuqori tartibli, xususan buziluvchan koeffitsientli tenglamalar va Riman–Liuvill hosilasiga ega aralash masalalar chuqur tadqiq etilgan[8]. Uning ishlarida kasr tartibli hosilaga ega yuqori tartibli tenglamalarning qo‘yilishigacha bo‘lgan masalalar, fundamental yechimlar, Vaqt funksiyalari yordamida maxsus yechimlar qurilishi hamda spektoral metodlar asosida mavjudlik va yagona yechim teoremalari o‘rganilgan.

Mazkur ishlarning aksariyatida tenglama to‘rtinchi tartibda yoki yuqorisi bo‘lgan bo‘lsa-da, buziluvchan koeffitsientga ega bo‘lgan oltinchi tartibli kasr tartibli tenglamalar kam o‘rganilgan. Ko‘p hollarda buzilishning kuchi, uning fazoviy o‘zgarish xususiyatlari yoki chegaraviy shartlaridagi noaniqliklar to‘liq tadqiq qilinmagan. Shuningdek, aralash masala uchun qo‘yilgan shartlarning mosligi va Riman–Liuvill hosilasi ishtirokida yechimning yetarli silliqligi (regularity) masalasi haligacha ochiq bo‘lib qolayotgan muammolardan biridir.

Endi Riman-Liuvill ma’nosidagi hosilaga ta’rif keltiraylik.

$\Omega = [a,b] (-\infty < a < b < \infty) \mathbb{R}$ haqiqiy sonlar o‘qida chegaralangan interval bo‘lsin. $\alpha \in C (\Re(\alpha) > 0)$ tartibli $I_{a+}^{\alpha} f$ va $I_{b-}^{\alpha} f$ Riman-Liuvill kasr tartibli integrallari mos ravishda quyidagicha ifodalanadi:

$$\left(I_{a+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \left(x > a; \Re(\alpha) > 0\right),$$

$$\left(I_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}} \left(x < b; \Re(\alpha) > 0\right).$$

bu yerda $\Gamma(\alpha)$ Gamma funksiyasi. Ushbu integrallar chap tomondan va o‘ng tomondan kasr tartibli integrallar deb ataladi.

Asosiy qism. Agar $\Omega = \Omega_x \times \Omega_y, \Omega_x = \{0 < x < 1\}, \Omega_y = \{0 < y < 1\}$ sohada

$$D_{0x}^{\alpha} u(x,y) + x^{\beta} \frac{\partial^6 u(x,y)}{\partial y^6} = 0, \qquad (1)$$

tenglama berilgan, bu yerda $0 < \alpha < 1, -\alpha < \beta \in R,$

$$D_{0x}^{\alpha}u(x,y)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{u(\tau,y)d\tau}{(x-\tau)^{\alpha}}$$

- Riman-Liuvill ma’nosidagi kasr tartibli hosila[9].

H masala. Ω sohada berilgan (1) tenglamani va quyidagi shartlarni bajaruvchi *u(x,y)* funksiya topilsin:

$$D_{0x}^{\alpha}u(x,y)\in C(\Omega),\, D_{0x}^{\alpha-1}u(x,y)\in C(\overline{\Omega}),$$

$$\frac{\partial^5 u(x,y)}{\partial y^5}\in C(\overline{\Omega}_y),\, \frac{\partial^6 u(x,y)}{\partial y^6}\in C(\Omega),$$

$$\begin{cases} a_0u(x,0)+a_1u_x(x,0)+a_2u_{xx}(x,0)+a_3u_{xxx}(x,0)+a_4u_{xxxx}(x,0)+a_5u_{xxxxx}(x,0)=0\\ b_0u(x,0)+b_1u_x(x,0)+b_2u_{xx}(x,0)+b_3u_{xxx}(x,0)+b_4u_{xxxx}(x,0)+b_5u_{xxxxx}(x,0)=0\\ d_0u(x,0)+d_1u_x(x,0)+d_2u_{xx}(x,0)+d_3u_{xxx}(x,0)+d_4u_{xxxx}(x,0)+d_5u_{xxxxx}(x,0)=0\end{cases}$$

$$\begin{cases} l_0u(x,1)+l_1u_x(x,1)+l_2u_{xx}(x,1)+l_3u_{xxx}(x,1)+l_4u_{xxxx}(x,1)+l_5u_{xxxxx}(x,1)=0\\ m_0u(x,1)+m_1u_x(x,1)+m_2u_{xx}(x,1)+m_3u_{xxx}(x,1)+m_4u_{xxxx}(x,1)+m_5u_{xxxxx}(x,1)=0\\ n_0u(x,1)+n_1u_x(x,1)+n_2u_{xx}(x,1)+n_3u_{xxx}(x,1)+n_4u_{xxxx}(x,1)+n_5u_{xxxxx}(x,1)=0\end{cases} \quad (2)$$

$$D_{0x}^{\alpha-1}u(0,y)=\varphi(y),$$

bu yerda

$$rang\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5\\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5\\ d_0 & d_1 & d_2 & d_3 & d_4 & d_5\end{pmatrix}=rang\begin{pmatrix} l_0 & l_1 & l_2 & l_3 & l_4 & l_5\\ m_0 & m_1 & m_2 & m_3 & m_4 & m_5\\ n_0 & n_1 & n_2 & n_3 & n_4 & n_5\end{pmatrix}=3,$$

$\varphi(y)$ - yetarlicha silliq funksiya[9].

H masalaning yechimini qator ko‘rinishida izlaymiz

$$u(x,y)=\sum_{n=1}^{\infty}X_n(x)Y_n(y), \quad (3)$$

bu yerda

$$\begin{cases} Y_n^{(6)}(y)=-\lambda_nY_n(y),\\ a_0Y(0)+a_1Y'(0)+a_2Y''(0)+a_3Y'''(0)+a_4Y^{(4)}(0)+a_5Y^{(5)}(0)=0\\ b_0Y(0)+b_1Y'(0)+b_2Y''(0)+b_3Y'''(0)+b_4Y^{(4)}(0)+b_5Y^{(5)}(0)=0\\ d_0Y(0)+d_1Y'(0)+d_2Y''(0)+d_3Y'''(0)+d_4Y^{(4)}(0)+d_5Y^{(5)}(0)=0\\ l_0Y(1)+l_1Y'(1)+l_2Y''(1)+l_3Y'''(1)+l_4Y^{(4)}(1)+l_5Y^{(5)}(1)=0\\ m_0Y(1)+m_1Y'(1)+m_2Y''(1)+m_3Y'''(1)+m_4Y^{(4)}(1)+m_5Y^{(5)}(1)=0\\ n_0Y(1)+n_1Y'(1)+n_2Y''(1)+n_3Y'''(1)+n_4Y^{(4)}(1)+n_5Y^{(5)}(1)=0\end{cases} \quad (4)$$

$Y_n(y)$ funksiyalar masalaning xos funksiyalari, $X_n(x)$ funksiyalar quyidagi masalaning yechimi:

$$\begin{cases} D_{0x}^{\alpha} X_n(x) = -\lambda_n x^{\beta} X_n(x), \\ D_{0x}^{\alpha-1} X_n(0) = \varphi_n = \int\limits_0^1 \varphi(y) Y_n(y)dy, n = 1, 2, ... \end{cases} \tag{5}$$

Lemma. Agar

$$\begin{vmatrix} a_0 & a_2 & a_3 \\ b_0 & b_2 & b_3 \\ d_0 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_4 \\ b_0 & b_1 & b_4 \\ d_0 & d_1 & d_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} = \begin{vmatrix} a_0 & a_2 & a_5 \\ b_0 & b_2 & b_5 \\ d_0 & d_2 & d_5 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_5 \\ b_0 & b_1 & b_5 \\ d_0 & d_1 & d_5 \end{vmatrix},$$

$$\begin{vmatrix} l_0 & l_2 & l_3 \\ m_0 & m_2 & m_3 \\ n_0 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} l_0 & l_1 & l_4 \\ m_0 & m_1 & m_4 \\ n_0 & n_1 & n_4 \end{vmatrix}, \begin{vmatrix} l_1 & l_2 & l_4 \\ m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \end{vmatrix} = \begin{vmatrix} l_0 & l_2 & l_5 \\ m_0 & m_2 & m_5 \\ n_0 & n_2 & n_5 \end{vmatrix}, \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} l_0 & l_1 & l_5 \\ m_0 & m_1 & m_5 \\ n_0 & n_1 & n_5 \end{vmatrix}$$

shartlar bajarilsa, quyidagi differensial operator

$$L(Y) = \frac{d^6}{dy^6} Y,$$

o‘z-o‘ziga qo‘shma bo‘ladi. Bu yerda *L* operatorning aniqlanish sohasi quyidagicha:

$$D(L) = \left(Y : Y^{(6)}(y) \in C(0,1), Y^{(5)}(y) \in C[0,1]\right) \cap (4).$$

Isbot. Agar *L* operator uchun $(LY, Z) = (Y, LZ)$ shart bajarilsa, u holda *L* – operator o‘z-o‘ziga qo‘shma operator deyiladi, bu yerda $Y, Z \in D(\mathrm{L})$.

Faraz qilaylik

$$\Delta = \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ d_0 & d_1 & d_2 \end{vmatrix} \neq 0$$

(boshqa holatlar ham xuddi shunday ko‘rib chiqiladi). *L* – operatorning o‘z-o‘ziga qo‘shma bo‘lish shartlarini topamiz:

$$\int\limits_0^1 Y^{(6)}(y)Z(y)dy = Y^{(5)}Z\Big|_{y=0}^{y=1} + Y^{(4)}Z'\Big|_{y=0}^{y=1} + Y'''Z''\Big|_{y=0}^{y=1} + Y''Z'''\Big|_{y=0}^{y=1} + Y'Z^{(4)}\Big|_{y=0}^{y=1} + YZ^{(5)}\Big|_{y=0}^{y=1} + \int\limits_0^1 YZ^{(5)}dy$$

$y = 0$ nuqtada ifodani olamiz ($y = 1$ nuqta uchun ham shu usuldan foydalanamiz)

$$B = Y^{(5)}(0)Z(0) - Y^{(4)}(0)Z'(0) + Y'''(0)Z''(0) - Y''(0)Z'''(0) + Y'(0)Z^{(4)}(0) - Y(0)Z^{(5)}(0).$$

H masalaning (2) shartidan foydalanib, quyidagi sistemalarni hosil qilamiz:

$$\begin{cases} a_0 Y(0) + a_1 Y'(0) + a_2 Y''(0) + a_3 Y'''(0) + a_4 Y^{(4)}(0) + a_5 Y^{(5)}(0) = 0 \\ b_0 Y(0) + b_1 Y'(0) + b_2 Y''(0) + b_3 Y'''(0) + b_4 Y^{(4)}(0) + b_5 Y^{(5)}(0) = 0 \\ d_0 Y(0) + d_1 Y'(0) + d_2 Y''(0) + d_3 Y'''(0) + d_4 Y^{(4)}(0) + d_5 Y^{(5)}(0) = 0 \end{cases}$$

$$\begin{cases} a_0 Z(0) + a_1 Z'(0) + a_2 Z''(0) + a_3 Z'''(0) + a_4 Z^{(4)}(0) + a_5 Z^{(5)}(0) = 0 \\ b_0 Z(0) + b_1 Z'(0) + b_2 Z''(0) + b_3 Z'''(0) + b_4 Z^{(4)}(0) + b_5 Z^{(5)}(0) = 0 \\ d_0 Z(0) + d_1 Z'(0) + d_2 Z''(0) + d_3 Z'''(0) + d_4 Z^{(4)}(0) + d_5 Z^{(5)}(0) = 0 \end{cases}$$

bu yerda

$$\Delta = \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ d_0 & d_1 & d_2 \end{vmatrix} \neq 0$$

agar

$$-f_0^1 = a_3 Y'''(0) + a_4 Y^{(4)}(0) + a_5 Y^{(5)}(0)$$

$$-f_1^1 = b_3 Y'''(0) + b_4 Y^{(4)}(0) + b_5 Y^{(5)}(0)$$

$$-f_2^1 = d_3 Y'''(0) + d_4 Y^{(4)}(0) + d_5 Y^{(5)}(0)$$

$$-f_0^2 = a_3 Z'''(0) + a_4 Z^{(4)}(0) + a_5 Z^{(5)}(0)$$

$$-f_1^2 = b_3 Z'''(0) + b_4 Z^{(4)}(0) + b_5 Z^{(5)}(0)$$

$$-f_2^2 = d_3 Z'''(0) + d_4 Z^{(4)}(0) + d_5 Z^{(5)}(0)$$

belgilashlar kiritsak,

$$Y(0) = \frac{\begin{vmatrix} f_0^1 & a_1 & a_2 \\ f_1^1 & b_1 & b_2 \\ f_2^1 & d_1 & d_2 \end{vmatrix}}{\Delta}, \quad Y'(0) = \frac{\begin{vmatrix} a_0 & f_0^1 & a_2 \\ b_0 & f_1^1 & b_2 \\ d_0 & f_2^1 & d_2 \end{vmatrix}}{\Delta}, \quad Y''(0) = \frac{\begin{vmatrix} a_0 & a_1 & f_0^1 \\ b_0 & b_1 & f_1^1 \\ d_0 & d_1 & f_2^1 \end{vmatrix}}{\Delta},$$

$$Z(0) = \frac{\begin{vmatrix} f_0^2 & a_1 & a_2 \\ f_1^2 & b_1 & b_2 \\ f_2^2 & d_1 & d_2 \end{vmatrix}}{\Delta}, \quad Z'(0) = \frac{\begin{vmatrix} a_0 & f_0^2 & a_2 \\ b_0 & f_1^2 & b_2 \\ d_0 & f_2^2 & d_2 \end{vmatrix}}{\Delta}, \quad Z''(0) = \frac{\begin{vmatrix} a_0 & a_1 & f_0^2 \\ b_0 & b_1 & f_1^2 \\ d_0 & d_1 & f_2^2 \end{vmatrix}}{\Delta}.$$

bo’ladi.

Yuqoridagilarni hisobga olib,

$$B = Y^{(5)}(0)\begin{vmatrix} f_0^2 & a_1 & a_2 \\ f_1^2 & b_1 & b_2 \\ f_2^2 & d_1 & d_2 \end{vmatrix} - Y^{(4)}(0)\begin{vmatrix} a_0 & f_0^2 & a_2 \\ b_0 & f_1^2 & b_2 \\ d_0 & f_2^2 & d_2 \end{vmatrix} + Y'''(0)\begin{vmatrix} a_0 & a_1 & f_0^2 \\ b_0 & b_1 & f_1^2 \\ d_0 & d_1 & f_2^2 \end{vmatrix} - Z'''(0)\begin{vmatrix} a_0 & a_1 & f_0^1 \\ b_0 & b_1 & f_1^1 \\ d_0 & d_1 & f_2^1 \end{vmatrix}$$

$$+ Z^{(4)}(0)\begin{vmatrix} a_0 & f_0^2 & a_2 \\ b_0 & f_1^2 & b_2 \\ d_0 & f_2^2 & d_2 \end{vmatrix} - Z^{(5)}(0)\begin{vmatrix} f_0^2 & a_1 & a_2 \\ f_1^2 & b_1 & b_2 \\ f_2^2 & d_1 & d_2 \end{vmatrix} = Y^{(5)}(0)\begin{vmatrix} f_0^2 & a_1 & a_2 \\ f_1^2 & b_1 & b_2 \\ f_2^2 & d_1 & d_2 \end{vmatrix} + Y^{(4)}(0)\begin{vmatrix} f_0^2 & a_0 & a_2 \\ f_1^2 & b_0 & b_2 \\ f_2^2 & d_0 & d_2 \end{vmatrix}$$

$$+ Y'''(0)\begin{vmatrix} f_0^2 & a_0 & a_1 \\ f_1^2 & b_0 & b_1 \\ f_2^2 & d_0 & d_1 \end{vmatrix} - Z'''(0)\begin{vmatrix} a_0 & a_1 & f_0^1 \\ b_0 & b_1 & f_1^1 \\ d_0 & d_1 & f_2^1 \end{vmatrix} - Z^{(4)}(0)\begin{vmatrix} a_0 & a_2 & f_0^2 \\ b_0 & b_2 & f_1^2 \\ d_0 & d_2 & f_2^2 \end{vmatrix} - Z^{(5)}(0)\begin{vmatrix} a_1 & a_2 & f_0^2 \\ b_1 & b_2 & f_1^2 \\ d_1 & d_2 & f_2^2 \end{vmatrix} = 0$$

Elementar almashtirishlar yordamida

$$B = Y^{(5)}(0)\begin{vmatrix} f_0^2 & a_1 & a_2 \\ f_1^2 & b_1 & b_2 \\ f_2^2 & d_1 & d_2 \end{vmatrix} - Y^{(4)}(0)\begin{vmatrix} a_0 & f_0^2 & a_2 \\ b_0 & f_1^2 & b_2 \\ d_0 & f_2^2 & d_2 \end{vmatrix} + Y'''(0)\begin{vmatrix} a_0 & a_1 & f_0^2 \\ b_0 & b_1 & f_1^2 \\ d_0 & d_1 & f_2^2 \end{vmatrix} - Z'''(0)\begin{vmatrix} a_0 & a_1 & f_0^1 \\ b_0 & b_1 & f_1^1 \\ d_0 & d_1 & f_2^1 \end{vmatrix}$$

$$+ Z^{(4)}(0)\begin{vmatrix} a_0 & f_0^2 & a_2 \\ b_0 & f_1^2 & b_2 \\ d_0 & f_2^2 & d_2 \end{vmatrix} - Z^{(5)}(0)\begin{vmatrix} f_0^2 & a_1 & a_2 \\ f_1^2 & b_1 & b_2 \\ f_2^2 & d_1 & d_2 \end{vmatrix} =$$

$$+ \left(Y^{(4)}(0)Z^{(5)}(0) - Y^{(5)}(0)Z^{(4)}(0)\right)\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 & a_5 \\ b_0 & b_2 & b_5 \\ d_0 & d_2 & d_5 \end{vmatrix}\right) +$$

$$\left(Y'''(0)Z^{(4)}(0) - Y^{(4)}(0)Z'''(0)\right)\left(\begin{vmatrix} a_0 & a_2 & a_3 \\ b_0 & b_2 & b_3 \\ d_0 & d_2 & d_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_1 & a_4 \\ b_0 & b_1 & b_4 \\ d_0 & d_1 & d_4 \end{vmatrix}\right) +$$

$$+ \left(Y'''(0)Z^{(5)}(0) - Y^{(5)}(0)Z'''(0)\right)\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_1 & a_5 \\ b_0 & b_1 & b_5 \\ d_0 & d_1 & d_5 \end{vmatrix}\right) = 0$$

ga ega bo’lamiz. **Lemma isbotlandi.**

Faraz qilaylik, 1-lemma shartlari bajarilsin va $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$ bo‘lsin. Misol uchun

$$Y^{(6)}(y) = -\lambda Y(y),$$

$$\begin{cases} Y(0) - Y^{V}(0) = 0 \\ Y'(0) + Y^{IV}(0) = 0 \\ Y''(0) - Y'''(0) = 0 \\ Y(1) + Y^{V}(1) = 0 \\ Y'(1) + Y^{IV}(1) = 0 \\ Y''(1) + Y'''(1) = 0 \end{cases}$$

Koeffisiyentlaridan tuzilgan matritsaning rangi 3 ga teng, chunki

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ d_0 & d_1 & d_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

$$\begin{vmatrix} l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \\ n_0 & n_1 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

.

1-lemmaga ko‘ra,

$$\begin{vmatrix} a_0 & a_2 & a_3 \\ b_0 & b_2 & b_3 \\ d_0 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} a_0 & a_1 & a_4 \\ b_0 & b_1 & b_4 \\ d_0 & d_1 & d_4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_0 & a_2 & a_5 \\ b_0 & b_2 & b_5 \\ d_0 & d_2 & d_5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} a_0 & a_1 & a_5 \\ b_0 & b_1 & b_5 \\ d_0 & d_1 & d_5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} l_0 & l_2 & l_3 \\ m_0 & m_2 & m_3 \\ n_0 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} l_0 & l_1 & l_4 \\ m_0 & m_1 & m_4 \\ n_0 & n_1 & n_4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} l_1 & l_2 & l_4 \\ m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} l_0 & l_2 & l_5 \\ m_0 & m_2 & m_5 \\ n_0 & n_2 & n_5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} l_0 & l_1 & l_5 \\ m_0 & m_1 & m_5 \\ n_0 & n_1 & n_5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Demak, berilgan misoldagi masala o’z-o’ziga qo’shma.

Endi $\lambda > 0$ bo’lishini ko’rsatamiz. Buning uchun tenglamani ikkala tarafini $Y(y)$ ga ko’paytirib, (0,1) oraliqda integrallaymiz

$$\int_0^1 Y(y) Y^{(6)}(y)\,dy = -\lambda \int_0^1 Y^2(y)\,dy,$$

bundan

$$Y^{V}Y\Big|_0^1 - Y^{IV}Y'\Big|_0^1 + Y'''Y''\Big|_0^1 - \int_0^1 (Y'''(y))^2\,dy = -\lambda\int_0^1 Y^2(y)\,dy.$$

Integrallar ichidagi funksiyalar kvadrat shaklida ekanligidan, $\lambda > 0$ bo'lishi uchun

$$Y^{V}(1)Y(1) - Y^{V}(0)Y(0) - Y^{IV}(1)Y'(1) + Y^{IV}(0)Y'(0) + Y'''(1)Y''(1) - Y'''(0)Y''(0) \le 0$$

yetarli.

Berilgan masalaning chegaraviy shartlariga ko'ra,

$$\begin{cases} Y(0) = Y^{V}(0) \\ Y'(0) = -Y^{IV}(0) \\ Y''(0) = Y'''(0) \\ Y(1) = -Y^{V}(1) \\ Y'(1) = -Y^{IV}(1) \\ Y''(1) = -Y'''(1) \end{cases}$$

ekanligidan,

$$Y^{V}(1)Y(1) - Y^{V}(0)Y(0) - Y^{IV}(1)Y'(1) + Y^{IV}(0)Y'(0) + Y'''(1)Y''(1) - Y'''(0)Y''(0) =$$

$$-Y^2(1) - Y^2(0) - (Y'(1))^2 - (Y'(0))^2 - (Y''(1))^2 - (Y''(0))^2 \le 0$$

kelib chiqadi. Bundan $\lambda \ge 0$. Endi $\lambda > 0$ ekanligini ko'rsatamiz. Faraz qilaylik, $\lambda = 0$ bo'lsin. Bundan

$$\begin{cases} Y(1) = 0, Y'(1) = 0, Y''(1) = 0 \\ Y(0) = 0, Y'(0) = 0, Y''(0) = 0 \end{cases}$$

shartlar hosil bo'ldi, bu shartlardan

$$Y^{V}Y\Big|_0^1 - Y^{IV}Y'\Big|_0^1 + Y'''Y''\Big|_0^1 - \int_0^1 (Y'''(y))^2\,dy = -\int_0^1 (Y'''(y))^2\,dy$$

$$Y'''Y\Big|_0^1 - Y''Y'\Big|_0^1 + \int_0^1 (Y''(y))^2\,dy = \int_0^1 (Y''(y))^2\,dy,$$

$$\int_0^1 (Y'''(y))^2\,dx = 0 \Rightarrow Y(y) = c_1y^2 + c_2y + c_3.$$

$$Y(0) = c_3 = 0, Y'(0) = c_2 = 0, Y(1) = c_1 = 0$$

$Y(y) = 0$ ligi kelib chiqadi, demak $\lambda > 0$.

(4) spektral masala quyidagi integral tenglamaga ekvivalent

$$Y_n(y) = \lambda_n\int_0^1 Y_n(y)G(y,\xi)\,d\xi \quad , \qquad (6)$$

bu yerda $G(y,\xi)=G(\xi,y)$ - Grin funksiyasi. $\lambda \neq 0$ ligidan Grin funksiyasi mavjud. (6) bu simmetrik yadroga ega integral tenglama. Integral tenglamalar nazariyasidan (6) tenglamaning $Y_n(y)$ xos funksiyalari va $\lambda_n, n=1,2,\ldots,$ xos sonlari mavjud. Bessel tengsizligiga ko‘ra

$$\sum_{n=1}^{\infty}\left(\frac{Y_n(y)}{\lambda_n}\right)^2 \leq \int_0^1 G^2(y,\xi)d\xi < \infty. \tag{7}$$

Ixtiyoriy $\varphi(y)$ funksiyani (3) masalaning xos funksiyalari orqali qatorga yoyish shartlarini topamiz.

1-Teorema. Agar $\varphi(y)\in A(L); L(\varphi)\in L_2[0,1]$ shartlar bajarilsa, u holda $\varphi(y)$ funksiyani (4) masalaning $Y_n(y)$ xos funksiyalari orqali tekis va absolyut yaqinlashuvchi Furye qatoriga yoyish mumkin.

Isbot. $l(\varphi(y))=g(y)$ bo‘lsin, bundan

$$\varphi(y)=\int_0^1 G(y,\xi)g(\xi)d\xi.$$

Hilbert-Shimdt teoremasidan foydalanib, quyidagiga kelamiz:

$$\varphi(y)=\sum_{n=1}^{\infty}c_n Y_n(y) \tag{8}$$

bu yerda:

$$c_n=\int_0^1 \varphi(y)Y_n(y)dy.$$

1-teorema isbotlandi.

Endi (5) masalani tadqiq etishni boshlaymiz. Masalani yechimini

$$X_n(x)=\sum_{k=1}^{\infty}c_k x^{ak+b}, a,b=?, c_k=? \tag{9}$$

qator ko‘rinishida izlaymiz. Bu qatorni (5) tenglamaga olib boramiz.

$$D_{0x}^{\alpha}X_n(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{\sum_{k=1}^{\infty}c_k\tau^{ak+b}}{(x-\tau)^{\alpha}}d\tau=\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{\infty}c_k\frac{d}{dx}\int_0^x\frac{\tau^{ak+b}}{(x-\tau)^{\alpha}}d\tau.$$

Ifodada $\tau = xt, d\tau = xdt$ almashtirish bajaramiz

$$D_{0x}^{\alpha} X_n(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{\infty} c_k \frac{d}{dx} \int_0^1 \frac{(xt)^{ak+b}}{(x-xt)^{\alpha}} xdt = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{\infty} c_k \frac{d}{dx} (x^{ak+b-\alpha+1}) \int_0^1 \frac{t^{ak+b}}{(1-t)^{\alpha}} dt =$$

$$= \sum_{k=1}^{\infty} c_k \frac{(ak+b-\alpha+1)}{\Gamma(1-\alpha)} x^{ak+b-\alpha} \frac{\Gamma(ak+b+1)\Gamma(1-\alpha)}{\Gamma(ak+b-\alpha+2)}.$$

Bundan

$$D_{0x}^{\alpha} X_n(x) = \sum_{k=1}^{\infty} c_k \frac{(ak+b+1-\alpha) x^{ak+b-\alpha} \Gamma(ak+b+1)}{\Gamma(ak+b-\alpha+2)}$$

kelib chiqadi. Tenglikdan noma’lum koeffitsiyentlarni topamiz.
Ushbu tenglamani yechishda Kilbas-Saigo funksiyasidan foydalanamiz

$$E_{\alpha,m,l}(z) = \sum_{n=0}^{\infty} c_n z^n,$$

bu yerda

$$c_k = \prod_{i=0}^{n-1} \frac{\Gamma(\alpha(im+l)+1)}{\Gamma(\alpha(im+l)+1)}.$$

Ushbu

$$\sum_{k=0}^{\infty} \frac{c_k (ak+b+1-\alpha) x^{ak-(\alpha+\beta)} \Gamma(ak+b+1)}{\Gamma(ak+b-\alpha+2)} = -\lambda_n \sum_{k=0}^{\infty} c_k x^{ak}$$

ifodadan koeffitsiyentlarni tenglash usulidan $a = \alpha + \beta$, $b = \alpha - 1$ ni aniqlaymiz. O‘rniga olib borib qo‘yamiz:

$$\sum_{k=0}^{\infty} \frac{c_k akx^{a(k-1)} \Gamma(ak+\alpha)}{\Gamma(ak+1)} = -\lambda_n \sum_{k=0}^{\infty} c_k x^{ak}.$$

c_k koeffitsiyentni izlashni boshlaymiz.

$$\frac{c_k akx^{a(k-1)} \Gamma(ak+\alpha)}{\Gamma(ak+1)} = -\lambda_n c_{k-1},$$

$$c_k = (-\lambda_n) \frac{\Gamma(ak+1)}{ak\Gamma(ak+\alpha)} c_{k-1} = \ldots = (-\lambda_n)^k c_0 \prod_{i=1}^{k} \frac{\Gamma(ai)}{\Gamma(ai+\alpha)}.$$

Xususan, tenglikda $c_0 = 1$ deb olib, quyidagilarni hosil qilamiz:

$$(-\lambda_n)^k c_0 \prod_{i=1}^{k} \frac{\Gamma(ai)}{\Gamma(ai+\alpha)} = (-\lambda_n)^k \prod_{i=0}^{k} \frac{\Gamma((\alpha+\beta)(i+1))}{\Gamma((\alpha+\beta)(i+1)+\alpha)} =$$

$$(-\lambda_n)^k\prod_{i=0}^{k-1}\frac{\Gamma(\alpha(i+1)+\beta(i+1)-1+1)}{\Gamma(\alpha(i+1)+\beta(i+1)+\alpha-1+1)}=(-\lambda_n)^k\prod_{i=0}^{k-1}\frac{\Gamma\left(\alpha(i+1+\frac{\beta}{\alpha}(i+1)-\frac{1}{\alpha})+1\right)}{\Gamma\left(\alpha(i+1+\frac{\beta}{\alpha}(i+1)+1-\frac{1}{\alpha})+1\right)},$$

$$(-\lambda_n)^k\prod_{i=0}^{k-1}\frac{\Gamma\left(\alpha((1+\frac{\beta}{\alpha})i+1+\frac{\beta-1}{\alpha})+1\right)}{\Gamma\left(\alpha((1+\frac{\beta}{\alpha})i+1)+1+\frac{\beta-1}{\alpha}+1)+1\right)}=(-\lambda_n)^k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}.$$

Koeffitsiyentlarni (9) ga qo‘yamiz

$$X_n(x)=Cx^{\alpha-1}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\lambda_n x^{\alpha+\beta}) \tag{10}$$

yoki

$$X_n(x)=Cx^{\alpha-1}\sum_{k=0}^{\infty}c_k(-\lambda_n x^{\alpha+\beta})^k. \tag{11}$$

(10) qatordan *C* noma’lumni topish uchun (5) masaladan foydalanamiz

$$D_{0x}^{\alpha-1}x^{(\alpha+\beta)k+\alpha-1}=\frac{1}{\Gamma(1-\alpha)}\int_0^x\frac{\tau^{(\alpha+\beta)k+\alpha-1}}{(x-\tau)^{\alpha}}d\tau.$$

Endi $\tau=xt, d\tau=xdt$ almashtirish bajarib, ifodani integrallaymiz

$$D_{0x}^{\alpha-1}x^{(\alpha+\beta)k+\alpha-1}=\frac{\Gamma((\alpha+\beta)k)}{\Gamma((\alpha+\beta)k+1-\alpha)}x^{(\alpha+\beta)k}.$$

Bundan,

$$D_{0x}^{\alpha}X_n(0)=\varphi_n=\frac{C}{\Gamma(1-\alpha)}\Rightarrow C=\varphi_n\Gamma(1-\alpha) \tag{12}$$

kelib chiqadi. (12) ifodani (11) ga qo‘yamiz:

$$X_n(x)=\varphi_n\Gamma(1-\alpha)x^{\alpha-1}\sum_{k=0}^{\infty}c_k(-\lambda_n x^{\alpha+\beta})^k. \tag{13}$$

(13) ni quyidagicha ham yozishimiz mumkin:

$$X_n(x)=\varphi_n\Gamma(1-\alpha)x^{\alpha-1}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\lambda_n x^{\alpha+\beta}). \tag{14}$$

Yetarlicha katta λ_n lar uchun quyidagi tengsizlik o‘rinli [8]:

$$|X_n(x)|\le M|\varphi_n|x^{\alpha-1}. \tag{15}$$

Endi (3) qatorni **H** masalaning yechimi bo‘lishi uchun quyidagi teorema o‘rinli.

2-Teorema. Agar:

1). $\varphi(y)\in D(\mathrm{L})$;

2). $\varphi^{(6)}(y)\in D(L), \varphi^{(12)}(y)\in L_2(0,1)$

shartlar bajarilsa, **H** masalaning yechimi mavjud.

Isbot: (3) qator (1) tenglamani qanoatlantiradi. Bu qatorni differensiallash mumkinligini ko‘rsataylik. (15) ni hisobga olib, quyidagini ko‘rsatamiz

$$|x^{1-\alpha}u(x,y)|\leq M\sum_{n=1}^{\infty}|\varphi_n||Y_n(y)|.$$

Koshi-Bunyakovskiy tengsizligidan foydalanib,

$$\sum_{n=1}^{\infty}|\varphi_n||Y_n(y)|\leq\sqrt{\sum_{n=1}^{\infty}\left(\frac{Y_n(y)}{\lambda_n}\right)^2}\sqrt{\sum_{n=1}^{\infty}(\lambda_n\varphi_n)^2}$$

ega bo‘lamiz. Oxirgi tengsizlikda birinchi bo‘linmaning yaqinlashuvchiligi (7) dan kelib chiqadi. Ikkinchi ko‘paytmani hisoblaymiz. Bizda

$$\varphi_n=\int_0^1\varphi(y)Y_n(y)dy=-\frac{1}{\lambda_n}\int_0^1\varphi(y)Y_n^{(6)}(y)dy$$

tenglik ma’lum. Bo‘laklab integrallab,

$$\varphi_n=-\frac{1}{\lambda_n}\int_0^1\varphi^{(6)}(y)Y_n(y)dy \tag{16}$$

hosil qilamiz. Shunday qilib,

$$-\lambda_n\varphi_n=\int_0^1\varphi^{(6)}(y)Y_n(y)dy$$

bo‘ladi. Natijada, $-\lambda_n\varphi_n$ ketma-ketlik $\varphi^{(6)}(y)$ funksiyaning Furye koeffitsiyentlari. Bessel tengsizligiga ko‘ra

$$\sum_{n=1}^{\infty}\lambda_n^2|\varphi_n|^2\leq\int_0^1\left(\varphi^{(6)}(y)\right)^2dy<\infty. \tag{17}$$

(7) va (17) dan (3) qatorning yaqinlashuvchiligi kelib chiqadi. Hosilalarga o‘tamiz. (5) tenglamaning berilishiga ko‘ra, bizda quyidagi bor

$$\left|D_{0x}^{\alpha}u(x,y)\right|\leq\sum_{n=1}^{\infty}\left|D_{0x}^{\alpha}X_n(x)\right|\left|Y_n(y)\right|\leq Mx^{\alpha+\beta-1}\sum_{n=1}^{\infty}|\varphi_n||Y_n(y)|\leq$$
$$\leq Mx^{\alpha+\beta-1}\sqrt{\sum_{n=1}^{\infty}\left(\frac{Y_n(y)}{\lambda_n}\right)^2}\sqrt{\sum_{n=1}^{\infty}\left(\lambda_n^2\varphi_n\right)^2}. \tag{18}$$

Yana, (16) ni esda tutib,

$$\varphi_n=\left(\frac{1}{\lambda_n}\right)^2\int_0^1\varphi^{(6)}(y)Y_n^{(6)}(y)dy=\left(\frac{1}{\lambda_n}\right)^2\int_0^1\varphi^{(12)}(y)Y_n(y)dy$$

kelamiz. Endi, Bessel tengsizligini qo‘llaymiz

$$\sum_{n=1}^{\infty}\left(\lambda_n^2\varphi_n\right)^2 \le \int_0^1\left|\varphi^{(12)}(y)\right|^2 dy < \infty. \qquad (19)$$

(18) ifodaning yaqinlashuvchiligi (7) va (19) dan kelib chiqadi. Demak, quyidagi

$$D_{0x}^{\alpha}u(x,y)=\sum_{n=1}^{\infty} D_{0x}^{\alpha}X_n(x)Y_n(y).$$

qator tekis yaqinlashuvchi ekan. Keyingi

$$u^{(6)}(x,y)=\sum_{n=1}^{\infty} X_n(x)Y_n^{(6)}(y)=-\sum_{n=1}^{\infty}\lambda_n X_n(x)Y_n(y)$$

qatorning tekis yaqinlashuvchiligi ham shunday ko'rsatiladi.

2-teorema isbotlandi.

Yechimni yagonaligini ko'rsatamiz. Quyidagi teorema o'rinli.

3-teorema. Agar **H** masalaning yechimi mavjud bo'lsa, u holda bu yechim yagonadir.

Isbot: Boshlang'ich va chegaraviy shartlar bilan berilgan **H** masalaning yechimi $u(x,y)$ funksiya bo'lsin. Uning Furye koeffitsiyentlarini (4) masalaning xos funksiyalari sistemasining nuqtai nazaridan ko'rib chiqamiz

$$u_n(x)=\int_0^1 u(x,y)Y_n(y)dy, \quad n=1,2,\ldots$$

$u_n(x)$ ni quyidagi masalaga

$$\begin{cases} D_{0x}^{\alpha}u_n(x)=-\lambda_n x^{\beta}u_n(x), \\ D_{0x}^{\alpha-1}u_n(0)=\varphi_n=\int_0^1\varphi(y)Y_n(y)dy, n=1,2,.... \end{cases}$$

yechim ekanini ko'rsatish oddiy. Bu masalaning trivial yechimi [9] da keltirilgan, masalan

$$\int_0^1 u(x,y)Y_n(y)dy=0, \quad n=1,2,\ldots$$

$\overline{G}(y,\xi)$ funksiyani uzluksiz simmetrikligi va quyidagi integrallarning

$$\int_0^1\overline{G}^2(y,\xi)d\xi<\infty, \quad \int_0^1\overline{G}^2(y,\xi)dy<\infty, \quad \int_0^1\int_0^1\overline{G}^2(y,\xi)dyd\xi<\infty, \quad \lambda_n>0, \forall n$$

yaqinlashuvchiligi sababli, Merser teoremasidan

$$\overline{G}(y,\xi)=\sum_{n=1}^{\infty}\frac{\overline{Y_n}(y)\overline{Y_n}(\xi)}{\lambda_n}$$

kelib chiqadi. Bizda

$$u(x,y)=\int_0^1 \overline{G}(y,\xi)u^{(6)}(x,\xi)d\xi=\int_0^1\sum_{n=1}^{\infty}\frac{\overline{Y_n}(y)\overline{Y_n}(\xi)}{\lambda_n}u^{(6)}(x,\xi)d\xi=$$

$$=\sum_{n=1}^{\infty}\frac{Y_n(y)}{\lambda_n}\int_0^1 Y_n(\xi)u^{(6)}(x,\xi)d\xi$$

tenglik hosil bo‘ladi. ketma-ketlik tekis yaqinlashuvchi ekan, integrallash va yig‘indini qayta tartibga solish mumkin. Xuddi shunday, oxirgi natija

$$\sum_{n=1}^{\infty}\frac{Y_n(y)}{\lambda_n}\int_0^1 Y_n(\xi)u^{(6)}(x,\xi)d\xi=\sum_{n=1}^{\infty}\frac{Y_n(y)}{\lambda_n}\int_0^1 Y_n^{(6)}(\xi)u(x,\xi)d\xi=$$

$$=\sum_{n=1}^{\infty}Y_n(y)\int_0^1 Y_n(\xi)u(x,\xi)d\xi=0$$

ga o‘zgaradi. Shuning uchun,

$$u(x,y)\equiv 0.$$

3-teorema isbotlandi.

Xulosa. Ushbu maqolada Riman–Liuvill ma’nosidagi kasr tartibli hosila qatnashgan buzilishga ega oltinchi tartibli differensial tenglama uchun aralash masala o‘rganildi. Tenglamaga mos spektral masala tahlil qilinib, operatorning o‘z-o‘ziga qo‘shmaligi isbotlandi hamda xos funksiyalar yordamida yechimni qator ko‘rinishida ifodalash imkoniyati asoslab berildi. Kilbas–Saigo funksiyasidan foydalanilgan holda koeffitsiyentlarning analitik ko‘rinishi aniqlandi.

Qatorlarning tekis va absolyut yaqinlashuvchanligi Hilbert–Shmidt va Bessel turidagi baholar orqali tasdiqlanib, aralash masalaning yechimining mavjudligi va yagonaligi bo‘yicha tegishli teoremalar isbotlandi. Olingan natijalar buziluvchan koeffitsientli yuqori tartibli kasr differensial tenglamalar uchun aralash masalalar nazariyasini rivojlantiradi va ushbu sinfga mansub masalalarni keyingi tahlillarda qo‘llash uchun nazariy asos yaratadi.

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