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APPLICATIONS OF THE THEORY OF INVARIANTS

Annotation.*In this article, the main types of classical matrix groups, their algebraic structure, geometric interpretation, and significance within the theory of symmetries are comprehensively discussed. First, the fundamental groups the special linear group SL(n, R), the unipotent group T(n, R), and the orthogonal group O(n, R) are defined, and their characteristic determinant conditions, the number of free parameters, and structural properties are examined in detail. The interaction of these groups with geometric figures such as the preservation of lengths, volume, general shape, or spectral characteristics is presented through the corresponding invariants.*

Keywords: *special linear group, unitary group, orthogonal group, invariant.*

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Farg‘ona davlat universiteti dotsenti

INVARIANTLAR NAZARIYASINING QO‘LLANISHI

Annotatsiya. *Ushbu maqolada klassik matritsaviy gruppalarning asosiy turlari, уларнинг algebraik tuzilishi, geometrik talqini ва simmetriyalar nazariyasidagi ahamiyati atroflicha bayon etiladi. Avvalo, maxsus chiziqli gruppa SL(n,R), унитар gruppa T(n,R) ва ортогонал gruppa O(n,R) kabi fundamental gruppalarning aniqlanishi, ular uchun xoc bo‘lgan determinant shartlari, erkin parametrlar soni ва tuzilish xususiyatlari batafsil ko‘rib chiqilgan. Ushbu gruppalarning geometrik figuralar bilan o‘zaro ta’siri, masalan, uzunliklar, hajm, shaklning umumiy ko‘rinishi yoki spektral xususiyatlarning saqlanishi invariantlar orqali ko‘rsatilgan.*

Калит сўзлар: *maxsus chiziqli gruppa, унитар gruppa, ортогонал gruppa, invariant.*

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ПРИМЕНЕНИЕ ТЕОРИИ ИНВАРИАНТОВ

Аннотация. *В данной статье всесторонне рассматриваются основные типы классических матричных групп, их алгебраическая структура, геометрическая интерпретация и значение в теории симметрий. Сначала определяются фундаментальные группы специальная линейная группа SL(n, R), унитарная группа T(n, R) и ортогональная группа O(n, R), а также подробно исследуются их характерные условия на определитель, количество свободных параметров и структурные свойства. Взаимодействие этих групп с геометрическими фигурами, такими как сохранение длин, объема, общей формы или спектральных характеристик, представлено через соответствующие инварианты.*

Ключевые слова: *специальная линейная группа, унитарная группа, ортогональная группа, инвариант.*

Introduction

The notion of invariance occupies an important place in science, as it reflects the fundamental properties of objects and phenomena that remain unchanged under various transformations. In the broad sense, invariance means constancy — the preservation of certain characteristics of a system during transformations. This concept is especially well developed in mathematics, where it encompasses such fields as geometry, algebra, topology, group theory, and number theory.

Cayley was the first to formulate the general problem of finding a basis of algebraic invariants, that is, a finite set of invariants $J_1,...,J_N$ such that any other invariant J can be expressed as a polynomial in them $J = P(J_1,...,J_N)$.

Over the following decades, intensive research was carried out aimed at finding invariants of various forms and their combinations under identical linear transformations. Numerous partial results were obtained, but the question posed by Cayley — the existence of a finite basis of invariants in the general case —

remained open. It was David Hilbert who managed to solve it in the years 1890–1893. He proved two fundamental theorems that completely resolved Cayley's problem.

One of the most difficult tasks in this field later became Hilbert's 14th problem, formulated by him in 1900 at the Second International Congress of Mathematicians. The problem asked to prove or disprove the hypothesis about the finiteness of a rational basis of invariants for any subgroup of the general linear group. This problem remained unsolved for more than fifty years, until in 1958 the Japanese mathematician Nagata constructed a counterexample that disproved the hypothesis.

Preliminaries

Let R^n be an n-dimensional vector space. We will represent elements of R^n as n-dimensional column vectors.

We denote by $GL(n,R)$ the group of all invertible linear transformations of the space R^n, and by $O(n,R)$ the subgroup of all orthogonal transformations, i.e.,

$$O(n,R)=\{g\in GL(n,R): g^T g=E\},$$

where g^T is the transpose of the matrix g, and E is the identity matrix.

Let G be an arbitrary subgroup of $GL(n,R)$. Consider the natural action $(g,x)\to gx$ of the group G on the vector space R^n, defined as the usual multiplication of a matrix by a column vector.

A continuous mapping $x(t)$ from $[0,1]$ into R^n is called a *path*. If the coordinates $x_j(t),\ j=\overline{1,n}$ of the path $x(t)$ are infinitely differentiable functions on $[0,1]$, then $x(t)$ is called an R^∞-path. In what follows, we consider only R^∞.

The r-th derivative of a path $x(t)=\{x_j(t)\}_{j=1}^n$ is called the vector $\overset{(r)}{x}(t)=\{\overset{(r)}{x_j}(t)\}_{j=1}^n$, where $\overset{(r)}{x_j}(t)$ is the r-th derivative of the coordinate function $x_j(t)$, $j=\overline{1,n}$.

For each path $x(t)$ consider the $n\times n$ matrix $M(x)(t)$, whose r-th column consists of the coordinates of the vector $\overset{(r-1)}{x}$, $r=\overline{1,n}$.

If for the path $x(t)$ the determinant $\det M(x)(t)$ is nonzero for all $t \in [0,1]$ then the path $x(t)$ is called regular.

The following theorem provides necessary and sufficient conditions for the $O(n,R)$-equivalence of two paths in terms of the matrices $M(x)$.

Theorem. Two paths $x(t)$ and $y(t)$ are $O(n,R)$-equivalent if and only if the following equalities hold:

1. $(M(x))^{-1}(t)\, M'(x)(t) = (M(y))^{-1}(t) M'(y)(t)$ for all $t \in [0,1]$;
2. $M^T(x)(t)\ M(x)(t) = M^T(y)(t)\, M(y)(t)$ for all $t \in [0,1]$.

Proof. If the paths $x(t)$ and $y(t)$ are $O(n,R)$-equivalent, then there exists $g \in O(n,R)$ such that $y(t) = g\, x(t)$. According to the construction of the matrix $M(x)$ we have $M(y) = gM(x)$. Therefore, the validity of equalities (1) and (2) is easily verified:

$$(M(y))^{-1} M'(y) = (g\, M(x))^{-1} (gM(x))' = (M(y))^{-1} g^{-1} gM'(x) = (M(x))^{-1} M'(x);$$

$$M^T(y) M(y) = (g\, M(x))^T\, gM(x) = M^T(x) g^T\, g\, M(x) = M^T(x)\ M(x).$$

Conversely, suppose that for the paths $x(t)$ and $y(t)$ equalities (1) and (2) hold. If $A = A(t)$ is an invertible matrix, then from $AA^{-1} = E$ it follows that

$A' A^{-1} + A(A^{-1})' = 0$ or $(A^{-1})' = -A^{-1} A' A^{-1}$. Using this equality, we rewrite (1) and (2) as follows:

1') $(M(y)M^{-1}(x))' = 0$;

2') $(M(y)M^{-1}(x))^T M(y) M^{-1}(x) = E$.

From (1′) it follows that $M(y)M^{-1}(x) = g \in GL(n,R)$, i.e. $M(y) = gM(x)$ and therefore $y = gx$.

From this and (2′) it follows that $g \in O(n,R)$. Thus, the paths $x(t)$ and $y(t)$ are $O(n,R)$-equivalent.

Classical Groups of Linear Transformations

The variety of problems in the theory of invariants is determined not only by the diversity of the objects under investigation, but also by the diversity of the transformations used (that is, by considering different groups G). Here we will also restrict ourselves to examining four standard cases, focusing on the four most commonly used groups of real linear transformations.

The full linear group $GL(n, R)$, also called the centro-affine group consists of all non-degenerate square matrices G of order n that is, matrices satisfying the condition $|G| \neq 0$. The number of free parameters of the group N equals the number of elements of the matrix G i.e. $N = n^2$. Such a group is called an n^2-parametric group.

The action of a centro-affine transformation on plane geometric figures is illustrated in Fig.1. It does not preserve angles, lengths, or areas, but it does preserve parallelism and the general form of a geometric figure triangles are mapped to triangles, parallelograms to parallelograms, ellipses to ellipses, and so on.

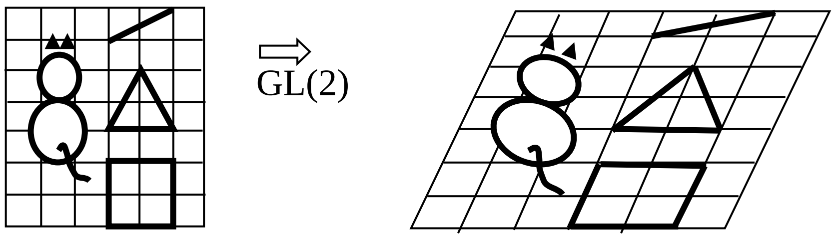

Fig.1

The special linear group SL(n,R), also called the unimodular group differs from the group GL(n,R) by replacing the condition $|G| \neq 0$ with the condition $|G| = 1$. Thus, it consists of matrices with determinant equal to one. Due to this property, the group preserves the areas (or volumes) of geometric figures. The number of free parameters of this group is one less than that of the full linear group, that is $N = n^2 - 1$.

The unipotent group T(n,R) is formed by upper triangular matrices with a unit diagonal. The determinants of such matrices are equal to one, therefore unipotent matrices are also unimodular. The name of the group is related to the fact that the characteristic polynomial of any matrix $G \in T(n)$ has the form $(\lambda-1)^n$ that is, all of its eigenvalues are equal to one. The number of free parameters of this group equals the number of entries above the diagonal of the matrix, i.e. $N = n(n-1)/2$.

Each of the listed groups is a subgroup of the previous one: $T(n,R) \subset SL(n,R) \subset G(n,R)$. An example of the action of the group $T(n,R)$ on various geometric figures is shown in Fig.2.

α=-2

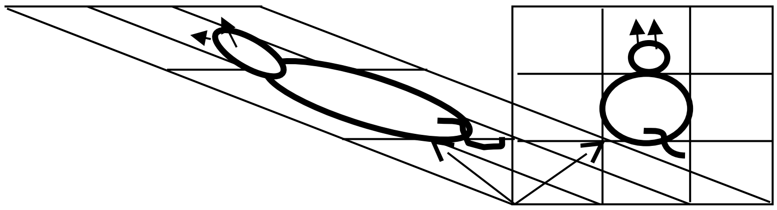

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Fig.2

The orthogonal group $O(n,R)$ is formed by all orthogonal matrices of order *n*, i.e. matrices that satisfy the condition $G^T G = E$. The determinants of such matrices are either +1 or –1. From a geometric point of view, this group performs various rotations of space and mirror reflections, that is, transformations that do not change the lengths of vectors. The number of free parameters of the orthogonal group is the same as that of the unipotent group $N = n(n\text{-}l)/2$. In particular, for the group $O(3,R)$ we obtain $N=3$ which corresponds to the three Euler angles that characterize an arbitrary rotation of a body in three-dimensional space.

In addition to those listed above, the classical groups also include the special orthogonal group, the unitary group, the symplectic group, and several others. One can also consider individual subgroups of the groups mentioned, for example, the group of scaling transformations defined by diagonal matrices; the group of homotheties (similarity transformations) defined by scalar matrices; the symmetric group defined by permutation matrices; and others.

Conclusion

The information presented in the article shows that classical matrix groups play an important role in studying the symmetries of geometric objects. Each of them preserves certain invariants $SL(n,R)$ preserves volume, $O(n,R)$ preserves lengths and T(n,R) preserves spectral properties. The hierarchy among these groups demonstrates the gradual expansion of symmetry levels. Moreover, subgroups such as pseudo-orthogonal, special pseudo-orthogonal and symmetric groups are also widely used and serve many practical and theoretical purposes.

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