VAQT BO‘YICHA BESSEL OPERATORI QATNASHGAN CHEGARAVIY MASALANI TAQRIBIY YECHISH

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Annotatsiya. *Ushbu ishda vaqt bo‘yicha singulyar koeffisient qatnashgan chegaraviy masalani almashtirish operatori yordamida sodda ko‘rinishga keltirib taqribiy hisoblash o‘rganildi. Unda hususiy xosilali differensial tenglamalarni taqribiy yechishda yangicha yondashuv ko‘rsatib o‘tildi. Sonli hisoblash algoritmi batafsil yoritib berildi. Usulning yaqinlashish baxosi olindi. Sonli natijalar grafik ko‘rinishda e‘lon qilindi.*

Kalit so‘zlar. *Bessel operatori, Erdelyi-Kober operatori, hususiy xosilali differensial tenglama, to‘lqin tenglamasi, singulyar koeffisient, almashtirish operatori, taqribiy hisoblash, to‘rlar usuli, absolyut xatolik.*

ПРИБЛИЖЕННОЕ РЕШЕНИЕ КРАЕВОЙ ЗАДАЧИ С ОПЕРАТОРОМ БЕССЕЛЯ ПО ВРЕМЕНИ

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Аннотация. *В данной работе исследуется приближенное решение краевой задачи с сингулярным коэффициентом по времени с использованием оператором преобразования, который упрощает уравнение. Представлен новый подход к приближенному решению дифференциальных уравнений в частных производных. Подробно описан алгоритм численного расчета и получена оценка сходимости метода. Численные результаты представлены в графической формах.*

Ключевые слова: *Оператор Бесселя, оператор Эрдейи-Кобера, дифференциальное уравнение в частных производных, волновое уравнение, сингулярный коэффициент, оператор замены, приближенные вычисления, метод сеток, абсолютная погрешность.*

APPROXIMATE SOLUTION OF A BOUNDARY VALUE PROBLEM WITH A BESSEL OPERATOR IN TIME

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Annotation: *In this work, an approximate solution to the boundary problem with a singular time-dependent coefficient is studied using a transmumation operator that simplifies the equation. A new approach to the approximate solution of partial differential equations is presented. The numerical computation algorithm is described in detail, and an estimate of the method's convergence is provided. Numerical results are presented in graphical forms.*

Key words*: Bessel operator, Erdélyi-Kober operator, partial differential equation, wave equation, singular coefficient, transformation operator, approximate computation, grid method, absolute error.*

KIRISH.

Vaqt bo‘yicha singulyar koeffisientli to‘lqin tarqalish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad -\infty < k < \infty \tag{1}$$

dastlab L. Eylerning ishida [1], keyinchalik S.D. Puasson [2], G. Darbu [3] va B. Riman [4] tomonidan o‘rganilgan. (1) tenglamani dastlab o‘rgangan taniqli olimlar nomiga Eyler-Puasson-Darbu tenglamasi deb ataladi.

Matematika va fizikaning ko‘plab sohalaridagi masalalar (1) tenglamaga keltiriladi, masalan, sirtlar nazariyasiga oid masalalar [5], tovush tarqalishiga oid masalalar [6], tortishish to‘lqinlarining to‘qnashuvi [7] va boshqalar.

A.Vaynshteynning [8]-[12] ishlarida (1) tenglamaning ko‘p o‘lchovli holi uchun $k \in \mathbb{R}$ da birinchi boshlang‘ich shart noldan farqli bo‘lib, ikkinchi boshlang‘ich shart nolga teng bo‘lgan Koshi masalasi (2) ko‘rib chiqiladi.

$$u(x,0) = f(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0. \tag{2}$$

ADABIYOTLAR TAHLILI:

[8]-[13] adabiyotlarda (1)-(2) masalaning klassik ma’nodagi yechimi, [14]-[16] adabiyotlarda esa umumlashtirilgan masalaning yechimlari olingan. [17] da (1) ning ko‘p o‘lchovli holi uchun birinchi va ikkinchi boshlang‘ich shartlar noldan farqli bo‘lganda Tersenov Koshi masalasini yechdi. Eyler-Puasson-Darbu tenglamasi uchun bir jinsli simmetrik fazoda Koshi masalasining yechimi turg‘unligining zaruriy va yetarli shartlar [18] da keltirilgan.

(1) ga qo‘yilgan turli masalalarni analitik usulda yechish yaxshi o‘rganilgan bo‘lishiga qaramay ularni taqribiy yechish kam o‘rganilgan. Agar birinchi va ikkinchi

boshlang‘ich shartlar jadval ko‘rinishidagi funksiyalar orqali berilsa, yuqorida keltirilgan adabiyotlardagi analitik yechimlardan bevosita foydalanish ancha murakkabdir. Gilbert fazosida bir o‘zgaruvchi abstrakt Eyler-Puasson-Darbu tenglamasi uchun Koshi masalasining aniq yechimidan foydalanib taqribiy yechish algoritmi keltirilgan va xatolikni baxolash o‘rganilgan [19]. Boshlang‘ich shartlar singulyarlik chizig‘ida berilgani bois to‘g‘ridan-to‘g‘ri to‘rlar usulini qo‘llash imkonsizdir. Shunga qaramay singulyar koeffisient $\frac{k}{t}$ ni $\frac{k}{t+\varepsilon}, (\varepsilon > 0)$ bilan almashtirib (1) ga yaqin bo‘lgan o‘xshash tenglamani taqribiy yechish mumkin [20].

Ushbu ishda singulyar koeffisientli Eyler-Puasson-Darbu tenglamasi uchun singulyarlik chizig‘ida Koshi sharti bilan berilgan chegaraviy masalani almashtirish operatorlari yordamida taqribiy yechish o‘rganilgan. Bunda Erdelyi-Kober kasr tartibli integral operatorining Bessel operatoriga ta’siri natijasida singulyar koeffisientidan qutulish xossasidan [24]-[25] foydalanamiz.

TADQIQOT METODOLOGIYASI.

$\Omega = \{(x,t) : 0 < x < l, 0 < t < T\}$ sohada quyidagi

$$Lu \equiv u_{tt} + \frac{2\beta}{t} u_t - u_{xx} = 0, \tag{3}$$

tenglamaning

$$u(x,0) = \varphi(x), \ \lim_{t \to 0} t^{2\beta} u_t(x,t) = \psi(x), \tag{4}$$

$$u(0,t) = u(l,t) = 0, \tag{5}$$

shartlarni qanoatlantiruvchi yechimini topish masalasini qaraylik. Bu yerda $0 < \beta < \frac{1}{2}$.

L operator chiqizli ($L(u_1 + u_2) = Lu_1 + Lu_2$) bo’lgani uchun (3)-(5) masalaning yechimini $u(x,t) = u_1(x,t) + u_2(x,t)$ ko’rinishda qidirishimiz mumkin. Bunda $u_1(x,t)$ va $u_2(x,t)$ lar quyidagi masalalarning yechimlari bo’ladi:

$$Lu_1 \equiv \frac{\partial^2 u_1}{\partial t^2} + \frac{2\beta}{t}\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = 0, \tag{6}$$

$$u_1(x,0) = \varphi(x),\ \frac{\partial}{\partial t}u_1(x,t)\Big|_{t=0} = 0, \tag{7}$$

$$u_1(0,t) = u_1(l,t) = 0, \tag{8}$$

va

$$Lu_2 \equiv \frac{\partial^2 u_2}{\partial t^2} + \frac{2\beta}{t}\frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = 0, \tag{9}$$

$$u_2(x,0) = 0,\ t^{2\beta}\frac{\partial}{\partial t}u_2(x,t)\Big|_{t=0} = \psi(x), \tag{10}$$

$$u_2(0,t) = u_2(l,t) = 0, \tag{11}$$

Dastlab, (6) – (8) masalani yechishni ko’rib chiqamiz. (6) tenglamani Bessel operatori yordamida quyidagicha yozishimiz mumkin:

$$B^t_{\beta-1/2}u_1 - \frac{\partial^2 u_1}{\partial x^2} = 0. \tag{12}$$

(12) dagi Bessel operatori quyidagicha aniqlangan:

$$B^x_\eta = x^{2\eta-1}\frac{\partial}{\partial x}x^{2\eta+1}\frac{\partial}{\partial x} = \frac{\partial^2}{\partial x^2} + \frac{2\eta+1}{x}\frac{\partial}{\partial x}.$$

1-teorema. Agar $\alpha > 0, \eta > -1/2,\ f(x) \in C^2(0,b), b > 0$ bo’lib, $x^{2\eta+1}f(x)$ funksiya nol nuqtada integrallanuvchi va $\lim\limits_{x\to 0} x^{2\eta+1}f'(x) = 0$ bo‘lsa, u holda

$$B^x_{\eta+\alpha}I^x_{\eta,\alpha}f(x) = I^x_{\eta,\alpha}B^x_\eta f(x), \tag{13}$$

tenglik o’rinli bo’ladi.

Teoremadagi $I_{\eta,\alpha}f(x)$ - Erdelyi-Kober kasr tartibli integral operatori bo‘lib, u quyidagicha aniqlangan

$$I_{\eta,\alpha}f(x) = \frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^x (x^2 - t^2)^{\alpha-1} t^{2\eta+1}f(t)\,dt,$$

bu yerda $\alpha > 0,\ \eta > -1$. Teoremaning isboti [24] da keltirilgan.

(6) – (8) masalada $u_1(x,t)$ noma’lum funksiyani quyidagi ko‘rinishda qidiramiz:

$$u_1(x,t) \equiv I^t_{-1/2,\beta} v_1(x,t) = \frac{2t^{1-2\beta}}{\Gamma(\beta)} \int_0^t (t^2-\tau^2)^{\beta-1} v_1(x,\tau)\,d\tau . \tag{14}$$

(14) almashtirishni (12) tenglamaga qo’yaylik:

$$B^t_{\beta-1/2} I^t_{-1/2,\beta} v_1(x,t) - \frac{\partial^2}{\partial x^2}\left(I^t_{-1/2,\beta} v_1(x,t)\right) = 0 ,$$

(7) dagi ikkinchi shartlarga ko‘ra 1-teorema shartlari bajarilishi ma’lum. (14) tenglikni qo‘llab quyidagiga ega bo’lamiz:

$$I^t_{-1/2,\beta} B^t_{-1/2} v_1(x,t) - I^t_{-1/2,\beta} v_{1xx}(x,t) = 0 ,$$

kasr tartibli integrallarni birlashtiramiz

$$I^t_{-1/2,\beta}\left(B^t_{-1/2} v_1(x,t) - v_{1xx}(x,t)\right) = 0 .$$

Tenglikni ikkala tomoniga $\left(I^t_{-1/2,\beta}\right)^{-1}$ teskari opeatorni ta’sir ettirib quyidagiga ega bo‘lamiz:

$$v_{1tt} - v_{1xx} = 0 .$$

(14) almashtirishni (7) dagi birinchi shartga qo’yamiz:

$$u_1(x,0) = I^t_{-1/2,\beta} v_1(x,t)\Big|_{t=0} = \varphi(x).$$

Tenglikni har ikki tomoniga $\left(I^t_{-1/2,\beta}\right)^{-1}$ operatorni ta’sir ettiramiz va quyidagiga ega bo‘lamiz

$$v_1(x,0) = \frac{\Gamma(\beta+1/2)}{\sqrt{\pi}} \varphi(x) .$$

(14) almashtirishni (7) dagi ikkinchi shartga qo’yamiz va quyidagi shartga ega bo‘lamiz

$$v_{1t}(x,0) = 0.$$

Natijada (6) – (8) masala quyidagi ko’rinishga o’tdi:

$$v_{1tt} - v_{1xx} = 0,\ 0 < x < l,\ 0 < t \le T, \tag{15}$$

$$v_1(x,0) = \tilde{\varphi}(x),\ \ v_{1t}(x,0) = 0,\ 0 \le x \le l, \tag{16}$$

$$v_1(0,t) = v_1(l,t) = 0, \quad 0 \le t \le T, \tag{17}$$

bu yerda $\tilde{\varphi}(x) = \dfrac{\Gamma(\beta + 1/2)}{\sqrt{\pi}} \varphi(x)$.

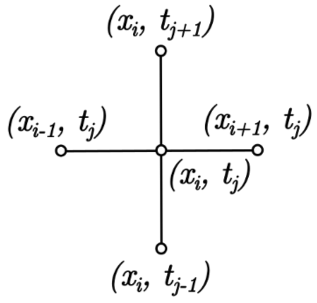
Ma'lumki, (15) - (17) masala korrekt qo'yilgan, uning yechimi mavjud va yagona bo'lib boshlang'ich va chgaraviy shartlarga uzviy bog'liqdir (turg'un) [26].

(15) – (17) masalani sonli yechishni ko'rib chiqamiz.

Ikki o'lchovli $\omega_{h\tau}$ to'rdan foydalanamiz. Bu yerda $\omega_{h\tau} = \omega_h \times \omega_\tau$ bo'lib,

$$\omega_h = \{x_i = ih, i = 0,1,\dots,n, hn = l\},$$
$$\omega_\tau = \{t_j = j\tau, j = 0,1,\dots,k, k\tau = T\}.$$

(15) tenglamani approksimasiya qilish uchun besh nuqtali shablon (andoza) ishlatiladi (1-rasm). Bunda uchta vaqtinchalik qatlamlardan ($j-1, j, j+1$) foydalanamiz. Bunday sxemalar uch qatlamli deb ataladi [23]. Ushbu sxemadan foydalanishda yuqori $j+1$ qatlamni hisoblash uchun j va $j-1$ qatlam qiymatlaridan foydalaniladi.



1-Rasm

(15) tenglama va (17) chegaraviy shartlar uchun oddiy ayirmali sxema quyidagicha tuziladi:

$$\frac{y_1^{i,j+1} - 2y_1^{i,j} + y_1^{i,j-1}}{\tau^2} - \frac{y_1^{i+1,j} - 2y_1^{i,j} + y_1^{i-1,j}}{h^2} = 0,$$
$$i = 1,2,\dots,n-1, \; j = 1,2,\dots,k-1, \tag{18}$$

$$v_{0,j+1} = 0, \; v_{n,j+1} = 0, \; j = 0,1,\dots,k-1.$$

(18) ayirmali sxema tenglamasi h va τ bo'yicha ikkinchi tartibli $O(h^2 + \tau^2)$ xatolikka ega [23]. $j+1$ qatlam qiymatlari oldingi qatlamlar qiymatlari orqali aniq ifodalanadi.

$$y_1^{i,j+1} = 2y_1^{i,j} - y_1^{i,j-1} + \frac{\tau^2}{h^2}\left(y_1^{i+1,j} - 2y_1^{i,j} + y_1^{i-1,j}\right),$$
$$i = 1,2,\dots,n-1, \quad j = 1,2,\dots,k-1. \tag{19}$$

(19) formula bo‘yicha hisoblashni boshlashimiz uchun avvalo bizga $y_1^{i,0}, y_1^{i,1}, i = 0,1,\dots,n-1,n$ qiymatlar kerak bo’ladi. (16) dagi birinchi boshlang’ich shartdan bevosita quyidagiga egamiz:

$$y_1^{i,0} = \varphi(x_i),\ i = 1,2,\dots,n-1.$$

(16) dagi ikkinchi shartdan esa quyidagiga egamiz:

$$\left.\frac{\partial v}{\partial t}\right|_{t=0} = \frac{y_1^{i,1} - y_1^{i,0}}{\tau} + O(\tau) = 0.$$

ushbu munosabatdan to‘rning vaqt bo‘yicha birinchi qatlamidagi qiymatlarini topishimiz mumkin:

$$y_1^{i,1} = y_1^{i,0} + O(\tau),\ i = 0,1,\dots,n \tag{20}$$

(20) formula τ bo’yicha birinchi tartibli yaqinlashishga ega. (19) formula esa ikkinchi tartibli yaqinlashishga ega. Masala shartlarining aproksimatsiyasi ham tenglama aproksimasiyasi bilan bir hil darajada yaqinlashish tartibiga ega bo’lgani ma’qul. Buning uchun quyidagi yoyilmadan foydalanamiz:

$$\frac{v_1(x,\tau) - v_1(x,0)}{\tau} = \frac{\partial v_1(x,0)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 v_1(x,0)}{\partial t^2} + O(\tau^2)$$

(15) tenglamaga ko‘ra quyidagiga egamiz:

$$\frac{\partial^2 v_1(x,0)}{\partial t^2} = \frac{\partial^2 v_1(x,0)}{\partial x^2} = \varphi''(x).$$

U holda

$$\frac{\partial v_1(x,0)}{\partial t} \approx \frac{v_1(x,\tau) - v_1(x,0)}{\tau} - \frac{\tau}{2}\varphi''(x)$$

bo’lib, $v_{1t}(x,0) = 0$ ekanligi hisobga olib, ayirmali tenglama shaklida yozsamiz:

$$\frac{y_1^{i,1} - y_1^{i,0}}{\tau} = \frac{\tau}{2}\varphi''(x_i),$$

bundan $y_1^{i,1}$ ni topamiz:

$$y_1^{i,1} = y_1^{i,0} + \frac{\tau^2}{2}\varphi''(x_i).$$

Yuqoridagi ifodani hisoblash uchun $\varphi''(x)$ hosilaning qiymati w_h to‘rda berilmagan bo’lsa, u holda

$$\varphi''(x_i) = \frac{\varphi(x_{i+1}) - 2\varphi(x_i) + \varphi(x_{i-1})}{h^2} + O(h^2)$$

formuladan foydalanamiz va quyidagiga ega bo’lamiz

$$\begin{aligned} y_1^{i,1} &= y_1^{i,0} + \frac{\tau^2}{2}\left(\frac{\varphi(x_{i+1}) - 2\varphi(x_i) + \varphi(x_{i-1})}{h^2}\right) \\ &= \varphi(x_i) + \frac{\tau^2}{2}\left(\frac{\varphi(x_{i+1}) - 2\varphi(x_i) + \varphi(x_{i-1})}{h^2}\right) \quad i = 1,\dots,n-1. \end{aligned}$$

$y_1^{i,0}$ va $y_1^{i,1}$ qiymatlardan foydalanib (19) rekkurent formula bo’yicha $j = 2,3,\dots,k$ qatlamlar uchun $y_1^{i,j}$ qiymatlar hisoblanadi.

Umuman olganda quyidagi uch qatlamli oshkor ayirmali sxemaga keldik:

$$\begin{gathered} y_1^{i,j+1} = 2y_1^{i,j} - y_1^{i,j-1} + \frac{\tau^2}{h^2}\left(y_1^{i+1,j} - 2y_1^{i,j} + y_1^{i-1,j}\right), \\ i = 1,2,\dots,n-1, \quad j = 1,2,\dots,k-1. \end{gathered} \tag{21}$$

$$y_{i,0} = \varphi(x_i),\ i = 1,2,\dots,n-1,$$

$$y_1^{i,1} = \varphi(x_i) + \frac{\tau^2}{2}\varphi''(x_i), \quad i = 1,\dots,n-1. \tag{22}$$

$$y_1^{0,j+1} = 0, \quad y_1^{n,j+1} = 0, \ j = 0,1,\dots,k-1. \tag{23}$$

Agar $\tau \le h$ shart bajarilsa yuqoridagi ayirmali sxemaning approksimasiya xatoligi $O(\tau^2 + h^2)$ bilan baxolanadi [23]. Bu esa quyidagini anglatadi, agar $v_1(x,t)$ (15)-(17) masalaning yechimi bo’lib, $y_1^{i,j}$ (21) - (23) ayirmali sxemaning yechimi bo‘lsa, u holda quyidagi o‘rinli bo‘ladi:

$$\|y_1(x,t) - v_1(x,t)\| = O(\tau^2 + h^2),$$

yoki

$$\|y_1(x,t) - v_1(x,t)\| \le M_1(\tau^2 + h^2), \ M_1 \ge 0.$$

normaning $\|x\| - \|y\| \le \|x \pm y\|$ xossasiga ko‘ra quyidagiga egamiz

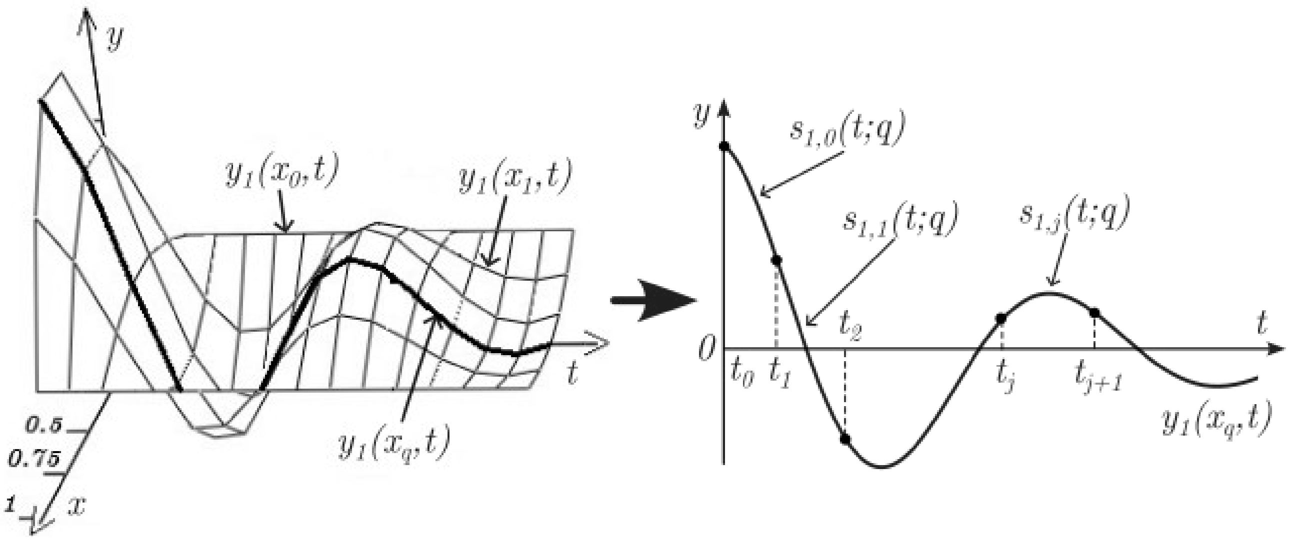
$$\|y_1(x,t)\| \le \|v_1(x,t)\| + M_1(t^2 + h^2),\quad M_1 \ge 0.$$

(15) – (17) masalaning w_{ht} to’r tugun nuqtalarida aniqlangan y_1 yechimidan foydalanib (6) – (8) masalaning taqribiy yechimi $\tilde{u}_1(x,t)$ ni qurishni ko‘rib chiqaylik.

w_{ht} to‘rda aniqlangan y_1 funksiyaning $x = x_q,\quad q = \overline{1, n-1}$ tayinlagan nuqtadagi $y_1(x_q,t)$ qiymatlarini t argumentli biror funksiyalarning qiymatlari deb qarashimiz mumkin. Demak biz

$$u_1(x_q,t) = I^t_{-1/2,b} v_1(x_q,t) \approx I^t_{-1/2,b} y_1(x_q,t), \tag{24}$$

larni hisoblashimiz kerak.



2-rasm

Ma‘lumki, har bir $y_1(x_q,t)$ lar $w_t = \{t_j = jt,\ j = 0,1,\ldots,k,\ kt = T\}$ to‘rda jadval shaklda berilgan funksiyalardir. Ushbu funksiyalarni mos ravishda $(t_0, y_1(x_q,t_0)), (t_1, y_1(x_q,t_1)), \ldots, (t_k, y_1(x_q,t_k))$ nuqtalar to‘plamlarida o‘zaro bog‘langan bo‘lakli $s_{1,j}(t;q)$ ko‘phadlar to‘plamidan iborat bo‘lgan interpolyatsion $s_1(t;q)$ splaynlar bilan almashtirib (bu yerda q parametr) quyidagicha belgilaymiz

$$y_1(x_q,t) \cong s_1(t;q) = \begin{cases} s_{10}(t;q), & agar\ t \in [t_0, t_1] \\ s_{11}(t;q), & agar\ t \in [t_1, t_2] \\ \ldots \\ s_{1,k-1}(t;q), & agar\ t \in [t_{k-1}, t_k] \end{cases} \qquad (25)$$

bu yerda $s_{1j}(t;q) = \sum_{m=0}^{3} C_{q,j,m}(t-t_j)^{3-m}$, $t \in [t_j, t_{j+1}]$, $j = 0,1,\ldots,k-1$ har bir kichik intervelda 3 – darajali ko‘phadlardir (2-rasm). $C_{q,j,m}$ esa i - spaynda $m = 0,1,2,3$ lar uchun j - kichik $[t_j, t_{j+1}]$ intervaldagi $s_{1j}(t;q)$ ko‘phadning no’malum koeffisientlaridir. Har bir splayndagi $4n$ tadan $C_{q,j,m}$ koeffisientlar qiymatlarini aniqlash uchun $4n$ ta tenglamadan iborat quyidagi sistemani yechishimiz kerak

$$\begin{cases} s_{1j}(t_j;q) = y_1(x_q,t_j), & j = 0,\ldots,k-1, \\ s_{1j}(t_{j+1};q) = y_1(x_q,t_{j+1}), & j = 0,\ldots,k-1, \\ s'_{1j}(t_{j+1};q) = s'_{1j+1}(t_{j+1};q), & j = 0,\ldots,k-2, \\ s''_{1j}(t_{j+1};q) = s''_{1j+1}(t_{j+1};q), & j = 0,\ldots,k-2, \\ s''_{10}(t_0;q) = y''_1(x_q,t_0), & s''_{1k-1}(t_k;q) = y''_1(x_q,t_k). \end{cases} \qquad (26)$$

Agar $y_1(x_q,t)$ funksiya hosilasining $t = t_0$ va $t = t_k$ nuqtalardagi qiymatlari ma’lum bo‘lmasa, quyidagi shartlardan foydalangan holda “Tabiiy splayn” deb nomlanuvchi splayn qurish mumkin:

$$\begin{cases} s_{1j}(t_j;q) = y_1(x_q,t_j), & j = 0,\ldots,k-1, \\ s_{1j}(t_{j+1};q) = y_1(x_q,t_{j+1}), & j = 0,\ldots,k-1, \\ s'_{1j}(t_{j+1};q) = s'_{1j+1}(t_{j+1};q), & j = 0,\ldots,k-2, \\ s''_{1j}(t_{j+1};q) = s''_{1j+1}(t_{j+1};q), & j = 0,\ldots,k-2, \\ s''_{10}(t_0;q) = 0, & s''_{1k-1}(t_k;q) = 0. \end{cases}$$

Yuqoridagi (26) tenglamalar sistemasini yechib, no’malum $C_{q,j,m}(q = 1,\ldots,n-1,\ j = 0,\ldots,k-1,\ m = 0,\ldots 3)$ koeffisientlarning qiymatlarini aniqlaymiz. $s_{1j}(t;q)$ $j = 0,\ldots,k-1$ ko‘phadlarni standart ko‘rinishda $s_{1j}(t;q) = A_{q,j,0}t^3 + A_{q,j,1}t^2 + A_{q,j,2}t + A_{q,j,3}$ ifodalash uchun $A_{q,j,m}$ $j = 0,1,\ldots,k-1$ koeffisientlar quyidagi formulalar yordamida hisoblanadi

$$
\begin{aligned}
A_{q,j,0} &= C_{q,j,0},\\
A_{q,j,1} &= C_{q,j,1} - 3C_{q,j,0}t_j,\\
A_{q,j,2} &= 3C_{q,j,0}t_j^2 - 2C_{q,j,1}t_j + C_{q,j,2},\\
A_{q,j,3} &= -C_{q,j,0}t_j^3 + C_{q,j,1}t_j^2 - C_{q,j,2}t_j + C_{q,j,3},\ j = 0,1,\ldots,k-1.
\end{aligned}
$$

Shuni aytib o‘tish o‘tish kerakki, hozirda mavjud zamonaviy matematik tizimlarning ko‘pida kubik splaynni hisoblash uchun tayyor metodlar ishlab chiqilgan. Ushbu metodlarga to‘rning tugun nuqtalarida funksiya qiymatlarini berish orqali kubik splaynni qurish, uning bo‘lak ko‘phadlarining koeffisientlarini olish va ulardan turli hisob-kitoblar uchun foydalanish mumkin.

Har bir $y_1(x_q,t)$ jadval ko‘rinishidagi funksiyani w_t to‘rda approksimatsiya qiluvchi $s_1(t,q)$ splaynlar qurilgan so‘ng (24) formulada integral osti funksiya o’rniga kubik splaynlar integrallanadi

$$u_1(x_q,t) = I^t_{-1/2,b}v_1(x_q,t) \approx I^t_{-1/2,b}y_1(x_q,t) \approx I^t_{-1/2,b}s_1(t,q). \tag{27}$$

yoki

$$u_1(x_q,t) \approx \tilde{u}_1(x_q,t) = I^t_{-1/2,b}s_1(t,q).$$

Barcha q lar uchun olingan natijalardan (6)-(8) masalaning w_{ht} to‘rdagi taqribiy yechimi $\tilde{u}_1(x,t)$ quriladi:

$$u_1(x,t) \approx \tilde{u}_1(x,t).$$

(27) kasr tartibli Erdelyi-Kober integralini tayinlangan $t = t_r, r = \overline{0,k}$ nuqtadagi qiymatini taqribiy hisoblash uchun [22] adabiyotda keltirilgan quyidagi formuladan foydalanamiz

$$I^t_{-1/2,b}s_1(t,q)\Big|_{t=t_r} = \sum_{j=0}^{r-1}\frac{1}{\Gamma(b)}\sum_{m=0}^{3}A_{q,j,m}t_r^{3-m}\left[B\left(\left(\frac{t_{j+1}}{t_r}\right)^2; g, b\right) - B\left(\left(\frac{t_j}{t_r}\right)^2; g, b\right)\right]$$

bu yerda $g = 2 - j/2$ va $B(a,b,c)$ - to‘lamas Beta funksiya.

Sonli usulning yaqinlashishini tekshirish uchun quyidagi teoremadan foydalanamiz.

2-teorema. $f(t) \in C^4[a,b]$ funksiya uchun $\{x_i;\ i=0,\dots,n,\ x_0=a<x_1<\dots<x_n=b\}$ to‘rda

$$\begin{cases} s_i(x_i)=f(x_i), & i=0,\dots,n-1,\\ s_i(x_{i+1})=f(x_{i+1}), & i=0,\dots,n-1,\\ s'_i(x_{i+1})=s'_{i+1}(x_{i+1}), & i=0,\dots,n-2,\\ s''_i(x_{i+1})=s''_{i+1}(x_{i+1}), & i=0,\dots,n-2,\\ s''_0(x_0)=f''(x_0),\quad s''_{n-1}(x_n)=f''(x_n). \end{cases}$$

shartlarni qanoatlantiruvchi $s(x)$ splayn qurilgan bo‘lsa, u holda quyidagi tengsizlik o‘rinli bo‘ladi

$$\left\| f^{(m)}(x)-s^{(m)}(x)\right\| \le C_n \left\| f^{(4)}(x)\right\| h^{4-m},\quad (m=0,1,2,3)$$

bu yerda

$$h=\max_{i=0,\dots,n-1}\left(x_{i+1}-x_i\right),\ C_0=\frac{5}{384},\ C_1=\frac{9+\sqrt{3}}{216},\ C_2=\frac{3b+1}{12},\ C_3=\frac{b^2+1}{2}.$$

b esa to‘rdagi eng katta qadamning eng kichik qadamga nisbatidir,

$$b=\frac{h}{\min\limits_{i=0,\dots,n-1}\left(x_{i+1}-x_i\right)}$$

. Teorema isboti [21] adabiyotda keltirilgan.

Aytaylik, $\omega_{h\tau}$ to‘rda quyidagi tenglik o‘rinli bo‘lsin

$$y_1(x_i,t_j)=v_1(x_i,t_j)+\varepsilon(x_i,t_j),\quad x_i=ih,\ i=0,\dots,n,\ t_j=j\tau,\ j=0,\dots,k$$

u holda

$$\left\| y_1(x,t)-v_1(x,t)\right\| \le M_1\left(\tau^2+h^2\right)$$

ekanligidan

$$\left\| \varepsilon(x,t)\right\| \le M_1\left(\tau^2+h^2\right)$$

kelib chiqadi.

3-teorema. Agar $u_1(x,t)$ (6)-(8) masalaning yechimi va $\tilde{u}_1(x,t)$ shu masalaning $\omega_{h\tau}$ to‘rdagi taqribiy yechimi hamda $v_1(x,t)$ (15)-(17) masalaning yechimi bo’lsa, u holda quyidagi tengsizlik

$$\left\| u_1(x,t) - \tilde{u}_1(x,t) \right\| \le \frac{5\sqrt{\pi}}{384\Gamma(\beta+1/2)}\left(\left\| \nu_1^{(4)}(x,t) \right\| \tau^4 + M_1\left(\tau^2 + h^2\right) \right)$$

o‘rinli bo‘ladi.

Isbot. Aytaylik, $\psi_1(x_q,t)$ funksiya yetarli darajada silliq va $C^4[0,T]$ sinfga tegishli bo‘lgan funksiyaning ω_τ to‘rdagi ifodasi bo‘lsin. U holda 2-teoremada $m=1$ bo‘lgan hol uchun quyidagi tengsizlik o‘rinli bo‘ladi

$$\left\| \psi_1(x_q,t) - s_1(t;q) \right\| \le \frac{5}{384}\left\| \psi_1^{(4)}(x_q,t) \right\| \tau^4,$$

$$\left\| \nu_1(x_q,t) + \varepsilon(x_q,t) - s_1(t;q) \right\| \le \frac{5}{384}\left\| \left(\nu_1(x_q,t) + \varepsilon(x_q,t) \right)^{(4)} \right\| \tau^4,$$

Normaning $\|x\| - \|y\| \le \|x \pm y\|$ xossasiga ko‘ra

$$\left\| \nu_1(x_q,t) - s_1(t;q) \right\| - \left\| \varepsilon(x_q,t) \right\| \le \frac{5}{384}\left\| \left(\nu_1(x_q,t) + \varepsilon(x_q,t) \right)^{(4)} \right\| \tau^4,$$

$$\left\| \nu_1(x_q,t) - s_1(t;q) \right\| \le \frac{5}{384}\left\| \left(\nu_1(x_q,t) + \varepsilon(x_q,t) \right)^{(4)} \right\| \tau^4 + \left\| \varepsilon(x_q,t) \right\|. \qquad (28)$$

(6) – (8) masalaning yechimi $u_1(x_q,t)$ bilan taqribiy yechim $\tilde{u}_1(x_q,t)$ ning farqini hisoblaylik

$$Err_q = \left\| u_1(x_q,t) - \tilde{u}_1(x_q,t) \right\|_{t=t_r} = \left\| u_1(x_q,t) - I^t_{-\frac{1}{2},\beta} s_1(t;q) \right\|_{t=t_r}$$

$$= \left\| I^t_{-1/2,\beta}\nu_1(x_q,t) - I^t_{-1/2,\beta}s_1(t;q) \right\|_{t=t_r} = \left\| I^t_{-1/2,\beta}\left(\nu_1(x_q,t) - s_1(t;q) \right) \right\|_{t=t_r}$$

$$= \left\| \frac{2t_r^{1-2\beta}}{\Gamma(\beta)} \int_0^{t_r} \left(t_r^2 - t^2 \right)^{\beta-1} \left(\nu_1(x_q,t) - s_1(t;q) \right) dt \right\|$$

$$\le \frac{2t_r^{1-2\beta}}{\Gamma(\beta)} \int_0^{t_r} \left(t_r^2 - t^2 \right)^{\beta-1} \left\| \nu_1(x_q,t) - s_1(t;q) \right\| dt$$

$$= \frac{2t_r^{1-2\beta}}{\Gamma(\beta)} \sum_{j=0}^{r} \int_{t_j}^{t_{j+1}} \left(t_r^2 - t^2 \right)^{\beta-1} \left\| \nu_1(x_q,t) - s_1(t;q) \right\| dt$$

$$\le \frac{2t_r^{1-2b}}{\Gamma(b)} \sum_{j=0}^{r} \left\| v_1(x_q,t) - s_1(\tilde t_j,q)\right\| \int_{t_j}^{t_{j+1}} \left(t_r^2 - t^2\right)^{b-1} dt$$

bu yerda $\tilde t_j \in \left[t_j, t_{j+1}\right]$.

Agar $\forall\tilde t \in [0,t_r]$ uchun $\left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\| = \max\limits_{j=0,1,\dots,r-1} \left\| v_1(x_q,\tilde t_j) - s_1(\tilde t_j,q)\right\|$ o‘rinli deb faraz qilsak, u holda Err_q xatoning keyingi bahosi quyidagi shaklni oladi

$$Err_q \le \frac{2t_r^{1-2b}}{\Gamma(b)} \left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\| \sum_{j=0}^{r-1} \int_{t_j}^{t_{j+1}} \left(t_r^2 - t^2\right)^{b-1} dt$$

$$= \frac{2t_r^{1-2b}}{\Gamma(b)} \left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\| \int_{0}^{t_r} \left(t_r^2 - t^2\right)^{b-1} dt.$$

Integral o‘zgaruvchisini almashtiramiz: $x = \left(\frac{t}{t_r}\right)^2, t = t_r\sqrt{x}, dt = \frac{t_r}{2\sqrt{x}}dx$. Natijada quyidagiga ega bo‘lamiz

$$Err_q \le \frac{2t_r^{1-2b}}{\Gamma(b)} \left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\| \int_{0}^{1} \left(t_r^2 - t_r^2 x\right)^{b-1} \frac{t_r}{2\sqrt{x}} dx$$

$$= \frac{1}{\Gamma(b)} \left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\| B\left(\frac{1}{2}, b\right) = \frac{\sqrt{\pi}}{\Gamma(b+1/2)} \left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\|.$$

(28) tengsizlikka ko‘ra yuqoridagi tengsizlikni yanada kuchaytirib qayta yozamiz:

$$Err_q \le \frac{\sqrt{\pi}}{\Gamma(b+1/2)} \left\| v_1(x_q,\tilde t) - s_1(\tilde t,q)\right\|$$

$$\le \frac{\sqrt{\pi}}{\Gamma(b+1/2)} \left(\frac{5}{384} \left\| \left(v_1(x_q,t) + e(x_q,t)\right)^{(4)} \right\| \tau^4 + \left\| e(x_q,t)\right\| \right)$$

Biz faqatgina w_τ to‘rning tugun nuqtalaridagina $\left\| u_1(x,t) - \tilde u_1(x,t)\right\|$ ni baholayotganimizni e’tiborga olsak, $e(x_q,t)$ funksiyalarni $\left(t_j, y_1(x_q,t_j) - v_1(x_q,t_j)\right)$ $j = 0,1,\dots,k$ nuqtalar to‘plamlarida o‘zaro bog‘langan bo‘lakli $e_p(x_q,t) = a_p t + b_p$ ($p = 0,\dots,k-1$) chiziqlar to‘plamidan iborat interpolyatsion birinchi darajali splaynlar deb olishimiz mumkin. U holda quyidagi tenglik o‘rinli

$$\left\| \frac{\partial^4}{\partial t^4} \varepsilon(x,t) \right\| = \max_{q=1,\dots,n-1} \left\| \frac{\partial^4}{\partial t^4} \varepsilon(x_q,t) \right\| = \max_{\substack{q=1,\dots,n-1 \\ p=0,\dots,k-1}} \left| \frac{\partial^4}{\partial t^4} \varepsilon_p(x_q,t) \right| = 0.$$

Yuqoridagi tenglikni va $\|\varepsilon(x,t)\| \le M_1(\tau^2 + h^2)$ ekanligini e’tiborga olib quyidagiga ega bo‘lamiz

$$Err_q \le \frac{\sqrt{\pi}}{\Gamma(\beta+1/2)} \left(\frac{5}{384} \left\| v_1^{(4)}(x_q,t) \right\| \tau^4 + M_1(\tau^2 + h^2) \right)$$

Barcha $q = \overline{1,n-1}$ lar uchun xatolikni baxolaylik

$$Err = \| u_1(x,t) - \tilde{u}_1(x,t) \| = \max_{q=1,2,\dots,n-1} Err_q$$

$$= \max_{q=1,2,\dots,n-1} \left(\frac{5\sqrt{\pi}}{384\Gamma(\beta+1/2)} \left\| v_1^{(4)}(x_q,t) \right\| \tau^4 + M_1(\tau^2 + h^2) \right)$$

$$\le \frac{5\sqrt{\pi}}{384\Gamma(\beta+1/2)} \left(\left\| v_1^{(4)}(x,t) \right\| \tau^4 + M_1(\tau^2 + h^2) \right).$$

Teorema isbotlandi.

Ushbu teoremadan xulosa qilish mumkinki, $\tau \le h$ tengsizlik bajarilganda $\lim\limits_{h \to 0} Err = 0$ tenglik o‘rinli bo‘ladi.

(9) – (11) masalani yechish uchun [25] adabiyotda keltirilgan quyidagi lemmadan foydalanamiz.

1-lemma. Agar $u_1(x,t;1-\beta)$ funksiya $L_{1-\beta}(u_1(x,t)) = 0$ tenglamani va (7) shartni qanoatlantirsa, u holda $u_2(x,t) = t^{1-2\beta} u_1(x,t;1-\beta)$ ($\beta < 1/2$) funksiya $L_\beta(u_2(x,t)) = 0$ tenglamani va

$$u_2(x,0) = 0; \quad \lim_{t \to +0} t^{2\beta} u_2(x,t) = (1-2\beta)\varphi(x)$$

shartlarni qanoatlantiradi.

1-lemmaga ko‘ra (6)-(8) masaladagi $\varphi(x)$ ni $\dfrac{\psi(x)}{1-2\beta}$ bilan hamda β ni esa $1-\beta$ bilan almashtirib $\tilde{u}_1(x,t)$ taqribiy yechim topilsa, u holda (9)-(11) masalaning yechimi $u_2(x,t) \approx t^{1-2\beta}\tilde{u}_1(x,t)$ ko‘rinishda aniqlanadi.

(6)-(8) va (9)-(11) masalalarning $u_1(x,t)$ va $u_2(x,t)$ yechimlardan foydalanib (3)-(5) masalaning yechimini $u(x,t) = u_1(x,t) + u_2(x,t)$ ko‘rinishda topamiz.

TAXLILLAR VA NATIJALAR

Sonli yechim natijalarini tekshirish qulay bo‘lishi uchun analitik yechimi sodda bo‘lgan masaladan foydalanamiz.

1-masala. $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t < 3\}$ sohada quyidagi

$$u_{tt} + \frac{0.8}{t}u_t - u_{xx} = 0, \tag{29}$$

tenglamaning

$$u(x,0) = \sin \pi x,\quad \lim_{t \to 0}\left(t^{4/5}\frac{\partial}{\partial t}u(x,t)\right) = \frac{1}{5}\sin \pi x, \tag{30}$$

$$u(0,t) = u(1,t) = 0, \tag{31}$$

shartlarni qanoatlantiruvchi taqribiy yechimini almashtirish operatorlari yordamida topish masalasini qaraylik.

(29)-(31) masalaning analitik yechimini Furyening o’zgaruvchilarni ajratish usuli topamiz. Yechim quyidagi ko‘rinishda bo‘ladi:

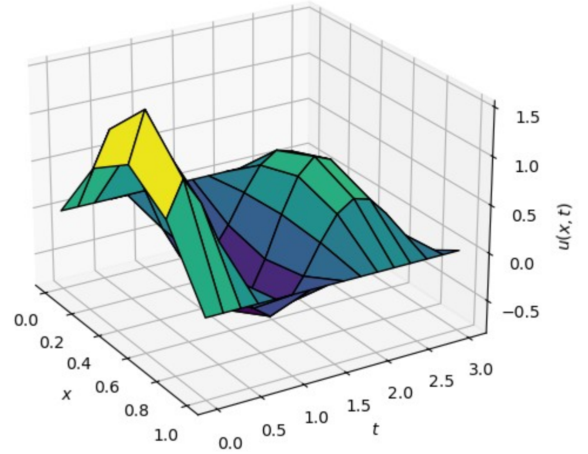
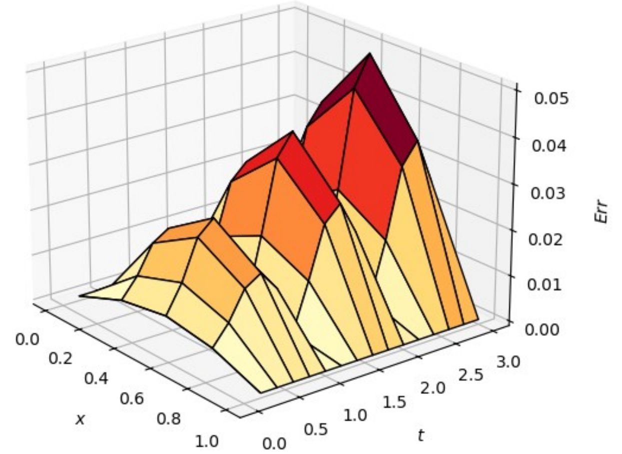
$$u(x,t) = \sin \pi x\left(t^{2/10}\bar{J}_{1/10}(\pi t) + \bar{J}_{-1/10}(\pi t)\right)$$

yoki $\bar{J}_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)$ formulaga ko’ra:

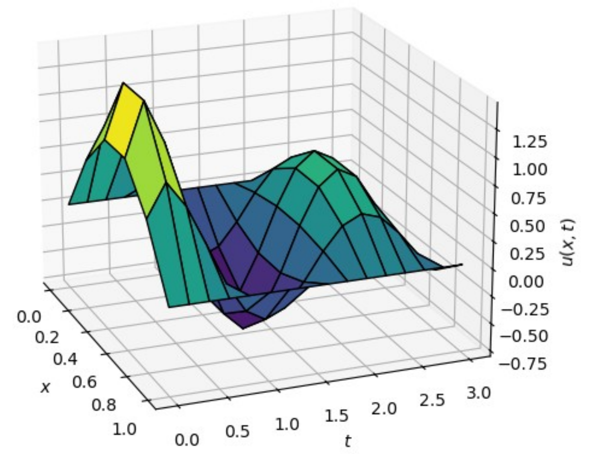
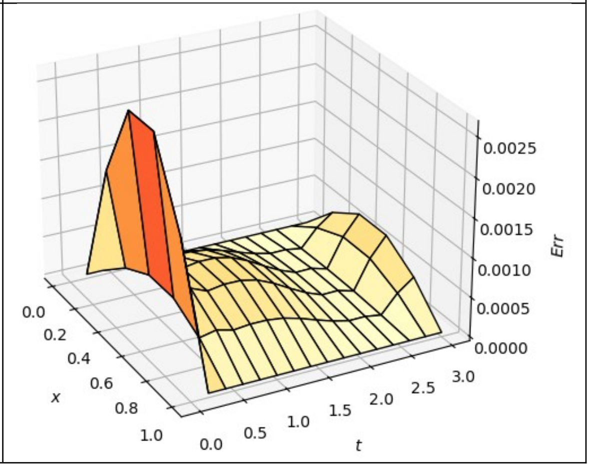
$$u(x,t) = \sin \pi x\begin{pmatrix} t^{1/10}\Gamma(11/10)(\pi/2)^{-1/10}J_{1/10}(\pi t) \\ +\Gamma(9/10)(\pi t/2)^{1/10}J_{-1/10}(\pi t) \end{pmatrix}$$

Ushbu topilgan yechimdan sonli hisoblash natijalarini taqqoslash uchun foydalanamiz. Quyida sonli yechish natijalari grafik shaklda keltirilgan. Barcha hisoblashlar Python dasturlash tilining numpy, scipy va math kutubxonalari yordamida float64 64 bitli suzuvchi nuqta turida amalga oshirildi.

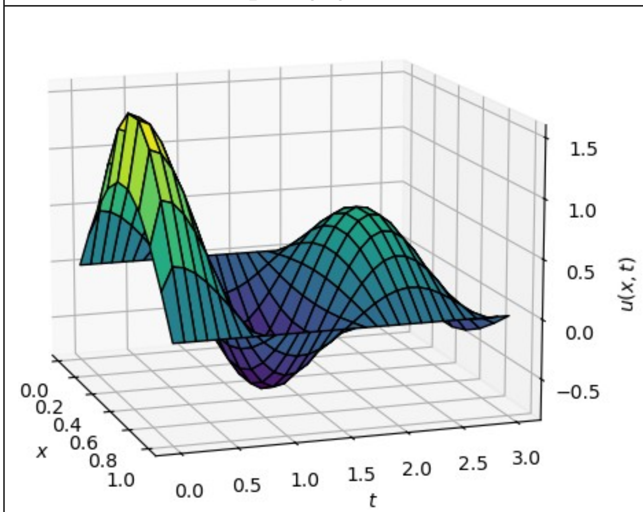
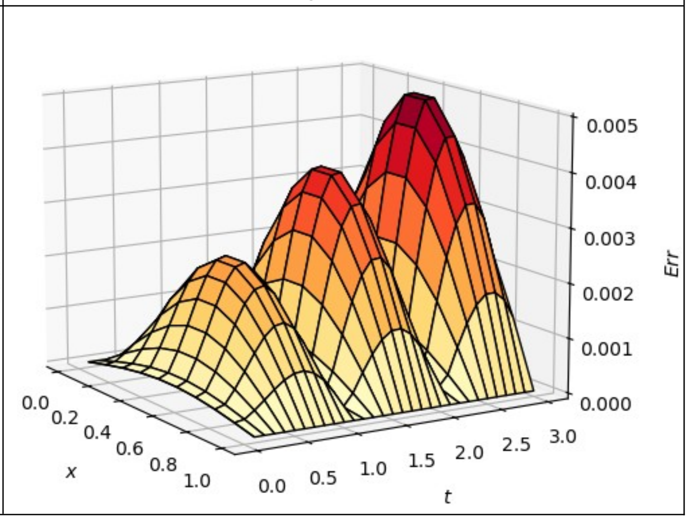
$t = 0,2$ va $h = 0,25$ bo‘lgan hol uchun olingan natija.

Taqribiy yechim	Absolyut xatolik
	

$t = 0,2$ va $h = 0,2$ bo‘lgan hol uchun olingan natija.

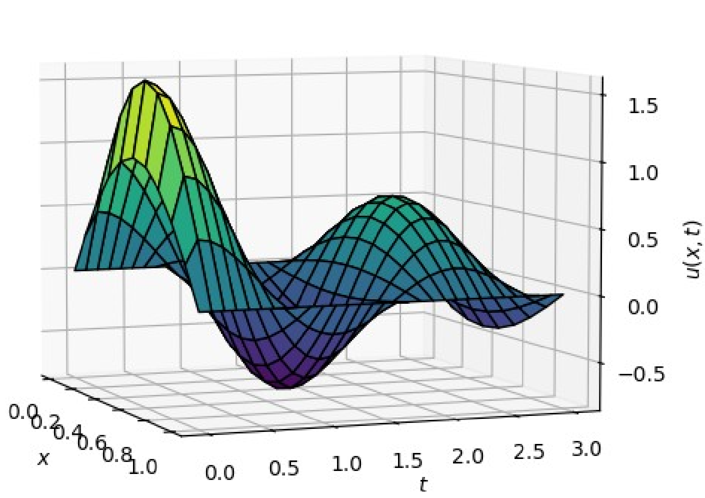
Taqribiy yechim	Absolyut xatolik
	

$t = 0,1$ va $h = 0,(1)$ bo‘lgan hol uchun olingan natija.

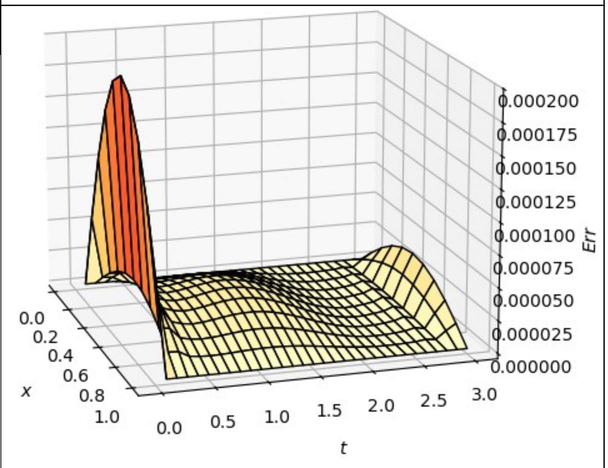
Taqribiy yechim	Absolyut xatolik
	

$t = 0{,}1$ va $h = 0{,}1$ bo‘lgan hol uchun olingan natija.

Taqribiy yechim

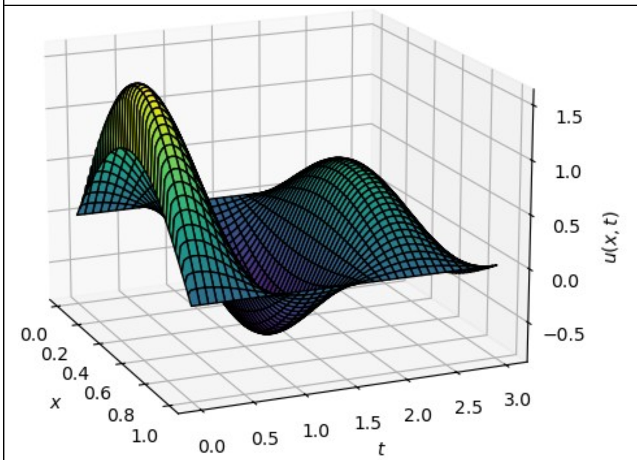


Absolyut xatolik



$t = 0{,}02$ va $h = 0{,}025$ bo‘lgan hol uchun olingan natija.

Taqribiy yechim



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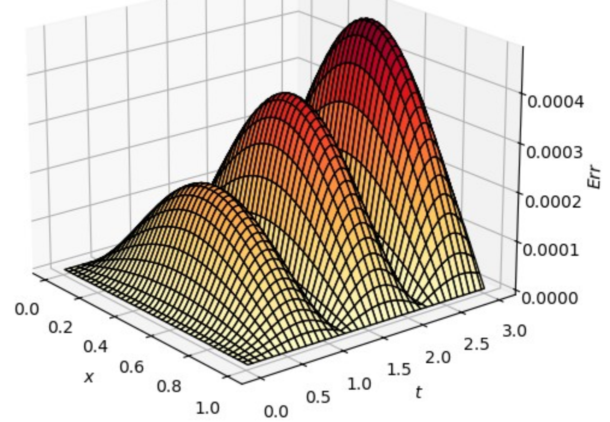
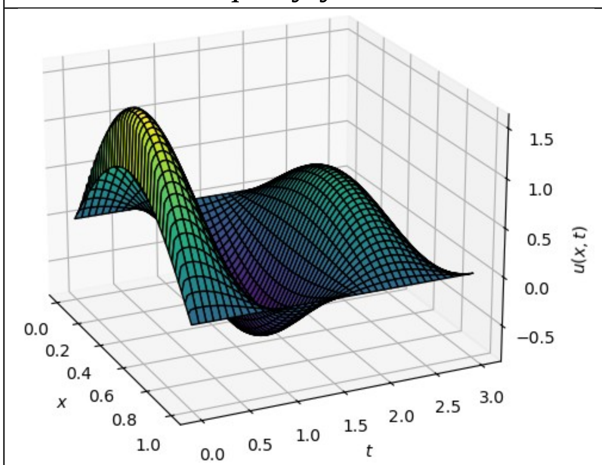


$t = 0{,}025$ va $h = 0{,}025$ bo‘lgan hol uchun olingan natija.

Taqribiy yechim



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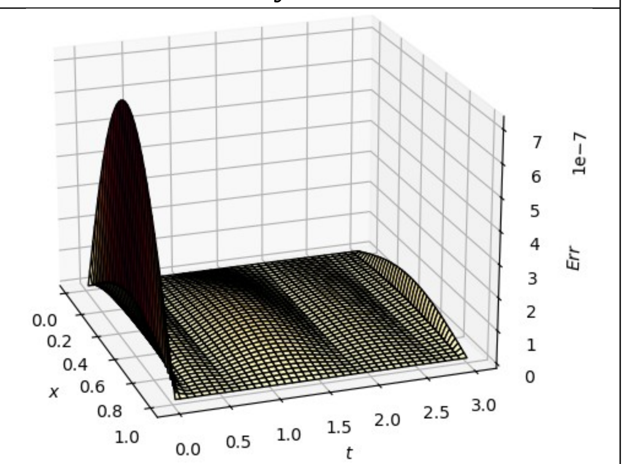

XULOSALAR

Ushbu ishda singulyar koeffisientli Eyler-Puasson-Darbu tenglamasi uchun chegaraviy masalani taqribiy yechish ko‘rib chiqildi. Masaladagi tenglama chiziqli ekanligidan foydalanib uni bittadan boshlang‘ich shartlari bir jinsli bo‘lgan ikkita masalaga keltiramiz. Birinchi masalaning yechimi vaqt bo‘yicha Erdelyi-Kober kasr tartibli integral operatori ta’siridagi funksiya ko‘rinishda qidirildi. Natijada masala singulyar koeffisientsiz yangi yordamchi masalaga keltirildi. Hosil bo‘lgan yordamchi masala to‘rlar usuli bilan taqribiy yechildi. To‘rning har bir ustunida yordamchi masala yechimini aproksimatsiya qiluvchi splaynlar qurildi. Splaynlarga Erdelyi-Kober operatorining ta’siri hisoblanib birinchi masala yechimi hosil qilindi. Ikkinchi masala 1-lemmada keltirilgan xossa asosida xuddi birinchi masala kabi taqribiy yechildi. Ikkala masalaning taqribiy yechimlarini qo‘shib asosiy masalaning taqribiy yechimi hosil qilindi.

Maslaning yechimida xatolikni kamaytirish uchun Erdelyi-Kober operatorini kubik splayn usuli yordamida hisoblandi [22]. Erdelyi-Kober operatorini yana to‘g‘ri to‘rtburchaklar, trapetsiyalar yoki o‘rta qiymat usulidan foydalanib ham hisoblash mumkin [27].

Olingan sonli natijalar shuni ko‘rsatdiki w_{ht} to‘rni $t < h$ shart bajariladigan qilib qurish masalaning taqribiy yechimini aniq yechimga sekin yaqinalashishiga sabab bo‘ladi. Agar to‘r $t = h$ shart bajariladigan qilib qurilsa taqribiy yechim aniq yechimga juda tez yaqinlashar ekan.

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