Kasr tartibli operator qatnashgan aralash turdagi tenglama uchun Bitsadze-Samarskiy tipidagi masala haqida

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Ushbu ishda mualliflar Bitsadze-Samarskiy tipidagi nolokal masalani to'rtburchak sohada Kaputo kasr tartibli differensial operatori qatnashgan ikkinchi tartibli aralash tipdagi tenglama uchun ko'rib chiqishgan. Bu masala o'ng chegaradagi noma'lum funksiyaning qiymatini sohaning belgilangan nuqtasidagi funksiya qiymati bilan bog'langan. Shuningdek, ikkinchi tartibli oddiy differensial tenglama uchun Bitsadze-Samarskiy tipidagi masalaning spektral xossalarini o'rganib chiqilgan va quyidagi natijalarga erishilgan: olingan xos qiymatlar va mos keladigan xos funksiyalar ularning to'laligi va bazis xususiyatini isbotlangan. Bundan tashqari, qo’yilgan masalaga qo'shma masala ham o'rganilgan. Masala yechimining yagonaligi va mavjudligi spektral usul yordamida isbotlangan.

Kalit so‘zlar: Bitsadze-Samarskiy tipidagi masala, kasr tartibli tenglama, xos sonlar, xos va ergashgan funksiyalar, to‘lalik, biortogonallik, Riss bazis.

Об одной задаче типа Бицадзе-Самарского для уравнения смешанного типа с оператором дробного порядка

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В данной работе рассматривается нелокальная задача типа Бицадзе-Самарского для уравнения второго порядка смешанного типа с дробно-дифференциальным оператором Капуто в прямоугольной области. В этой задаче значение неизвестной функции на правой границе связывается со значением функции в неподвижной точке области. Также исследованы спектральные свойства задачи типа Бицадзе-Самарского для обыкновенного дифференциального уравнения второго порядка и получены следующие результаты: получены собственные значения и соответствующие им собственные функции, доказана их полнота и базисность. Кроме того, исследована сопряженная задача к основной задаче. Единственность и существование решения задачи доказаны спектральным методом.

Ключевые слова: задача типа Бицадзе-Самарского, уравнение дробного порядка, собственные числа, собственные и присоединенные функции, полнота, биортогональность, базис Рисса.

On the Bitsadze-Samarsky type problem for a mixed-type equation involving a fractional-order operator.

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Abstract. In this work, the authors consider a Bitsadze-Samarskii-type non-local problem for a second-order mixed-type equation with the Caputo fractional differential operator in a rectangular domain. This problem related the value of the unknown function on the right boundary to the value of the function at the fixed point in the domain. The uniqueness and the existence of a solution to the problem are proved by the spectral method. Also, we researched the spectral properties of the Bitsadze-Samarskii type problem for an ordinary differential equation of the second order, and the following results were obtained: the eigenvalues and the corresponding eigenfunctions obtained proved their completeness and basis property. Moreover, the adjoint problem to the main problem was researched.

Key words: Bitsadze-Samarskii type problem, fractional order equation, eigenvalues, eigen and associated functions, completeness, biorthogonality, Riesz basis.

INTRODUCTION

In recent years, the theory of nonlocal boundary value problems for differential equations has become one of the rapidly evolving sections of the general theory of differential equations. The monograph [1] provides basic information about nonlocal problems. A.A. Samarskii and A.V. Bitsadze [2] researched a new type of nonlocal boundary value problem for an elliptic differential equation.

In [3-7], further research was discussed on similar problems involving operators of fractional or integer order, including Caputo and Riemann-Liouville. Additionally, studies on parabolic systems have been conducted in [8,9]. Non-local problems of the Bitsadze-Samarskii (BS) type for mixed-type equations incorporating fractional differential operators have been analyzed in the indicated works [10-12]. Specifically, in [10] examines a non-local boundary value problem for a fractional-order parabolic equation, while in [11] investigates a similar problem for a mixed fractional-order parabolic-hyperbolic equation. In this context, a non-local Bitsadze-Samarskii type condition is imposed as the boundary condition, making the corresponding differential operator inherently non-local. The completeness of the eigenfunctions and the main properties of such operators are studied. And for mixed fractional-order parabolic-hyperbolic and parabolic equations, the eigenvalues to the nonlocal boundary value problem and the corresponding root functions were found for case of $x_0 = \frac{p}{q}$, $0 < \frac{p}{q} < 1,\ q - p = 1$.

In our work, we find the eigenvalues of a nonlocal boundary value problem and the corresponding root functions for a mixed parabolic-hyperbolic equation of fractional order under conditions $x_0=\frac{p}{q}$, $0<\frac{p}{q}<1,\ q-p=2$. The existence and uniqueness of a solution to this problem was proved.

FORMULATION OF THE PROBLEM.

Let $\Omega=\{(x,t):0<x<1,-a<t<b\},a,b\in R^+,\Omega^-=\Omega\cap(t<0)$, $\Omega^+=\Omega\cap(t>0)$. We consider the following nonlocal problem in the domain Ω.

Problem BS. To find a function $u(x,t)$ from the class

$$u\in C^1(\overline{\Omega}^-)\cap C^{2,2}_{x,t}(\Omega^-),\ u,u_x\in C(\overline{\Omega}^+),\ {}_C D^{\alpha}_{0t}u,u_{xx}\in C(\Omega^+), \tag{1}$$

which satisfies the following equation in $\Omega^+\cup\Omega^-$

$$0=\begin{cases} {}_C D^{\alpha}_{0t}u(x,t)-\dfrac{\partial^2 u(x,t)}{\partial x^2},\ t>0,\\ \dfrac{\partial^2 u(x,t)}{\partial t^2}-\dfrac{\partial^2 u(x,t)}{\partial x^2},\ t<0,\end{cases} \tag{2}$$

with the following conditions

$$u(0,t)=0,\quad \frac{\partial u(1,t)}{\partial x}=\frac{\partial u(x_0,t)}{\partial x},\quad t\in[-a,b], \tag{3}$$

$$u(x,-a)-h\cdot u(x,b)=\psi(x),\quad x\in[0,1], \tag{4}$$

$$\lim_{t\to+0}{}_C D^{\alpha}_{0t}u(x,t)=\lim_{t\to-0}\frac{\partial u(x,t)}{\partial t}. \tag{5}$$

Here $\psi(x)$ is a given function, $h\in R$, $x_0=\frac{p}{q}$ is a given rational number and $0<\frac{p}{q}<1,\ q-p=2$. ${}_C D^{\alpha}_{0t}$ is the Caputo fractional operator of the order $0<\alpha<1$ with respect to variable t [13].

SPECTRAL PROPERTIES OF PROBLEM BS.

To solve the problem, we use the method of separation of variables. As usual, we are looking for a solution in the form $u(x,t)=X(x)\cdot u(t)$. Substituting this expression into equation (2) and boundary conditions (3) we obtain the following spectral problem:

$$X''(x)+\lambda X(x)=0,\ x\in(0,1), \tag{6}$$

$$X(0)=0,\ X'(1)=X'(x_0),\ x_0\in(0,1), \tag{7}$$

where $x_0=\frac{p}{q}$ and $0<\frac{p}{q}<1,\ q-p=2$.

Here we consider three cases: $\lambda<0$, $\lambda=0$, $\lambda>0$, which depends the general solution of the equation (6).

First case: $\lambda=0$. In this case the (6) equation has the following solution: $X(x)=A_1x+A_2$.

According to condition (7) , we obtain $A_1\neq 0, A_2=0$.

Consequently, we have $X(x)=x$, this eigenfunction associated to the single eigenvalue $\lambda_0=0$.

Second case $\lambda < 0$. The equation (6) has a solution on the following form: $X(x)=B_1e^{-\sqrt{\lambda}x}+B_2e^{\sqrt{\lambda}x}$.

Based on (7) condition, we get $B_1=0, B_2=0$.

Hence, the problem (6)–(7) has only a trivial solution for all $\lambda < 0$.

Finally, we consider the third case $\lambda > 0$. In this case (6) equation get the following solution $X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$.

According to condition (7), we obtain the following two series of eigenvalues $\lambda_{1n}=\left(\frac{2nq\pi}{p+q}\right)^2$, $\lambda_{2k}=(kq\pi)^2,\ n,k\in N$ and the corresponding eigenfunctions are $X_{1n}(x)=\sin\sqrt{\lambda_{1n}}x,\ X_{2k}(x)=\sin\sqrt{\lambda_{2k}}x$.

Hence, the set of eigenvalues of problem (6)–(7) are in the form

$$\lambda_0=0,\ \lambda_{1n}=\left(\frac{2nq\pi}{p+q}\right)^2,\ \lambda_{2k}=(kq\pi)^2,\ n,k\in N, \tag{8}$$

the corresponding eigenfunctions are

$$X_0(x)=x,\quad X_{1n}(x)=\sin\sqrt{\lambda_{1n}}x,\quad X_{2k}(x)=\sin\sqrt{\lambda_{2k}}x. \tag{9}$$

Corollary 1. *Let x_0 be a rational number such as $x_0=\frac{p}{q}$, $q-p=2$, p and q be coprime natural numbers. Then for two sets of eigenvalues $\{\lambda_{1n}\}_{n=1}^{\infty}$ and $\{\lambda_{2k}\}_{k=1}^{\infty}$ from (8), the inclusion $\{\lambda_{2k}\}_{k=1}^{\infty}\subset\{\lambda_{1n}\}_{n=1}^{\infty}$ takes place, namely, the set of eigenvalues $\{\lambda_{2k}\}_{k=1}^{\infty}$ is contained in the set $\{\lambda_{1n}\}_{n=1}^{\infty}$.*

During the consider (6), (7) problem, we also consider the Problem adjoint to it. It is easy to determine that the following problem will be adjoint to problem (6), (7):

$$Y''(x)+\lambda Y(x)=0,\quad x\in(0,x_0)\cup(x_0,1), \tag{10}$$

$$Y(0)=0,\ \ Y'(1)=0, \tag{11}$$

$$Y'(x_0-0)=Y'(x_0+0),\ Y(1)=Y(x_0+0)-Y(x_0-0). \tag{12}$$

Spectral problems of types (6), (7) and (10)–(12) were studied in [14], where the necessary and sufficient condition for the Riesz basis property of root (eigenfunctions and associated) functions of Bitsadze–Samarskii type problems for a second-order ordinary differential operator was formulated in a more general setting than conditions (7) and (11), (12).

Problem (10)-(12) also has identical eigenvalues with (8). Its eigenfunctions of the are found as the same way with the problem (6)-(7), on the following form:

$$\{Y_0(x);Y_{1n}(x);Y_{2n}(x)\},n\in N, \tag{13}$$

where

$$Y_0(x)=\begin{cases}0, & 0\le x<x_0,\\ \dfrac{2}{1-x_0^2}, & x_0<x\le 1,\end{cases}\quad Y_{2n}(x)=\begin{cases}0, & 0\le x<x_0,\\ \dfrac{4\cos\sqrt{\lambda_{2n}}x}{1-x_0^2}, & x_0<x\le 1,\end{cases}$$

$$Y_{1n}(x)=\begin{cases}\dfrac{4\sin\sqrt{\lambda_{1n}}x}{1+x_0}, & 0\le x<x_0,\\ \dfrac{2\cos\sqrt{\lambda_{1n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1n}}}, & x_0<x\le 1.\end{cases}$$

By solving problem (6), (7), we obtain that the eigenvalues of this problem have the form (8), they correspond to eigenfunctions (9), and for each value of λ_{2k}, there are associated functions of the form $\tilde{X}_{2k}(x)=x\cos\sqrt{\lambda_{2k}}x$.

In the same way, we find the root functions of the adjoint problem (10)–(12). As a result, we obtain

No.	Eigenvalues	Root functions of the main problem	Root functions of the adjoint problem
1	$\lambda_0=0$	$X_0(x)=x,\ x\in[0,1]$	$Y_0(x)=\begin{cases}0, x\in[0,x_0)\\ \dfrac{2}{1-x_0^2}, x\in(x_0,1]\end{cases}$
2	$\lambda_{1n}=\left(\dfrac{2q\pi n}{q+p}\right)^2$ $n\ne k\dfrac{q+p}{2}$ $k,n\in N$	$X_{1n}(x)=\sin\sqrt{\lambda_{1n}}x$ $x\in[0,1]$	$Y_{1n}(x)=\begin{cases}\dfrac{4\sin\sqrt{\lambda_{1n}}x}{1+x_0}, x\in[0,x_0)\\ \dfrac{2\cos\sqrt{\lambda_{1n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1n}}}, x\in(x_0,1]\end{cases}$
3	$\lambda_{2n}=(qn\pi)^2$ $n\in N$	$X_{2n}(x)=\sin\sqrt{\lambda_{2n}}x$ $x\in[0,1]$	$Y_{2n}(x)=\begin{cases}0, x\in[0,x_0)\\ \dfrac{4\cos\sqrt{\lambda_{2n}}x}{1-x_0^2}, x\in(x_0,1]\end{cases}$
4		$\tilde{X}_{2n}(x)=x\cos\sqrt{\lambda_{2n}}x$ $x\in[0,1]$	$\tilde{Y}_{2n}(x)=\begin{cases}\dfrac{4\sin\sqrt{\lambda_{2n}}x}{1+x_0}, x\in[0,x_0)\\ \dfrac{4(1-x)\sin\sqrt{\lambda_{2n}}x}{1-x_0^2}, x\in(x_0,1]\end{cases}$

The following takes place.

Lemma 1. The systems of functions in Table 1 are biorthogonal, i.e.

$$(X_0(x),Y_0(x))_{L_2(0,1)}=1,$$

$$(X_{in}(x),Y_0(x))_{L_2(0,1)}=(X_0(x),Y_{in}(x))_{L_2(0,1)}=0, i=1,2,$$

$$(X_0(x),\tilde{Y}_{2n}(x))_{L_2(0,1)}=(\tilde{X}_{2n}(x),Y_0(x))_{L_2(0,1)}=0,$$

$$\begin{matrix}(X_{1k}(x),Y_{2n}(x))_{L_2(0,1)}=(X_{2k}(x),Y_{2n}(x))_{L_2(0,1)}=0,\\ (X_{1k}(x),\tilde{Y}_{2n}(x))_{L_2(0,1)}=(\tilde{X}_{2k}(x),Y_{1n}(x))_{L_2(0,1)}=0,\end{matrix}\quad (X_{1k}(x),Y_{1n}(x))_{L_2(0,1)}=\begin{cases}1, & k=n,\\ 0, & k\ne n,\end{cases}$$

$$(X_{2k}(x),\tilde{Y}_{2n}(x))_{L_2(0,1)}=\begin{cases}1, & k=n,\\ 0, & k\neq n,\end{cases}\quad (\tilde{X}_{2k}(x),Y_{2n}(x))_{L_2(0,1)}=\begin{cases}1, & k=n,\\ 0, & k\neq n.\end{cases}$$

Proof. We will show that the last inequality is true. To do this, we first calculate the corresponding integral for the case $k=n$.

$$\int_0^1 \tilde{X}_{2n}(x)Y_{2n}(x)dx=\frac{4}{1-x_0^2}\int_{x_0}^1 x\cos^2\sqrt{\lambda_{2n}}x dx=\frac{2}{1-x_0^2}\int_{x_0}^1 x(1+\cos 2\sqrt{\lambda_{2n}}x)dx=\left.\frac{x^2}{1-x_0^2}\right|_{x_0}^1+$$

$$+\frac{2}{1-x_0^2}\int_{x_0}^1 x\cos 2\sqrt{\lambda_{2n}}x dx=1+\left.\frac{x\sin 2\sqrt{\lambda_{2n}}x}{\sqrt{\lambda_{2n}}(1-x_0^2)}\right|_{x_0}^1-\frac{1}{\sqrt{\lambda_{2n}}(1-x_0^2)}\int_{x_0}^1\sin 2\sqrt{\lambda_{2n}}x dx=1.$$

Now we calculate the corresponding integral for the case $k\neq n$.

$$\int_0^1 \tilde{X}_{2k}(x)Y_{2n}(x)dx=\frac{4}{1-x_0^2}\int_{x_0}^1 x\cos\sqrt{\lambda_{2k}}x\cos\sqrt{\lambda_{2n}}x dx=$$

$$=\frac{2}{1-x_0^2}\int_{x_0}^1 x\left[\cos(\sqrt{\lambda_{2k}}+\sqrt{\lambda_{2n}})x+\cos(\sqrt{\lambda_{2k}}-\sqrt{\lambda_{2n}})x\right]dx=$$

$$=\frac{2}{1-x_0^2}x\left.\left[\frac{\sin(\sqrt{\lambda_{2k}}+\sqrt{\lambda_{2n}})x}{\sqrt{\lambda_{2k}}+\sqrt{\lambda_{2n}}}+\frac{\sin(\sqrt{\lambda_{2k}}-\sqrt{\lambda_{2n}})x}{\sqrt{\lambda_{2k}}-\sqrt{\lambda_{2n}}}\right]\right|_{x_0}^1-$$

$$-\frac{2}{1-x_0^2}\int_{x_0}^1\frac{\sin(\sqrt{\lambda_{2k}}+\sqrt{\lambda_{2n}})x}{\sqrt{\lambda_{2k}}+\sqrt{\lambda_{2n}}}+\frac{\sin(\sqrt{\lambda_{2k}}-\sqrt{\lambda_{2n}})x}{\sqrt{\lambda_{2k}}-\sqrt{\lambda_{2n}}}dx=0.$$

Therefore, the functions $\tilde{X}_{2k}(x)$ and $Y_{2n}(x)$ are mutually biorthonormal. The remaining relations are proven similarly.

The following statement holds:

Theorem 1. *The system* $\{X_0(x),X_{in}(x),\tilde{X}_{2n}(x)\}$ *is complete in* $L_2(0,1)$.

It is easy to know that for reader, proof of the Theorem, obtained based on the Theorem 1 in [10].

Lemma 2. *Let* $x_0=\dfrac{p}{q}$ *be any rational number from the interval* $(0,1)$, *such that* $q-p=2$. *Then, the systems of root functions from Table 1 form a Riesz basis in the space* $L_2(0,1)$.

To prove Lemma 2 we will use the following theorem.

Theorem 2 [15]. *Let* H *be a Hilbert space. The following three statements are equivalent:*

1) the system $\{\varphi_j\}_1^\infty$ *form the Riesz basis in* H;

2) the system $\{\varphi_j\}_1^\infty$ *is minimal and complete in H, the systems* $\{\varphi_j\}_1^\infty$ *and its biorthogonal* $\{\psi_j\}_1^\infty$ *are Bessel in* H;

3) the system $\{\varphi_j\}_1^\infty$ *is uniformly minimal, Bessel and Hilbert in* H.

Proof. As it follows from item 2 of Theorem 2, to prove the Riesz basis property of systems $\left\{X_0(x), X_{in}(x), \tilde{X}_{2n}(x)\right\}$ and $\left\{Y_0(x), Y_{in}(x), \tilde{Y}_{2n}(x)\right\}$, $i=1,2$ it suffices to establish the completeness and minimality of system $\left\{X_0(x), X_{in}(x), \tilde{X}_{2n}(x)\right\}$, as well as the Bessel properties of both systems. The completeness of system $\left\{X_0(x), X_{in}(x), \tilde{X}_{2n}(x)\right\}$ follows from Theorem 1, and the minimality follows from the same Theorem 1 and Lemma 1 in [10]. The Bessel property of these systems follows from the Bessel property of systems of type $\{\cos(an+b)x\}$ and $\{\sin(an+b)x\}$ on any finite interval for any real $a \neq 0, b$ [15]. Lemma 2 is proven.

EXISTENCE AND UNIQUENESS OF A SOLUTION TO PROBLEM BS

Now, we are going to show the existence and uniqueness of the $u(x,t)$. For this aim, we will look $u(x,t)$ as follows:

$$u(x,t)=\begin{cases} u_0^+(t)X_0(x)+\sum\limits_{n=1}^{\infty *} u_{1n}^+(t)\cdot X_{1n}(x)+\sum\limits_{n=1}^{\infty}\left(u_{2n}^+(t)\cdot X_{2n}(x)+\tilde{u}_{2n}^+(t)\cdot \tilde{X}_{2n}(x)\right), t>0, \\ u_0^-(t)X_0(x)+\sum\limits_{n=1}^{\infty *} u_{1n}^-(t)\cdot X_{1n}(x)+\sum\limits_{n=1}^{\infty}\left(u_{2n}^-(t)\cdot X_{2n}(x)+\tilde{u}_{2n}^-(t)\cdot \tilde{X}_{2n}(x)\right), t<0, \end{cases} \tag{14}$$

where $u_0^{\pm}(t), u_{in}^{\pm}(t), \tilde{u}_{2n}^{\pm}(t), i=1,2$ are unknown functions.

Taking into account that the system of root functions in Table 1 is a Riesz basis, from (14) we obtain

$$\begin{aligned} &u_0^{\pm}(t)=\left(u(x,t),Y_0(x)\right), u_{1n}^{\pm}(t)=\left(u(x,t),Y_{1n}(x)\right), \\ &u_{2n}^{\pm}(t)=\left(u(x,t),\tilde{Y}_{2n}(x)\right), \tilde{u}_{2n}^{\pm}(t)=\left(u(x,t),Y_{2n}(x)\right). \end{aligned} \tag{15}$$

Hence, taking into account equation (2) and condition (3), we found the following systems of equations:

$$\begin{cases} {}_C D_{0t}^{\alpha} u_0^+(t)=0, \\ u_0''^{\pm}(t)=0, \end{cases} \qquad \begin{cases} {}_C D_{0t}^{\alpha} u_{1n}^+(t)+\lambda_{1n}u_{1n}^+(t)=0, \\ u_{1n}''^-(t)(t)+\lambda_{1n}u_{1n}^-(t)=0, \end{cases}$$

$$\begin{cases} {}_C D_{0t}^{\alpha} \tilde{u}_{2n}^+(t)+\lambda_{2n}\tilde{u}_{2n}^+(t)=0, \\ \tilde{u}_{2n}''^-(t)+\lambda_{2n}\tilde{u}_{2n}^-(t)=0, \end{cases} \qquad \begin{cases} {}_C D_{0t}^{\alpha} u_{2n}^+(t)+\lambda_{2n}u_{2n}^+(t)=-\sqrt{\lambda_{2n}}\,\tilde{u}_{2n}^+(t), \\ u_{2n}''^-(t)+\lambda_{2n}u_{2n}^-(t)=-\sqrt{\lambda_{2n}}\,\tilde{u}_{2n}^-(t), \end{cases}$$

whose general solutions have the form

$$\begin{cases} u_0^+(t)=A_0, \\ u_0^-(t)=B_0t+L_0, \end{cases} \qquad \begin{cases} u_{1n}^+(t)(t)=A_{1n}E_{\alpha}(-\lambda_{1n}t^{\alpha}), \\ u_{1n}^-=B_{1n}\sin\sqrt{\lambda_{1n}}t+L_{1n}\cos\sqrt{\lambda_{1n}}t, \end{cases} \tag{16}$$

$$\begin{cases} \tilde{u}_{2n}^+=\tilde{A}_{2n}E_{\alpha}(-\lambda_{2n}t^{\alpha}), \\ \tilde{u}_{2n}^-=\tilde{B}_{2n}\sin\sqrt{\lambda_{2n}}t+\tilde{L}_{2n}\cos\sqrt{\lambda_{2n}}t, \end{cases} \qquad \begin{cases} u_{2n}^+(t)=A_{2n}E_{\alpha}(-\lambda_{2n}t^{\alpha})-\dfrac{2\sqrt{\lambda_{2n}}\tilde{\psi}_{2n}}{\Delta_{2n}}F_n(t), \\ u_{2n}^-(t)=B_{2n}\sin\sqrt{\lambda_{2n}}t+L_{2n}\cos\sqrt{\lambda_{2n}}t-\dfrac{2\tilde{\psi}_{2n}}{\Delta_{2n}}G_n(t), \end{cases} \tag{17}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function, $A_0, B_0, L_0, A_{in}, B_{in}, L_{in}, \tilde{A}_{2n}, \tilde{B}_{2n}, \tilde{L}_{2n}, i=1,2$ are arbitrary constants, and

$$F_n(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{2n}(t-\tau)^{\alpha}) E_{\alpha}(-\lambda_{2n}\tau^{\alpha}) d\tau,$$

$$G_n(t) = \int_t^0 (\cos\sqrt{\lambda_{2n}}\tau - \sqrt{\lambda_{2n}}\sin\sqrt{\lambda_{2n}}\tau)\sin\sqrt{\lambda_{2n}}(\tau - t) d\tau.$$

From (15), and using (1), (4), (5), conditions, we get that the functions $u_0^{\pm}(t), u_{in}^{\pm}(t), \tilde{u}_{2n}^{\pm}(t), i=1,2$ must satisfy the following conditions:

$$\lim_{t\to+0} u_0^+(t) = \lim_{t\to-0} u_0^-(t),\ \lim_{t\to+0} {}_C D_{0t}^{\alpha} u_0^+(t) = \lim_{t\to-0} u_0^{'-}(t), \tag{18}$$

$$\lim_{t\to+0} u_{in}^+(t) = \lim_{t\to-0} u_{in}^-(t),\ \lim_{t\to+0} {}_C D_{0t}^{\alpha} u_{in}^+(t) = \lim_{t\to-0} u_{in}^{'-}(t),\ i=1,2, \tag{19}$$

$$\lim_{t\to+0} \tilde{u}_{2n}^+(t) = \lim_{t\to-0} \tilde{u}_{2n}^-(t),\ \lim_{t\to+0} {}_C D_{0t}^{\alpha} \tilde{u}_{2n}^+(t) = \lim_{t\to-0} \tilde{u}_{2n}^-(t), \tag{20}$$

$$u_0^-(-a) - hu_0^+(b) = \psi_0,\ u_{in}^-(-a) - hu_{in}^+(b) = \psi_{in}, u_{2n}^-(-a) - hu_{2n}^+(b) = \tilde{\psi}_{2n}, i=1,2, \tag{21}$$

where $\psi_0, \psi_{1n}, \psi_{2n}, \tilde{\psi}_{2n}$ are the expansion coefficients, i.e.

$$\psi_0 = \int_0^1 \psi(x) Y_0(x) dx, \psi_{1n} = \int_0^1 \psi(x) Y_{1n}(x) dx, \varphi_{2n} = \int_0^1 \psi(x) \tilde{Y}_{2n}(x) dx, \tilde{\psi}_{2n} = \int_0^1 \psi(x) Y_{2n}(x) dx, \tag{22}$$

where $Y_0(x),\ Y_{in}(x),\ \tilde{Y}_{2n}(x), i=1,2$ are functions defined in Table 1.

Then, by virtue of (18)-(21), to obtain constants $A_0, B_0, L_0, A_{in}, B_{in}, L_{in}, \tilde{A}_{2n}, \tilde{B}_{2n}, \tilde{L}_{2n}$, $i=1,2$ we have the following equations:

$$\begin{cases} L_0 = A_0, B_0 = 0, \\ L_0 - B_0 a = hA_0 + \psi_0, \end{cases} \begin{cases} L_{1n} = A_{1n}, B_{1n} = -\sqrt{\lambda_{1n}} A_{1n}, \\ L_{1n}\cos\sqrt{\lambda_{1n}}a - B_{1n}\sin\sqrt{\lambda_{1n}}a = A_{1n} h E_{\alpha}(-\lambda_{1n} b^{\alpha}) + \psi_{1n}, \end{cases}$$

$$\begin{cases} \tilde{L}_{2n} = \tilde{A}_{2n}, \tilde{B}_{2n} = -\sqrt{\lambda_{2n}}\tilde{A}_{2n}, \\ \tilde{L}_{2n}\cos\sqrt{\lambda_{2n}}a - \tilde{B}_{2n}\sin\sqrt{\lambda_{2n}}a = \tilde{A}_{2n} h E_{\alpha}(-\lambda_{2n} b^{\alpha}) + \tilde{\psi}_{2n}, \end{cases}$$

$$\begin{cases} L_{2n} = A_{2n}, B_{2n} = -\sqrt{\lambda_{2n}} A_{2n} - \dfrac{2\tilde{\psi}_{2n}}{\Delta_{2n}}, \\ L_{2n}\cos\sqrt{\lambda_{2n}}a - B_{2n}\sin\sqrt{\lambda_{2n}}a - \dfrac{2\tilde{\psi}_{2n}}{\Delta_{2n}} G_{2n}(-a) = \\ = h\left[A_{2n} E_{\alpha}(-\lambda_{2n} b^{\alpha}) - \dfrac{\sqrt{\lambda_{2n}}\tilde{\psi}_{2n}}{\Delta_{2n}} F_{2n}(b) \right] + \psi_{2n}. \end{cases}$$

This system has a unique solution,

$$L_0 = A_0, B_0 = 0, A_0 = \frac{1}{1-h}\psi_0,\ L_{1n} = A_{1n}, B_{1n} = -\sqrt{\lambda_{1n}} A_{1n}, A_{1n} = \frac{1}{\Delta_{1n}}\psi_{1n},$$

$$\tilde{L}_{2n} = \tilde{A}_{2n}, \tilde{B}_{2n} = -\sqrt{\lambda_{2n}}\tilde{A}_{2n}, \tilde{A}_{2n} = \frac{1}{\Delta_{2n}}\tilde{\psi}_{2n}, L_{2n} = A_{2n}, B_{2n} = -\sqrt{\lambda_{2n}} A_{2n} - \frac{2}{\Delta_{2n}}\tilde{\psi}_{2n},$$

$$L_{2n} = \frac{1}{\Delta_{2n}}\left(\psi_{2n} - \frac{h\sqrt{\lambda_{2n}} F_n(b) - G_n(-a)}{\Delta_{2n}}\tilde{\psi}_{2n} - \frac{2\sin\sqrt{\lambda_{2n}}a}{\Delta_{2n}^2}\tilde{\psi}_{2n} \right),$$

under the conditions that for all $n=1,2...$ takes place the following

$$\Delta_{in}=\cos\sqrt{\lambda_{in}}a+\sqrt{\lambda_{in}}\sin\sqrt{\lambda_{in}}a-hE_{\alpha}(-\lambda_{in}b^{\alpha})\neq 0,\ i=1,2,\ \Delta_0=1-h\neq 0. \quad (23)$$

Substituting the found solutions into (16) and (17), we finally have

$$\begin{cases} u_0^+(t)=\dfrac{\psi_0}{1-h}, \\ u_0^-(t)=\dfrac{\psi_0}{1-h}, \end{cases} \quad \begin{cases} u_{1n}^+(t)=\dfrac{\psi_{1n}}{\Delta_{1n}}E_{\alpha}(-\lambda_{1n}t^{\alpha}), \\ u_{1n}^-(t)=\dfrac{\psi_{1n}}{\Delta_{1n}}(\cos\sqrt{\lambda_{1n}}t-\sqrt{\lambda_{1n}}\sin\sqrt{\lambda_{1n}}t), \end{cases} \quad (24)$$

$$\begin{cases} \tilde{u}_{2n}^+(t)=\dfrac{\tilde{\psi}_{2n}}{\Delta_{2n}}E_{\alpha}(-\lambda_{2n}t^{\alpha}), \\ \tilde{u}_{2n}^-(t)=\dfrac{\tilde{\psi}_{2n}}{\Delta_{2n}}(\cos\sqrt{\lambda_{2n}}t-\sqrt{\lambda_{2n}}\sin\sqrt{\lambda_{2n}}t), \end{cases} \quad (25)$$

$$\begin{cases} u_{2n}^+(t)=\left(\psi_{2n}-\dfrac{h\sqrt{\lambda_{2n}}F_n(b)-G_n(-a)}{\Delta_{2n}}\tilde{\psi}_{2n}-\dfrac{2\sin\sqrt{\lambda_{2n}}a}{\Delta_{2n}}\tilde{\psi}_{2n}\right)\times \\ \qquad\times\dfrac{E_{\alpha}(-\lambda_{2n}t^{\alpha})}{\Delta_{2n}}-\dfrac{\sqrt{\lambda_{2n}}F_n(t)}{\Delta_{2n}}\tilde{\psi}_{2n}, \\ u_{2n}^-(t)=\left(\psi_{2n}-\dfrac{h\sqrt{\lambda_{2n}}F_n(b)-G_n(-a)}{\Delta_{2n}}\tilde{\psi}_{2n}-\dfrac{2\sin\sqrt{\lambda_{2n}}a}{\Delta_{2n}}\tilde{\psi}_{2n}\right)\times \\ \qquad\times\dfrac{\cos\sqrt{\lambda_{2n}}t-\sqrt{\lambda_{2n}}\sin\sqrt{\lambda_{2n}}t}{\Delta_{2n}}-\dfrac{2\sin\sqrt{\lambda_{2n}}t}{\Delta_{2n}}\tilde{\psi}_{2n}-\dfrac{G_n(t)}{\Delta_{2n}}\tilde{\psi}_{2n}. \end{cases} \quad (26)$$

Now we will show that the solution to the problem is unique. Let us, assume that Problem BS have the two solutions as follows: $u_1(x,t)$, $u_2(x,t)$. According to properties of linear operators, $u(x,t)=u_1(x,t)-u_2(x,t)$ is the solution for the homogeneous Problem BS. Since that and from (4) condition we have $\psi(x)\equiv 0$. If condition (23) is valid, then and since $\psi(x)\equiv 0$ we obtain $\psi_0=0, \psi_{in}=0, \tilde{\psi}_{2n}=0, i=1,2$. According to formulas (24)-(26), $u_0^{\mp}(t)=0, u_{in}^{\pm}(t)=0, \tilde{u}_{2n}^{\pm}(t)=0,\ i=1,2$ and based on (15)

$$\int_0^1 u(x,t)Y_0(x)dx=0,\ \int_0^1 u(x,t)Y_{in}(x)dx=0,\ \int_0^1 u(x,t)\tilde{Y}_{2n}(x)dx=0,\ i=1,2,\ n\in N.$$

Since the system $\{Y_0(x),Y_{in}(x),\tilde{Y}_{2n}(x)\}$ is complete in the space $L_2(0,1)$, we obtain, $u(x,t)=0$ equality is valid almost everywhere on $[0,1]$ for any fixed $t\in[-a,b]$. Since, by condition (1), the function $u(x,t)$ are continuous on $\overline{\Omega}$, respectively, we have $u(x,t)\equiv 0$ in $\overline{\Omega}$.

Thus, the following theorem holds:

Theorem 3. If exists a solution to problem BS and conditions (23) hold for all $n\in N\cup\{0\}$, then this solution is unique.

Now, represent the expression Δ_{in} in the form:

$$\Delta_{in}=\sqrt{1+\lambda_{in}}\sin\left(\sqrt{\lambda_{in}}a+\varepsilon_{in}\right)-hE_{\alpha}(-\lambda_{in}b^{\alpha}),$$

where $\varepsilon_{in}=\arcsin\left(1/\sqrt{1+\lambda_{in}}\right)\to 0$ at $n\to+\infty$. Since that, we can know that $\Delta_{in}=0$ is valid if only if

$$a=\frac{1}{\sqrt{\lambda_{in}}}\left[(-1)^k\arcsin\frac{hE_\alpha(-\lambda_{in}b^\alpha)}{\sqrt{1+\lambda_{in}}}+\pi k-\varepsilon_{in}\right],k=1,2,\ldots.$$

Since a and b are arbitrary numbers, it follows that the expression $\Delta_{in}, i=1,2$ can become sufficiently small for sufficiently large n; i.e., we encounter the problem of "small denominators." Therefore, to substantiate the existence of a solution to this problem, it is necessary to show the existence of numbers a and b such that, for sufficiently large n, $\Delta_{in}, i=1,2$ will be separated from zero [17,18].

Lemma 3. *Let $h\in R/\{1\}$, c,d be arbitrary natural numbers $(c,d)=1$, $a=\dfrac{c(q+p)}{2d}$. Then for large values of n, there exists a positive constant M_0 such that the estimate*

$$\left|\Delta_{in}\right|\geq M_0>0,\ i=1,2 \tag{27}$$

hold.

We did not present the proof of this lemma because it was proved similarly to the proof of Lemma 6 in [11].

Lemma 4. Let the condition (23) and the estimate (27) hold. Then

1) if $0\le t\le b$, then

$$|u_{1n}^+(t)|\le M_1|\psi_{1n}|,\ |{}_CD_{0t}^\alpha u_{1n}^+(t)|\le M_2n^2|\psi_{1n}|,\ |\tilde{u}_{2n}^+(t)|\le M_3|\tilde{\psi}_{2n}|,$$
$$|{}_CD_{0t}^\alpha\tilde{u}_{2n}^+(t)|\le M_4n^2|\tilde{\psi}_{2n}|,\ |u_{2n}^+(t)|\le M_5\left(|\psi_{2n}|+n|\tilde{\psi}_{2n}|\right),$$
$$|{}_CD_{0t}^\alpha u_{2n}^+(t)|\le M_6\left(n^2|\psi_{2n}|+n^3|\tilde{\psi}_{2n}|\right),$$

2. if $-a\le t\le 0$ then

$$\left|\frac{\partial^j u_{1n}^-(t)}{\partial t^j}\right|\le M_7n^{j+1}|\psi_{1n}|,\qquad \left|\frac{\partial^j\tilde{u}_{12}^-(t)}{\partial t^j}\right|\le M_8n^{j+1}|\tilde{\psi}_{2n}|,$$
$$\left|\frac{\partial^j u_{2n}^-(t)}{\partial t^j}\right|\le M_9\left(n^{j+1}|\psi_{2n}|+n^{j+2}|\tilde{\psi}_{2n}|\right),\quad j=0,1,2,$$

where $M_k, k=\overline{1,9}$ are positive constants.

We will show that these inequalities are valid in the two examples. For this, if we apply the operator ${}_CD_{0t}^\alpha$ to the solution of (26) for $t>0$, we get

$${}_CD_{0t}^\alpha u_{2n}^+(t)=\frac{-\lambda_{2n}}{\Delta_{2n}}\left(\frac{2\tilde{\psi}_{2n}}{\Delta_{2n}}\left(G_n(-a)-h\sqrt{\lambda_{2n}}F_n(b)-\sin\sqrt{\lambda_{2n}}a\right)+\psi_{2n}\right)E_\alpha(-\lambda_{2n}t^\alpha)+$$
$$+2\sqrt{\lambda_{2n}^3}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{2n}(t-\tau)^\alpha)\frac{\tilde{\psi}_{2n}}{\Delta_{2n}}E_\alpha(-\lambda_{2n}\tau^\alpha)d\tau-2\sqrt{\lambda_{2n}}\frac{\tilde{\psi}_{2n}}{\Delta_{2n}}E_\alpha(-\lambda_{2n}t^\alpha).$$

Using the estimate of the Mittag-Leffler function and the estimate (27),

$$\left|{}_C D_{0t}^{\alpha} u_{2n}^{+}(t)\right| \le \left|\frac{-\lambda_{2n}}{\Delta_{2n}}\left(\frac{2\tilde{\psi}_{2n}}{\Delta_{2n}}\left(G_n(-a) - h\sqrt{\lambda_{2n}}F_n(b) - \sin\sqrt{\lambda_{2n}}a\right) + \psi_{2n}\right)E_{\alpha}(-\lambda_{2n}t^{\alpha}) +\right.$$

$$\left.+2\sqrt{\lambda_{2n}^3}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{2n}(t-\tau)^{\alpha})\frac{\tilde{\psi}_{2n}}{\Delta_{2n}}E_{\alpha}(-\lambda_{2n}\tau^{\alpha})d\tau - 2\sqrt{\lambda_{2n}}\frac{\tilde{\psi}_{2n}}{\Delta_{2n}}E_{\alpha}(-\lambda_{2n}t^{\alpha})\right| \le$$

$$\le M_5\left(n^3|\tilde{\psi}_{2n}| + n^3|\tilde{\psi}_{2n}| + n^2|\tilde{\psi}_{2n}| + n^2|\psi_{2n}| + n^3|\tilde{\psi}_{2n}| + n|\tilde{\psi}_{2n}|\right) \le M_5\left(n^2|\psi_{2n}| + n^3|\tilde{\psi}_{2n}|\right)$$

is, which shows the validity of the sixth inequality.

Now we prove the last inequality. By differentiating relation (26) twice with respect to t for $t < 0$, we obtain

$$v_{2n}''(t) = \frac{1}{\Delta_{2n}}\left(\frac{2\tilde{\psi}_{2n}}{\Delta_{2n}}G_n(-a) - \frac{2h\tilde{\psi}_{2n}\sqrt{\lambda_{2n}}}{\Delta_{2n}}F_n(b) - \frac{2\tilde{\psi}_{2n}}{\Delta_{2n}}\sin\sqrt{\lambda_{2n}}a + \psi_{2n}\right)\times$$

$$\times\lambda_{2n}\left(-\cos\sqrt{\lambda_{2n}}t + \sqrt{\lambda_{2n}}\sin\sqrt{\lambda_{2n}}t\right) + \frac{2\tilde{\psi}_{2n}}{\Delta_{2n}}\lambda_{2n}\sin\sqrt{\lambda_{2n}}t - \frac{2\tilde{\psi}_{2n}}{\Delta_{2n}}G_n''(t),$$

$$v_{2n}''(t) \le M_{10}n^2\left(n^2|\tilde{\psi}_{2n}| + n^2|\tilde{\psi}_{2n}| + |\tilde{\psi}_{2n}| + n|\psi_{2n}| + |\tilde{\psi}_{2n}| + n|\tilde{\psi}_{2n}|\right) \le M_{11}n^3\left(|\psi_{2n}| + n|\tilde{\psi}_{2n}|\right)$$

can be evaluated in the form, which means that the inequality
$v_{2n}''(t) \le M_{11}n^3\left(|\psi_{2n}| + n|\tilde{\psi}_{2n}|\right)$ is valid. The remaining inequalities can be proved in the same way. The lemma is proved.

Theorem 4. Let the function $\psi(x)$ satisfy the following conditions

$$\psi(x) \in C^6[0,1],\ \psi^{2l}(0) = 0,\ \psi^{2l+1}(1) = \psi^{2l+1}(x_0),\ l = 0,1,2,$$

and conditions (23) and (27) are satisfied. Then there is a unique solution to the Problem BS and the solution is given by the series (14).

Proof. we find formal solution of Problem BS in domains Ω^+ and Ω^-, given by formulas (14), respectively, where functions $u_0^{\pm}(t), u_{in}^{\pm}(t), \tilde{u}_{2n}^{\pm}(t), i = 1,2$ are defined by formulas (24)-(26).

One can easily check that function $u(x,t)$ satisfy equation (2) and conditions (3)-(5). We only need to prove validity of these steps. For this aim we prove convergence of series (14) and

$$u_x(x,t) = u_0^{\pm}(t) + \sum_{n=1}^{\infty *} u_{1n}^{\pm}(t)\cdot X_{1n}'(x) + \sum_{n=1}^{\infty}\left(u_{2n}^{\pm}(t)\cdot X_{2n}'(x) + \tilde{u}_{2n}^{\pm}(t)\cdot \tilde{X}_{2n}'(x)\right), \quad (28)$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty *} u_{1n}^{\pm}(t)\cdot X_{1n}''(x) + \sum_{n=1}^{\infty}\left(u_{2n}^{\pm}(t)\cdot X_{2n}''(x) + \tilde{u}_{2n}^{\pm}(t)\cdot \tilde{X}_{2n}''(x)\right), \quad (29)$$

$${}_C D_{0t}^{\alpha}u(x,t) = \sum_{n=1}^{\infty *} {}_C D_{0t}^{\alpha}u_{1n}^{+}(t)X_{1n}(x) + \sum_{n=1}^{\infty}\left({}_C D_{0t}^{\alpha}u_{2n}^{+}(t)X_{2n}(x) + {}_C D_{0t}^{\alpha}\tilde{u}_{2n}^{+}(t)\tilde{X}_{2n}(x)\right), \quad (30)$$

$$u_{tt}(x,t) = \sum_{n=1}^{\infty *} u_{1n}''^{-}(t)X_{1n}(x) + \sum_{n=1}^{\infty}\left(u_{2n}''^{-}(t)X_{2n}(x) + \tilde{u}_{2n}''^{-}(t)\tilde{X}_{2n}(x)\right). \quad (31)$$

By virtue of Lemmas 3 and 4, it is easy to see that the series (28)-(31) is majorized by the series

$$\sum_{n=1}^{\infty *} n^3 |\psi_{1n}| + \sum_{n=1}^{\infty} n^3 (|\psi_{2n}| + n|\tilde{\psi}_{2n}|), \tag{32}$$

whose convergence easily follows from the conditions of the theorem imposed on the function $\psi(x)$. Indeed, given the ratio

$$\psi_{1n} = \frac{1}{\lambda_{1n}^2}\psi_{1n}^{(4)}, \psi_{1n}^{(4)} = \int_0^1 \psi^{(4)}(x) Y_{1n}(x)dx, \ \tilde{\psi}_{2n} = \frac{1}{\lambda_{2n}^3}\tilde{\psi}_{2n}^{(6)}, \tilde{\psi}_{2n}^{(6)} = \int_0^1 \psi^{(6)}(x) Y_{2n}(x)dx,$$

$$\psi_{2n} = \frac{1}{\lambda_{2n}^2}\psi_{2n}^{(4)} - \frac{1}{\lambda_{2n}^2\sqrt{\lambda_{2n}}}\tilde{\psi}_{2n}^{(4)}, \psi_{2n}^{(4)} = \int_0^1 \psi^{(4)}(x)\tilde{Y}_{2n}(x)dx,$$

we obtain that the series (32) is estimated by the series

$$\sum_{n=1}^{\infty *}\frac{1}{n}\left|\psi_{1n}^{(4)}\right| + \sum_{n=1}^{\infty}\frac{1}{n}\left|\psi_{2n}^{(4)}\right| + \frac{1}{n^2}\left|\tilde{\psi}_{2n}^{(6)}\right|,$$

whose convergence follows from the Cauchy-Schwarz inequality, as well as the basis property of the system $\left\{X_0(x), X_{in}(x), \tilde{X}_{2n}(x)\right\}, i = 1,2$. From the convergence of the last series, by virtue of the Weierstrass criterion, the series (14), (28) converge absolutely and uniformly in the closed domain $\bar{\Omega}$, the series (28)-(31) converge absolutely and uniformly, respectively, in the domains $\overline{\Omega}^{+}$ and $\overline{\Omega}^{-}$. Therefore, the function $u(x,t)$ defined by series (14) satisfies condition (3). Substituting, now, series (14) into equation (2) at $t \neq 0$, we are convinced that function (14) also satisfies equation (2). Theorem 4 is proved.

CONCLUSION

In this paper, the Bitsadze–Samarskii type a nonlocal problem for a mixed-type equation with the fractional operator was researched. While researching this problem, the adjoin problem was considered and the eigenvalues and the corresponding root functions of the problems were found. These root functions are different in the case of $q - p = 2$ than in the case of $q - p = 1$. The conditions for the problem to have a unique solution were found.

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