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COMPUTATIONAL EXPLORATION OF TWO CLOSED TRAJECTORIES IN QUADRATIC DYNAMICAL SYSTEMS

Abstract: *Quadratic dynamical systems can display a range of dynamic behaviors, including the presence of multiple stable limit cycles. This work focuses on systems exhibiting two such closed trajectories through a detailed computational approach. The computational framework provides a powerful tool for understanding the intricate dynamics of these systems.*

Keywords: *quadratic dynamical systems, closed trajectories, limit cycles, computational exploration, numerical simulation, approximate period.*

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КОМПЬЮТЕРНОЕ ИССЛЕДОВАНИЕ ДВУХ ЗАМКНУТЫХ ТРАЕКТОРИЙ В КВАДРАТИЧНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ

Аннотация: *Квадратичные динамические системы могут демонстрировать широкий спектр динамического поведения, включая наличие множественных устойчивых предельных циклов. Данная работа посвящена системам, демонстрирующим две такие замкнутые траектории, на основе подробного исследования с использованием компьютеров. Компьютерная платформа предоставляет мощный инструмент для понимания сложной динамики этих систем.*

***Ключевые слова**: квадратичные динамические системы, замкнутые траектории, предельные циклы, Компьютерное исследование, численное моделирование, приближенный период.*

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IKKITA YOPIQ TRAYEKTORIYAGA EGA KVADRATIK DINAMIK SISTEMALARNI KOMPYUTERDA TADQIQ ETISH

***Annotatsiya.** Kvadratik dinamik sistemalar turli dinamik holatlarni namoyish etishi mumkin, jumladan, ularda bir necha limit davralarning mavjudligi. Ushbu ish ikkita shunday yopiq trayektoriya ega sistemalarga bag‘ishlangan bo‘lib, ularni aniqlash algoritmi keltirilgan. Bu algoritm ushbu sistemalarning murakkab dinamikasini tushunish uchun samarali vosita hisoblanadi.*

***Kalit so‘zlar:** kvadratik dinamik sistemalar, yopiq traektoriyalar, limit davralar, kompyuter tadqiqoti, sonli modellashtirish, taqribiy davr.*

1. Introduction

Quadratic dynamical systems, characterized by their polynomial equations of degree two, represent a fundamental class of models in diverse scientific fields. While their mathematical simplicity might suggest predictable behavior, these systems can exhibit surprisingly complex dynamics, including the emergence of multiple stable states [1-6]. This article focuses on the computational exploration of two-dimensional quadratic systems that display the intriguing phenomenon of two coexisting closed trajectories, also known as limit cycles. This computational approach enables a detailed exploration of these systems’ dynamics, which might be challenging or impossible to achieve through purely analytical methods. Additionally, cubic dynamical systems that display closed trajectories have been thoroughly studied using this approach, as demonstrated extensively in reference [7].

We consider dynamical systems of the following general form:

$$z' = \sum_{¿\sigma \vee ¿1}^{2} \left(a_\sigma \cdot z^\sigma \right)$$

where $z \in R^3$, i.e: $z = (x, y, z)$; $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $|\sigma| = \sigma_1 + \sigma_2 + \sigma_3$, $a_\sigma$are constant coefficients.

Therefore, the expanded system of quadratic equations is:

$$x' = a_{000} + a_{100}x + a_{010}y + a_{001}z + a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{110}xy + a_{101}xz + a_{011}yz$$

$$y' = b_{000} + b_{100}x + b_{010}y + b_{001}z + b_{200}x^2 + b_{020}y^2 + b_{002}z^2 + b_{110}xy + b_{101}xz + b_{011}yz$$

$$z' = c_{000} + c_{100}x + c_{010}y + c_{001}z + c_{200}x^2 + c_{020}y^2 + c_{002}z^2 + c_{110}xy + c_{101}xz + c_{011}yz.$$

2. Main result

If we assume the coefficients of the system above are real numbers, one can observe that the space of such systems has the power of the continuum. From this viewpoint, it is interesting to consider the subclass of systems where the coefficients $a_{\sigma_1\sigma_2\sigma_3} \in \{-1, 0, 1\}$. The investigation (and potential solution) of systems within this subclass, particularly those exhibiting periodicity, is a worthwhile problem.

To represent the system above in a computer-readable format, we use the following coding method:

$$[b_1 b_2 b_3 b_4], [c_1 c_2 c_3 c_4], [d_1 d_2 d_3 d_4]$$

Here, x' is coded by $[b_1 b_2 b_3 b_4]$, y' by $[c_1 c_2 c_3 c_4]$, andz' by $[d_1 d_2 d_3 d_4]$.

For example, the code $[0111][1110][1101]$ corresponds to one of $3^7 \cdot 2^9 = 1119744$ systems, specifically:

$$\begin{cases} x' = y + z^2 - yz \\ y' = 1 - y^2 - y \\ z' = -1 + x - xy \end{cases}$$

To solve the systems, the Runge-Kutta method is used for approximate solution of differential equations.

The general form of dynamical systems corresponding to the codes $[0101][1010][0110]$ is as follows:

$$\begin{cases} x' = \{\pm x, \pm y, \pm z\} + \{\pm xy, \pm xz, \pm yz\} \\ y' = \{\pm 1\} + \{\pm x^2, \pm y^2 \pm z^2\} \\ z' = \{\pm x, \pm y, \pm z\} + \{\pm x^2, \pm y^2 \pm z^2\} \end{cases}.$$

The number of such systems is 9216 and one of the systems having two closed trajectories among them is:

$$\begin{cases} x' = x + xy \\ y' = 1 - x^2 \\ z' = z + y^2 \end{cases}$$

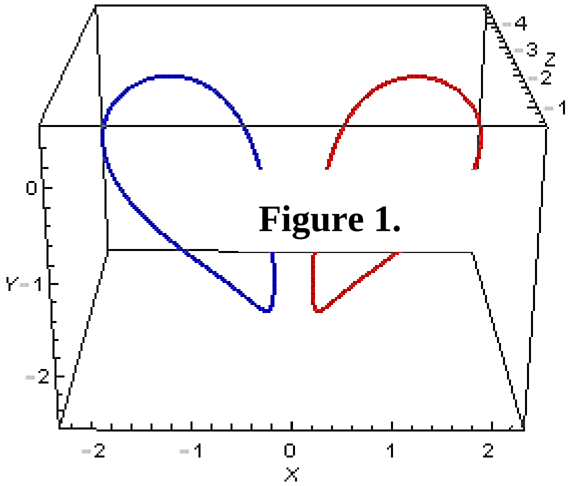

Figure 1.

with their initial point and approximate period (Figure 1): $T_b = T_r \approx 5.4$.

A point on the closed trajectory C_b: $(x_0, y_0, z_0) = (-0.69293, 0.47762, -1.65455)$.

And on the C_r: $(x_0, y_0, z_0) = (0.69293, 0.47762, -1.65455)$.

The parallelepiped containing the closed trajectories:

$$\Pi = \{(x, y, z) \in R^3 | -2.238 \le x \le 2.238, -2.475 \le y \le 0.478, -4.094 \le z \le -0.539\}$$

Further, general form of dynamical systems corresponding to the codes $[0101][1010][1001]$ is as follows:

$$\begin{cases} x' = \{\pm x, \pm y, \pm z\} + \{\pm xy, \pm xz, \pm yz\} \\ y' = \{\pm 1\} + \{\pm x^2, \pm y^2 \pm z^2\} \\ z' = \{\pm 1\} + \{\pm xy, \pm xz, \pm yz\} \end{cases}$$

the number of such systems is 3456 and one of the specific systems having two closed trajectories among them is :

$$\begin{cases} x' = -x - xy \\ y' = -1 + x^2 \\ z' = -1 - yz \end{cases}$$

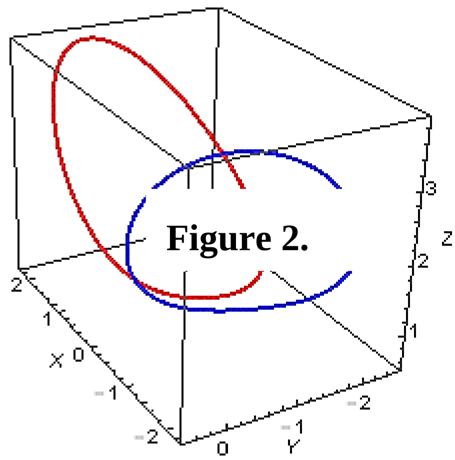

Figure 2.

with their initial point and approximate period (Figure 2) : $T_b = T_r \approx 5$.

A point on the closed trajectory $C_b(x_0, y_0, z_0) = (-0.18722, -1.10569, 0.65290)$.

A point on the closed trajectory C_r: $(x_0, y_0, z_0) = (0.18722, -1.10569, 0.65290)$.

The parallelepiped containing the closed trajectories:

$$\Pi=\{(x,y,z)\in R^3 \mid -1.609\le x\le 1.609, -2.479\le y\le 0.327, 0.54\le z\le 3.587\}$$

Any other general form of dynamical systems corresponding to the codes $[1012][1012][1012]$ is as follows:

$$\begin{cases} x'=\{\pm 1\}+\{\pm x^2, \pm y^2 \pm z^2\}+\{\pm xy \pm yz, \pm xz \pm yz, \pm xz \pm yz\} \\ y'=\{\pm 1\}+\{\pm x^2, \pm y^2 \pm z^2\}+\{\pm xy \pm yz, \pm xz \pm yz, \pm xz \pm yz\} \\ z'=\{\pm 1\}+\{\pm x^2, \pm y^2 \pm z^2\}+\{\pm xy \pm yz, \pm xz \pm yz, \pm xz \pm yz\} \end{cases}.$$

The number of such systems is $2^{12}\cdot 3^6=2985984$ and one of the systems having two closed trajectories among them is:

$$\begin{cases} x'=-1+y^2+xz-yz \\ y'=-1+z^2-xy+yz \\ z'=1-y^2-xy-xz \end{cases}$$

with their initial point and approximate period (Figure 3): $T_b=T_r\approx 9.3$

A point on the closed trajectory C_b:

$(x_0, y_0, z_0)=(-0.93685, -0.62400, -1.95723)$.

A point on the closed trajectory C_r:

$(x_0, y_0, z_0)=(0.93685, 0.62400, 1.95723)$.

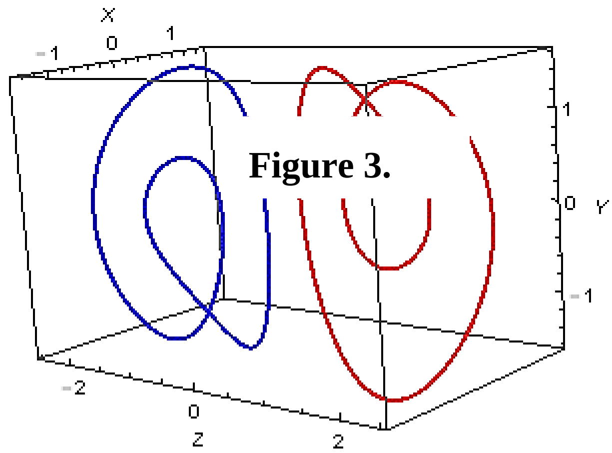

Figure 3.

The parallelepiped containing the closed trajectories:

$$\Pi=\{(x,y,z)\in R^3 \mid -0.937\le x\le 1.185, -1.246\le y\le 1.451, -2.474\le z\le 2.474\}$$

One more class of the systems corresponding to the codes $[1002][0201][0102]$ is:

$$\begin{cases} x'=\{\pm 1\}+\{\pm xy \pm yz, \pm xz \pm yz, \pm xz \pm yz\} \\ y'=\{\pm x \pm y, \pm x \pm z, \pm y \pm z\}+\{\pm xy, \pm xz, \pm yz\} \\ z'=\{\pm x, \pm y, \pm z\}+\{\pm xy \pm yz, \pm xz \pm yz, \pm xz \pm yz\} \end{cases}.$$

The number of such systems is $2^{10}\cdot 3^6=746496$ and one of the systems exhibiting two closed trajectories among them is:

$$\begin{cases} x' = -1 + xz + yz \\ y' = -x - z - xz \\ z' = z - xy - yz \end{cases}$$

with their initial point and approximate period (Figure 4): $T_b \approx 33.8; T_r \approx 4.1$.

A point on the closed trajectory C_b:

$(x_0, y_0, z_0) = (-0.31250, 1.98101, 0.59414)$.

A point on the closed trajectory C_r:

$(x_0, y_0, z_0) = (-1.01443, 2.17411, 1.52104)$.

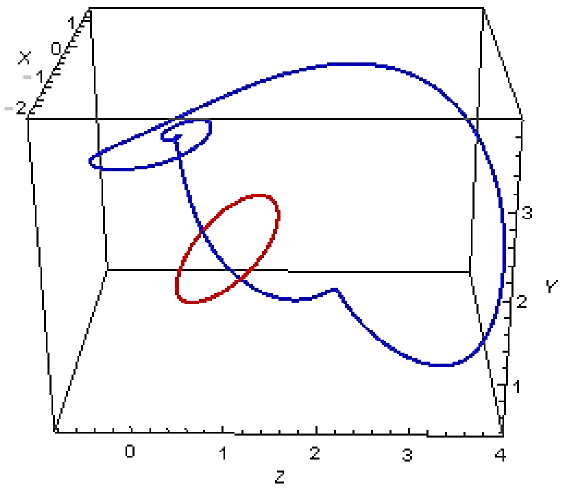

Figure 4.

The parallelepiped containing the closed trajectories:

$$\Pi = \{(x, y, z) \in R^3 | -1.015 \le x \le 0.752, 0.454 \le y \le 3.473, -0.491 \le z \le 3.189\}.$$

3. Conclusion

This study focused on quadratic dynamical systems with coefficients restricted to the set $\{-1, 0, 1\}$. We employed the Runge-Kutta method for approximate solutions of the differential equations and used Wolfram Mathematica to visualize systems exhibiting closed trajectories, specifying the interval in which these trajectories are located [8-10]. Specifically, we investigated quadratic dynamical systems and identified some of them with two closed trajectories. Determining the existence of closed trajectories is crucial in dynamical systems theory, as it allows for practical conclusions regarding the presence or absence of periodic behavior in the process modeled by the system.

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