Arzikulov Farhodjon Nematjonovich
Principal researcher of V.I.Romanovskiy Institute of Mathematics
of the Academy of Sciences of the Republic of Uzbekistan
e-mail: arzikulovfn@gmail.com

Qurbonov Abduqaxxor
Teacher of the Namangan State University
e-mail: qabduqaxxor@mail.ru

ON AUTOMORPHISMS OF FINITE-DIMENSIONAL NATURALLY GRADED ASSOCIATIVE ALGEBRAS

Abstract. In the present paper automorphisms of some five-dimensional naturally graded nilpotent associative algebras are studied. Namely, a general form of the matrices of automorphisms of these algebras is clarified. It turns out that the matrices of automorphisms of these algebras have the almost lower triangular matrix form.

Key words: associative algebra, automorphism, naturally graded algebra

Arziqulov Farxodjon Nematjonovich
O‘zbekiston Respublikasi Fanlar akademiyasi V.I.Romanovskiy
nomidagi matematika instituti bosh ilmiy xodimi.
e-mail: arzikulovfn@gmail.com

Qurbonov Abduqaxxor
Namangan davlat universiteti o‘qituvchisi
e-mail: qabduqaxxor@mail.ru

CHEKLI O’LCHAMLI TABIIY GRADUIRLANGAN ASSOTSIATIVE ALGEBRALARNING AVTOMORFIZMLARI HAQIDA

Annotatsiya. Ushbu maqolada ba'zi besh o'lchamli tabiiy graduirlangan nilpotent assotsiativ algebralarning avtomorfizmlari o'rganilgan. Aniqrog’i, shu algebralarning avtomorfizmlari matritsalarining umumiy shakli aniqlangan. Ma’lum bo‘lishicha, bu algebralardagi avtomorfizmlar matritsalari deyarli quyi uchburchakli matritsa ko’rinishiga ega.

Kalit so'zlar: assotsiativ algebra, avtomorfizm, tabiiy ravishda graduirlangan algebra.

Арзикулов Фарходжон Нематжонович
Главный научный сотрудник Математического института им.
В.И.Романовского Академии наук Республики Узбекистан
e-mail: arzikulovfn@gmail.com

Курбонов Абдукаххор

Преподаватель Наманганского государственного университета
e-mail: qabduqaxxor@mail.ru

ОБ АВТОМОРФИЗМАХ КОНЕЧНОМЕРНЫХ ЕСТЕСТВЕННЫМ ОБРАЗОМ ГРАДУИРОВАННЫХ АССОЦИАТИВНЫХ АЛГЕБР

Аннотация. В настоящей статье изучаются автоморфизмы некоторых пятимерных естественным образом градуированных нильпотентных ассоциативных алгебр. А именно, выясняется общий вид матриц автоморфизмов этих алгебр. Оказывается, матрицы автоморфизмов на этих алгебрах имеют почти нижний треугольный вид.

Ключевые слова: ассоциативная алгебра, автоморфизм, естественным образом градуированная алгебра.

Introduction

The study of automorphisms of associative algebras has deep roots in classical algebra and has evolved through interactions with many branches of mathematics, including ring theory, algebraic geometry, Lie theory, and mathematical physics. The concept of automorphism was first explored in the context of field extensions, leading to Galois Theory in the 19 th century. For associative algebras (particularly matrix algebras), understanding their automorphisms arose from the study of linear transformations and invariants. Wedderburn's Theorem (1908) characterized semisimple algebras as matrix rings over division rings, providing a structural foundation. Automorphisms of matrix algebras (e.g. $M_n(F)$) were then understood via inner automorphisms, thanks to the Skolem–Noether theorem. Skolem–Noether Theorem (1930s) describes automorphisms of central simple algebras: every automorphism is inner if the center is algebraically closed [4]. This theorem is crucial for understanding automorphisms in noncommutative settings. Nathan Jacobson's work [1] helped generalize structural theorems for algebras and understand derivations and automorphisms in noncommutative and nonassociative algebras. Automorphisms of noncommutative associative algebras such as Weyl algebras, quantum tori, and q-deformations became a central topic. Study of derivations and Poisson structures often goes hand-in-hand with automorphisms in deformation theory. In the 1970s–1980s, particularly through work of Shestakov and Umirbaev [2], distinctions were made between tame (composed of simple types) and wild automorphisms, especially for polynomial rings and free associative algebras [3]. These distinctions are essential in understanding the "complexity" of automorphism groups. Automorphisms of Hopf algebras and enveloping algebras are tied to Lie theory, quantum groups, and noncommutative geometry [5]. Classification results for automorphisms of certain classes of infinite-dimensional algebras have been achieved, especially in characteristic zero. Classification of automorphisms for specific classes of algebras, especially over fields of positive characteristic or for infinite-dimensional algebras, is ongoing.

In the present paper, automorphisms of five-dimensional naturally graded associative algebra with characteristic sequence $C(A)=(3,2)$ are studied. Namely, a general form of the matrices of automorphisms of these algebras is clarified. It turns out that the matrices of automorphisms of these algebras have the almost lower triangular matrix form.

1. Preliminaries

For an algebra A of an arbitrary variety, we consider the series

$$A^1, A^{i+1}=\sum_{k=1}^{i} A^k A^{i-k+1}, i\geq 1.$$

We say that an algebra A is *nilpotent* if $A^i=0$ for some $i\in N$. The smallest integer satisfying $A^i=0$ is called the index of nilpotency or nilindex of A.

An algebra A is called N-graded if there is a decomposition $A=\oplus_{i\in N} A_i$, $A_i A_j \subseteq A_{i+j}$, for all $i,j\in N$. Given an associative algebra A, one can consider its associated graded associative algebra $grA=\oplus_{i\in N}(A_i/A_{i+1})$ with the product

$$(x+A_{i+1})(y+A_{j+1})=xy+A_{i+j+1}, x\in A_i, y\in A_j.$$

Definition. Given a nilpotent associative algebra A, we have a natural grading on A induced by the series A_i, $A_i=A_i/A_{i+1}$, $1\leq i\leq k-1$, and $grA=A_1\oplus A_2\oplus\cdots\oplus A_k$. If grA and A are isomorphic, denoted by $grA\cong A$, we say that the algebra A is naturally graded. For any element x of A, we define the left multiplication operator as: $L_x: A\to A, z\to xz, z\in A$. For an element $x\in A\{A¿_2$, define the decreasing sequence $C(x)=(n_1,n_2,\dots,n_k)$ which consists of the dimensions of the Jordan blocks of the left multiplication operator L_x. Endow the set of these sequences with the lexicographic order, i.e., $C(x)=(n_1,n_2,\dots,n_k)\leq C(y)=(m_1,m_2,\dots,m_s)$, which means that there is an index $i\in N$, such that $n_j=m_j$ for all $j<m_i$.

Definition. The sequence $C(A)=max_{¿}¿C(x)$ is defined to be the characteristic sequence of the algebra A.

We assume throughout this section that F is an algebraically closed field of characteristic zero. Let A be a naturally graded n-dimensional quasi-filiform associative algebra. Then, there are two possibilities for the characteristic sequence, either $C(A)=(n-2,1,1)$ or $C(A)=(n-2,2)$.

The associative algebras

$$\pi_2:\begin{cases} e_1e_1=e_2, \\ e_1e_2=e_2e_1=e_3, \\ e_1e_4=e_4e_1=e_5, \\ e_4e_4=e_5 \end{cases} \pi_3:\begin{cases} e_1e_1=e_2, \\ e_1e_2=e_2e_1=e_3, \\ e_1e_4¿e_5, \\ e_4e_4=e_5 \end{cases}$$

are five-dimensional naturally graded associative algebras with characteristic sequence $C(A)=(3,2)$ (see Theorem 4.5 in Karimjonov-Ladra).

2. Automorphisms of five-dimensional naturally graded associative algebra with characteristic sequence $C(A)=(3,2)$

Now we give a description of automorphisms of the associative algebras π_2 and π_3.

Definition. Let A be an algebra. A bijective linear map $\Phi: A \to A$ is called an automorphism, if, for any elements $x, y \in A$, $\Phi(xy)=\Phi(x)\Phi(y)$.

Theorem 1. *A linear operator on the associative algebra π_2 is an automorphism if and only if the matrix of this linear operator has the following matrix form*

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{11}^2 & 0 & 0 & 0 \\ \alpha_{31} & 2\alpha_{11}\alpha_{21} & \alpha_{11}^3 & \alpha_{34} & 0 \\ \alpha_{41} & 0 & 0 & \alpha_{11}+\alpha_{41} & 0 \\ \alpha_{51} & 2\alpha_{11}\alpha_{41}+\alpha_{41}^2 & 0 & \alpha_{54} & (\alpha_{11}+\alpha_{41})^2 \end{pmatrix}, \qquad (1)$$

where $\alpha_{11}\neq 0$, $\alpha_{11}\neq -\alpha_{41}$.

Proof. Let ϕ be an automorphism on π_2 and $A=¿$ be its matrix. Then we compute the components of the matrix $A=¿$.

We have $\phi(e_1)=a_{1,1}e_1+a_{2,1}e_2+a_{3,1}e_3+a_{4,1}e_4+a_{5,1}e_5$, $\phi(e_2)=a_{1,2}e_1+a_{2,2}e_2+a_{3,2}e_3+a_{4,2}e_4+a_{5,2}e_5$.

Also, we have

$$\phi(e_1^2)=\phi(e_1)\phi(e_1)=¿$$

$$¿\, a_{1,1}^2 e_2+2a_{1,1}a_{2,1}e_3+2a_{1,1}a_{4,1}e_5+a_{4,1}^2 e_5.$$

At the same time,

$$\phi(e_1^2)=\phi(e_2)=a_{1,2}e_1+a_{2,2}e_2+a_{3,2}e_3+a_{4,2}e_4+a_{5,2}e_5.$$

Hence,

$$a_{1,2}=0,\ a_{2,2}=a_{1,1}^2,\ a_{3,2}=2a_{1,1}a_{2,1},\ a_{4,2}=0,\ a_{5,2}=2a_{1,1}a_{4,1}+a_{4,1}^2.$$

Similarly, from $\phi(e_3)=\phi(e_1)\phi(e_2)$ it follows that

$$a_{1,3}=0,\ a_{2,3}=a_{1,1}a_{1,2},\ a_{3,3}=a_{1,1}a_{2,2}+a_{1,2}a_{2,1},\ a_{4,3}=0,\ a_{5,3}=a_{1,1}a_{4,2}+a_{4,1}a_{1,2}+a_{4,1}a_{4,2},$$

from $\phi(e_5)=\phi(e_1)\phi(e_4)$ it follows that

$$a_{1,5}=0,\ a_{2,5}=a_{1,1}a_{1,4},\ a_{3,5}=a_{1,1}a_{2,4}+a_{1,4}a_{2,1},\ a_{4,5}=0,\ a_{5,5}=a_{1,1}a_{4,4}+a_{4,1}a_{1,4}+a_{4,1}a_{4,4},$$

from $\phi(e_5)=\phi(e_4)\phi(e_4)$ it follows that

$$a_{1,5}=0,\ a_{2,5}=a_{1,4}^2,\ a_{3,5}=2a_{1,4}a_{2,4},\ a_{4,5}=0,\ a_{5,5}=2a_{1,4}a_{4,4}+a_{4,4}^2,$$

from $\phi(e_1)\phi(e_3)=0$ it follows that

$$a_{1,1}a_{1,3}=0,\ a_{1,1}a_{2,3}+a_{1,3}a_{2,1}=0,\ a_{1,1}a_{4,3}+a_{4,1}a_{1,3}+a_{4,1}a_{4,3}=0,$$

from $\phi(e_1)\phi(e_5)=0$ it follows that

$$a_{1,1}a_{1,5}=0,\ a_{1,1}a_{2,5}+a_{1,5}a_{2,1}=0,\ a_{1,1}a_{4,5}+a_{4,1}a_{1,5}+a_{4,1}a_{4,5}=0,$$

from $\phi(e_2)\phi(e_2)=0$ it follows that

$$a_{1,2}=0,\ a_{1,2}a_{2,2}=0{,}2a_{1,2}a_{4,2}+a_{4,2}^2=0,$$

from $\phi(e_2)\phi(e_3)=0$ it follows that

$$a_{1,2}a_{1,3}=0,\ a_{1,2}a_{2,3}+a_{1,3}a_{2,2}=0,\ a_{1,2}a_{4,3}+a_{4,2}a_{1,3}+a_{4,2}a_{4,3}=0,$$

from $\phi(e_2)\phi(e_4)=0$ it follows that

$$a_{1,2}a_{1,4}=0, a_{1,2}a_{2,4}+a_{1,4}a_{2,2}=0, a_{1,2}a_{4,4}+a_{4,2}a_{1,4}+a_{4,2}a_{4,4}=0,$$

from $\phi(e_2)\phi(e_5)=0$ it follows that

$$a_{1,2}a_{1,5}=0, a_{1,2}a_{2,5}+a_{1,5}a_{2,2}=0, a_{1,2}a_{4,5}+a_{4,2}a_{1,5}+a_{4,2}a_{4,5}=0,$$

from $\phi(e_3)\phi(e_3)=0$ it follows that

$$a_{1,3}=0, a_{1,3}a_{2,3}=0, 2a_{1,3}a_{4,3}+a_{4,3}^2=0,$$

from $\phi(e_3)\phi(e_4)=0$ it follows that

$$a_{1,3}a_{1,4}=0, a_{1,3}a_{2,4}+a_{1,4}a_{2,3}=0, a_{1,3}a_{4,4}+a_{4,3}a_{1,4}+a_{4,3}a_{4,4}=0,$$

from $\phi(e_3)\phi(e_5)=0$ it follows that

$$a_{1,3}a_{1,5}=0, a_{1,3}a_{2,5}+a_{1,5}a_{2,3}=0, a_{1,3}a_{4,5}+a_{4,3}a_{1,5}+a_{4,3}a_{4,5}=0,$$

from $\phi(e_4)\phi(e_5)=0$ it follows that

$$a_{1,4}a_{1,5}=0, a_{1,4}a_{2,5}+a_{1,5}a_{2,4}=0, a_{1,4}a_{4,5}+a_{4,4}a_{1,5}+a_{4,4}a_{4,5}=0,$$

from $\phi(e_5)\phi(e_5)=0$ it follows that

$$a_{1,5}=0, a_{1,5}a_{2,5}=0, 2a_{1,5}a_{4,5}+a_{4,5}^2=0.$$

Additionally, from $a_{5,5}=a_{1,1}a_{4,4}+a_{4,1}a_{1,4}+a_{4,1}a_{4,4}$, $a_{5,5}=2a_{1,4}a_{4,4}+a_{4,4}^2$ it follows that $a_{5,5}=a_{1,1}a_{4,4}+a_{4,1}a_{4,4}=a_{4,4}^2$. Since $a_{5,5}\neq 0$, we have $a_{4,4}\neq 0$. Hence, from $a_{1,1}a_{4,4}+a_{4,1}a_{4,4}=a_{4,4}^2$ it follows that $a_{1,1}+a_{4,1}=a_{4,4}$. So, $a_{4,4}=a_{1,1}+a_{4,1}$ and $a_{5,5}=¿$. This ends the proof.

The following theorem is proven similar to the proof of Theorem 1.

Theorem 2. *A linear operator on the associative algebra $\pi_3(\alpha)$ is an automorphism if and only if the matrix of this linear operator has the following matrix form*

¿ (2)

where $\alpha_{11}\neq 0$.

Proof. Let ϕ be an automorphism on π_3 and $A=¿$ be its matrix. We compute the components of the matrix $A=¿$. We have $\phi(e_1)=a_{1,1}e_1+a_{2,1}e_2+a_{3,1}e_3+a_{4,1}e_4+a_{5,1}e_5$, $\phi(e_2)=a_{1,2}e_1+a_{2,2}e_2+a_{3,2}e_3+a_{4,2}e_4+a_{5,2}e_5$. Also, we have

$$\phi(e_1^2)=\phi(e_1)\phi(e_1)=¿$$

$$¿a_{1,1}^2e_2+2a_{1,1}a_{2,1}e_3+a_{1,1}a_{4,1}e_5+a_{4,1}^2e_5.$$

At the same time,

$$\phi(e_1^2)=\phi(e_2)=a_{1,2}e_1+a_{2,2}e_2+a_{3,2}e_3+a_{4,2}e_4+a_{5,2}e_5.$$

Hence,

$$a_{1,2}=0, a_{2,2}=a_{1,1}^2, a_{3,2}=2a_{1,1}a_{2,1}, a_{4,2}=0, a_{5,2}=a_{1,1}a_{4,1}+a_{4,1}^2.$$

We have $\phi(e_3)=a_{1,3}e_1+a_{2,3}e_2+a_{3,3}e_3+a_{4,3}e_4+a_{5,3}e_5$. Also, we have

$$\phi(e_1e_2)=\phi(e_1)\phi(e_2)$$

$$¿(a_{1,1}e_1+a_{2,1}e_2+a_{3,1}e_3+a_{4,1}e_4+a_{5,1}e_5)(a_{1,2}e_1+a_{2,2}e_2+a_{3,2}e_3+a_{4,2}e_4+a_{5,2}e_5)$$

$$¿a_{1,1}a_{1,2}e_2+(a_{1,1}a_{2,2}+a_{1,2}a_{2,1})e_3+a_{1,1}a_{4,2}e_5+a_{4,1}a_{4,2}e_5.$$

At the same time,

$$\phi(e_1e_2)=\phi(e_3)=a_{1,3}e_1+a_{2,3}e_2+a_{3,3}e_3+a_{4,3}e_4+a_{5,3}e_5.$$

Hence,

$$a_{1,3}=0, a_{2,3}=a_{1,1}a_{1,2}, a_{3,3}=a_{1,1}a_{2,2}+a_{1,2}a_{2,1}, a_{4,3}=0, a_{5,3}=a_{1,1}a_{4,2}+a_{4,1}a_{4,2}.$$

Similarly, from

$$\phi(e_2 e_1)=\phi(e_2)\phi(e_1)=\phi(e_3)$$
it follows that
$$a_{1,3}=0, a_{2,3}=a_{1,1}a_{1,2}, a_{3,3}=a_{1,1}a_{2,2}+a_{1,2}a_{2,1}, a_{4,3}=0, a_{5,3}=a_{1,2}a_{4,1}+a_{4,1}a_{4,2},$$
from
$$\phi(e_1 e_4)=\phi(e_1)\phi(e_4)=\phi(e_5)$$
it follows that
$$a_{1,5}=0, a_{2,5}=a_{1,1}a_{1,4}, a_{3,5}=a_{1,1}a_{2,4}+a_{1,4}a_{2,1}, a_{4,5}=0, a_{5,5}=a_{1,1}a_{4,4}+a_{4,1}a_{4,4}.$$
Since the table of the multiplication of π_2 is not symmetric, we have to use $\phi(e_4 e_1)=\phi(0)=0$. Similarly, from
$$\phi(e_4 e_1)=\phi(e_4)\phi(e_1)=\phi(0)=0$$
it follows that
$$a_{1,4}a_{1,1}=0, a_{1,4}a_{2,1}+a_{2,4}a_{1,1}=0, a_{1,4}a_{4,1}+a_{4,1}a_{4,4}=0,$$
from
$$\phi(e_4^2)=\phi(e_4)\phi(e_4)=\phi(e_5)$$
it follows that
$$a_{1,5}=0, a_{2,5}=a_{1,4}^2, a_{3,5}=2a_{1,4}a_{2,4}, a_{4,5}=0, a_{5,5}=a_{1,4}a_{4,4}+a_{4,4}^2,$$
from
$$\phi(e_1 e_3)=\phi(e_1)\phi(e_3)=\phi(0)=0$$
it follows that
$$a_{1,1}a_{1,3}=0, a_{1,1}a_{2,3}+a_{1,3}a_{2,1}=0, a_{1,1}a_{4,3}+a_{4,1}a_{4,3}=0,$$
Since $a_{1,1}\neq 0$, we have $a_{1,3}=0$, $a_{2,3}=0$.

From
$$\phi(e_1 e_5)=\phi(e_1)\phi(e_5)=\phi(0)=0.$$
it follows that
$$a_{1,1}a_{1,5}=0, a_{1,1}a_{2,5}+a_{1,5}a_{2,1}=0, a_{1,1}a_{4,5}+a_{4,1}a_{4,5}=0.$$
Since $a_{1,1}\neq 0$, we have $a_{2,5}=0$.

From
$$\phi(e_2^2)=\phi(e_2)\phi(e_2)=\phi(0)=0$$
it follows that
$$a_{1,2}=0, a_{1,2}a_{2,2}=0, a_{1,2}a_{4,2}+a_{4,2}^2=0,$$
from
$$\phi(e_2 e_3)=\phi(e_2)\phi(e_3)=\phi(0)=0$$
it follows that
$$a_{1,2}a_{1,3}=0, a_{1,2}a_{2,3}+a_{1,3}a_{2,2}=0, a_{1,2}a_{4,3}+a_{4,2}a_{4,3}=0,$$
from
$$\phi(e_2 e_4)=\phi(e_2)\phi(e_4)=\phi(0)=0$$
it follows that
$$a_{1,2}a_{1,4}=0, a_{1,2}a_{2,4}+a_{1,4}a_{2,2}=0, a_{1,2}a_{4,4}+a_{4,2}a_{4,4}=0.$$
Since $a_{2,2}\neq 0$, we have $a_{1,4}=0$.

Similarly, from
$$\phi(e_2 e_5)=\phi(e_2)\phi(e_5)=\phi(0)=0$$
it follows that
$$a_{1,2}a_{1,5}=0, a_{1,2}a_{2,5}+a_{1,5}a_{2,2}=0, a_{1,2}a_{4,5}+a_{4,2}a_{4,5}=0.$$
Since $a_{2,2}\neq 0$, we have $a_{1,4}=0$.

We have

$$\phi(e_3^2)=\phi(e_3)\phi(e_3)=\text{¿}$$
$$\text{¿}a_{1,3}^2 e_2+2a_{1,3}a_{2,3}e_3+a_{1,3}a_{4,3}e_5+a_{4,3}^2 e_5.$$

At the same time,

$$\phi(e_3^2)=\phi(0)=0.$$

Hence,

$$a_{1,3}=0,a_{1,3}a_{2,3}=0\,a_{1,3}a_{4,3}+a_{4,3}^2=0.$$

We have

$$\phi(e_3e_4)=\phi(e_3)\phi(e_4)$$
$$\text{¿}(a_{1,3}e_1+a_{2,3}e_2+a_{3,3}e_3+a_{4,3}e_4+a_{5,3}e_5)(a_{1,4}e_1+a_{2,4}e_2+a_{3,4}e_3+a_{4,4}e_4+a_{5,4}e_5)$$

$$\text{¿}a_{1,3}a_{1,4}e_2+(a_{1,3}a_{2,4}+a_{1,4}a_{2,3})e_3+a_{1,3}a_{4,4}e_5+a_{4,3}a_{4,4}e_5.$$

At the same time,

$$\phi(e_3e_4)=\phi(0)=0.$$

Hence,

$$a_{1,3}a_{1,4}=0,a_{1,3}a_{2,4}+a_{1,4}a_{2,3}=0,a_{1,3}a_{4,4}+a_{4,3}a_{4,4}=0.$$

Since $a_{2,2}\neq 0$, we have $a_{1,4}=0$.

We have

$$\phi(e_3e_5)=\phi(e_3)\phi(e_5)$$
$$\text{¿}(a_{1,3}e_1+a_{2,3}e_2+a_{3,3}e_3+a_{4,3}e_4+a_{5,3}e_5)(a_{1,5}e_1+a_{2,5}e_2+a_{3,5}e_3+a_{4,5}e_4+a_{5,5}e_5)$$
$$\text{¿}a_{1,3}a_{1,5}e_2+(a_{1,3}a_{2,5}+a_{1,5}a_{2,3})e_3+a_{1,3}a_{4,5}e_5+a_{4,3}a_{4,5}e_5.$$

At the same time,

$$\phi(e_3e_5)=\phi(0)=0.$$

Hence,

$$a_{1,3}a_{1,5}=0,a_{1,3}a_{2,5}+a_{1,5}a_{2,3}=0,a_{1,3}a_{4,5}+a_{4,3}a_{4,5}=0.$$

Since $a_{2,2}\neq 0$, we have $a_{1,4}=0$.

We have

$$\phi(e_4e_5)=\phi(e_4)\phi(e_5)$$
$$\text{¿}(a_{1,4}e_1+a_{2,4}e_2+a_{3,4}e_3+a_{4,4}e_4+a_{5,4}e_5)(a_{1,5}e_1+a_{2,5}e_2+a_{3,5}e_3+a_{4,5}e_4+a_{5,5}e_5)$$
$$\text{¿}a_{1,4}a_{1,5}e_2+(a_{1,4}a_{2,5}+a_{1,5}a_{2,4})e_3+a_{1,4}a_{4,5}e_5+a_{4,4}a_{4,5}e_5.$$

At the same time,

$$\phi(e_4e_5)=\phi(0)=0.$$

Hence,

$$a_{1,4}a_{1,5}=0,a_{1,4}a_{2,5}+a_{1,5}a_{2,4}=0,a_{1,4}a_{4,5}+a_{4,4}a_{4,5}=0.$$

Since $a_{2,2}\neq 0$, we have $a_{1,4}=0$.

We have

$$\phi(e_5^2)=\phi(e_5)\phi(e_5)$$
$$\text{¿¿}$$

$$\text{¿}a_{1,5}^2 e_2+2a_{1,5}a_{2,5}e_3+a_{1,5}a_{4,5}e_5+a_{4,5}^2 e_5.$$

At the same time,

$$\phi(e_5^2)=\phi(0)=0.$$

Hence,

$$a_{1,5}=0,a_{1,5}a_{2,5}=0,a_{1,5}a_{4,5}+a_{4,5}^2=0.$$

Since $a_{2,2}\neq 0$, we have $a_{1,4}=0$.

From $a_{1,1}a_{2,5}+a_{1,5}a_{2,1}=0$ it follows that $a_{2,5}=0$. We have $a_{1,4}=0$ since $a_{2,5}=a_{1,4}^2$ and $a_{2,5}=0$. We also have $a_{3,5}=2a_{1,4}a_{2,4}=a_{1,1}a_{2,4}$. Hence, $a_{3,5}=0$, $a_{2,4}=0$ since $a_{1,4}=0$.

From $a_{1,4}a_{4,1}+a_{4,1}a_{4,4}=0$ it follows that $a_{4,1}a_{4,4}=0$. Since $a_{4,4}\neq 0$, we have $a_{4,1}=0$.

So,

¿.

This ends the proof.

4. Conclusion

In the present paper a general form of the matrices of automorphisms of these algebras is clarified. It turns out that the matrices of automorphisms of these algebras have the almost lower triangular matrix form.

References

1. N. Jacobson. Structure of Rings. AMS Colloquium Publications, 1956.
2. I. Shestakov, U. Umirbaev. The tame and wild automorphisms of polynomial rings in three variables. Journal of the American Mathematical Society, 2004, 17(1), 197-227.
3. U. Umirbaev. Defining relations of the tame automorphism group of polynomial algebras in three variables. Journal für die reine und angewandte Mathematik, 2009, 2006(600), 203-235.
4. O. Ore. Structures and group theory in ring theory. Duke Mathematical Journal, 1935.
5. K.A. Brown and K.R. Goodearl. Lectures on Algebraic Quantum Groups. Birkhäuser, 2002.