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BESHINCHI TARTIBlI XUSUSIY HOSILALI TENGLAMA UCHUN ARALASH MASALA

Annotatsiya: *Maqolada beshinchi tartibli chiziqli xususiy hosilali tenglama Dirixle sharti asosida o‘rganilgan. Furye usuli bilan spektral masala yechilib, integral tenglamalar qatori hosil qilinadi. Qatorning yagona yechimga ega ekanligi isbotlanadi. Ketma-ket yaqinlashish va qisqartirib aks ettirish usullari qo‘llaniladi. Aralash masalaning yechimi Furye qatori shaklida topilib, uning yaqinlashishi asoslangan.*

Kalit so‘zlar: *Aralash masala, beshinchi tartibli tenglama, yagona yechimga ega bo‘lish.*

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СМЕШАННАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ПЯТОГО ПОРЯДКА

Аннотация: *В статье исследуется линейное частное дифференциальное уравнение пятого порядка с граничным условием Дирихле. С помощью метода Фурье решается спектральная задача и получен ряд интегральных уравнений. Доказывается существование единственного решения этого ряда. Применяются методы последовательных приближений и редукции. Решение смешанной задачи найдено в виде ряда Фурье, обоснована его сходимость.*

Ключевые слова: *Смешанная задача, уравнение пятого порядка, единственность решения.*

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MIXED PROBLEM FOR A FIFTH ORDER PARTIAL DIFFERENTIAL EQUATIONS

Abstract: *The article investigates a linear partial differential equation of the fifth order with Dirichlet boundary conditions. Using the Fourier method, a spectral problem is solved, and a series of integral equations is obtained. The existence of a unique solution to this series is proven. Methods of successive approximations and reduction are applied. The solution to the mixed problem is found in the form of a Fourier series, and its convergence is justified.*

Key words: *Mixed problem, fifth order equation, unique solvability.*

Formulation of the problem statement

In this paper, we study the solvability of the mixed problem for a fifth order differential equation. In the rectangular domain, $\Omega = \{0 < t < T,\ 0 < x < 1\}$ we consider the following partial differential equation:

$$\left[\frac{\partial^3}{\partial t^3} - \frac{\partial^5}{\partial t^3 \partial x^2} - \frac{\partial^2}{\partial x^2}\right] U(t,x) = a(t) U(t,x), \tag{1}$$

where $0 < a(t) \in C[0,T]$, T is given positive number.

In solving partial differential equation (1), with respect to spatial variable x we use Dirichlet type boundary value conditions

$$U(t,0) = U(t,1) = 0, \quad 0 \le t \le T. \tag{2}$$

With respect to time variable t we use initial value conditions

$$U(0,x) = \varphi_1(x), \ U_t(0,x) = \varphi_2(x), \ U_{tt}(0,x) = \varphi_3(x), \quad 0 \le x \le 1, \tag{3}$$

where $\varphi_i(x)$ is enough smooth on the segment $[0,1]$. For this function the following conditions are fulfilled $\varphi_i(0) = \varphi_i(1) = 0, \ i = 1,2,3.$

Problem statement. To find a function $U(t,x) \in C(\overline{\Omega}) \cap C^{3,2}_{t,x}(\Omega)$, which satisfies differential equation (1) and the specified conditions (2) and (3), where $\overline{\Omega} = \{ 0 \le t \le T, \ 0 \le x \le 1 \}$.

Formal solution of the problem (1)-(3)

The solution of the problem (1)-(3) we search in the form of Fourier series [1-4]

$$U(t,x) = \sum_{n=1}^{\infty} u_n(t) \vartheta_n(x), \tag{4}$$

where $\vartheta_n(x) = \sin\sqrt{\lambda_n}\, x, \ \lambda_n = (n\pi)^2$

$$u_n(t) = \int_0^1 U(t,x)\, \vartheta_n(x)\, dx. \tag{5}$$

Substituting the Fourier series (4) into the given differential equation (1), we obtain a countable system of third order ordinary differential equations

$$u_n'''(t) + \mu_n u_n(t) = \frac{1}{1+\lambda_n} a(t) u_n(t), \quad \mu_n = \frac{\lambda_n}{1+\lambda_n}. \tag{6}$$

The equation (6) we rewrite as

$$u_n'''(t) + \mu_n u_n(t) = f_n(t), \tag{7}$$

where

$$f_n(t) = \frac{1}{1+\lambda_n} a(t) u_n(t). \tag{8}$$

The characteristic equation $k^3 + \mu_n = 0$ for the homogeneous equation $u_n'''(t) + \mu_n u_n(t) = 0$ has the roots

$$k_1 = -\sqrt[3]{\mu_n}, \quad k_{2/3} = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right) \sqrt[3]{\mu_n}.$$

So, the general solution of the homogeneous equation presents as

$$u_n(t) = A_{1,n} u_{1,n}(t) + A_{2,n} u_{2,n}(t) + A_{3,n} u_{3,n}(t),$$

where A_{in} are yet arbitrary coefficients, which will be determined,

$$u_{1,n}(t) = e^{-\sqrt[3]{\mu_n} t}, \quad u_{2,n}(t) = e^{\frac{\sqrt[3]{\mu_n} t}{2}} \cos \frac{\sqrt{3}}{2} \sqrt[3]{\mu_n} t, \quad u_{3,n}(t) = e^{\frac{\sqrt[3]{\mu_n} t}{2}} \sin \frac{\sqrt{3}}{2} \sqrt[3]{\mu_n} t.$$

Denoting $2b_n = \sqrt[3]{\mu_n}$ we can write the set of fundamental solutions of the above equation in the form

$$u_{1,n}(t) = e^{-2b_n t}, \quad u_{2,n}(t) = e^{b_n t} \cos\left(\sqrt{3} b_n t\right), \quad u_{3,n}(t) = e^{b_n t} \sin\left(\sqrt{3} b_n t\right)$$

We will differentiate each of these functions two times

$$u_{1,n}'(t) = -2b_n u_{1,n}(t), \quad u_{1,n}''(t) = 4b_n^2 u_{1,n}(t);$$

$$\begin{pmatrix} u_{2,n}(t) \\ u_{3,n}(t) \end{pmatrix}' = \begin{pmatrix} b_n e^{b_n t} \cos(\sqrt{3} b_n t) - \sqrt{3} b_n e^{b_n t} \sin(\sqrt{3} b_n t) \\ b_n e^{b_n t} \sin(\sqrt{3} b_n t) + \sqrt{3} b_n e^{b_n t} \cos(\sqrt{3} b_n t) \end{pmatrix} = b_n \begin{pmatrix} u_{2,n}(t) - \sqrt{3} u_{3,n}(t) \\ \sqrt{3} u_{2,n}(t) + u_{3,n}(t) \end{pmatrix} =$$

$$= b_n \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} u_{2,n}(t) \\ u_{3,n}(t) \end{pmatrix} \Rightarrow \left[\begin{array}{l} u_{2,n}'(t) = b_n u_{2,n}(t) - \sqrt{3} b_n u_{3,n}(t) \\ u_{3,n}'(t) = \sqrt{3} b_n u_{2,n}(t) + b_n u_{3,n}(t) \end{array} \right.$$

$$\begin{pmatrix} u_{2,n}(t) \\ u_{3,n}(t) \end{pmatrix}'' = \left(\begin{pmatrix} u_{2,n}(t) \\ u_{3,n}(t) \end{pmatrix}' \right)' = b_n^2 \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}^2 \begin{pmatrix} u_{2,n}(t) \\ u_{3,n}(t) \end{pmatrix} =$$

$$= b_n^2 \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} \begin{pmatrix} u_{2,n}(t) \\ u_{3,n}(t) \end{pmatrix} \Rightarrow \left[\begin{array}{l} u_{2,n}''(t) = -2b_n^2 u_{2,n}(t) - 2\sqrt{3} b_n^2 u_{3,n}(t) \\ u_{3,n}''(t) = 2\sqrt{3} b_n^2 u_{2,n}(t) - 2b_n^2 u_{3,n}(t) \end{array} \right.$$

We search a special solution of the equation (7) as

$$\tilde{u}_n(t) = A_{1,n}(t)u_{1,n}(t) + A_{2,n}(t)u_{2,n}(t) + A_{3,n}(t)u_{3,n}(t). \tag{9}$$

Here we supposed that $A_{i,n}(t)$ are unknown functions. To find these functions we consider the following system of algebraic-differential equations

$$\begin{cases} A'_{1,n}(t)u_{1,n}(t) + A'_{2,n}(t)u_{2,n}(t) + A'_{3,n}(t)u_{3,n}(t) = 0, \\ A'_{1,n}(t)u'_{1,n}(t) + A'_{2,n}(t)u'_{2,n}(t) + A'_{3,n}(t)u'_{3,n}(t) = 0, \\ A'_{1,n}(t)u''_{1,n}(t) + A'_{2,n}(t)u''_{2,n}(t) + A'_{3,n}(t)u''_{3,n}(t) = f_n(t). \end{cases}$$

First, we solve the system as algebraic-functional equations by the Cramer rule:

$$W[u_{1,n}, u_{2,n}, u_{3,n}] = \begin{vmatrix} u_{1,n}(t) & u_{2,n}(t) & u_{3,n}(t) \\ u'_{1,n}(t) & u'_{2,n}(t) & u'_{3,n}(t) \\ u''_{1,n}(t) & u''_{2,n}(t) & u''_{3,n}(t) \end{vmatrix} =$$

$$= u_{1,n}\begin{vmatrix} u'_{2,n} & u'_{3,n} \\ u''_{2,n} & u''_{3,n} \end{vmatrix} - u'_{1,n}\begin{vmatrix} u_{2,n} & u_{3,n} \\ u''_{2,n} & u''_{3,n} \end{vmatrix} + u''_{1,n}\begin{vmatrix} u_{2,n} & u_{3,n} \\ u'_{2,n} & u'_{3,n} \end{vmatrix} =$$

$$= \left(b_n u_{1n} u_{2n} - \sqrt{3} b_n u_{1n} u_{3n}\right)\left(2\sqrt{3} b_n^2 u_{2n} - 2b_n^2 u_{3n}\right) +$$

$$+ 2b_n u_{1n} u_{3n}\left(2b_n^2 u_{2n} + 2\sqrt{3} b_n^2 u_{3n}\right) + 4b_n^2 u_{1n} u_{2n}\left(b_n\sqrt{3} u_{2n} + b_n u_{3n}\right) -$$

$$- 4b_n^2 u_{1n} u_{3n}\left(b_n u_{2n} - \sqrt{3} b_n u_{3n}\right) + \left(2b_n^2 u_{1n} u_{2n} + 2\sqrt{3} b_n^2 u_{1n} u_{3n}\right)\left(\sqrt{3} b_n u_{2n} + b_n u_{3n}\right) -$$

$$- 4b_n^2 u_{1n} u_{2n}\left(\sqrt{3} b_n u_{2n} + b_n u_{3n}\right) = 2\sqrt{3} b_n^3 u_{1n} u_{2n}^2 - 2b_n^3 u_{1n} u_{2n} u_{3n} - 6b_n^3 u_{1n} u_{2n} u_{3n} +$$

$$+ 2\sqrt{3} b_n^3 u_{1n} u_{3n}^2 + 4b_n^3 u_{1n} u_{2n} u_{3n} + 4\sqrt{3} b_n^3 u_{1n} u_{3n}^2 + 4\sqrt{3} b_n^3 u_{1n} u_{2n}^2 + 4b_n^3 u_{1n} u_{2n} u_{3n} -$$

$$- 4b_n^3 u_{1n} u_{2n} u_{3n} + 4\sqrt{3} b_n^3 u_{1n} u_{3n}^2 + 2\sqrt{3} b_n^3 u_{1n} u_{2n}^2 + 2b_n^3 u_{1n} u_{2n} u_{3n} + 6b_n^3 u_{1n} u_{2n} u_{3n} +$$

$$+ 2\sqrt{3} b_n^3 u_{1n} u_{3n}^2 - 4\sqrt{3} b_n^3 u_{1n} u_{2n}^2 - 4b_n^3 u_{1n} u_{2n} u_{3n} = 12\sqrt{3} b_n^3$$

$$\Delta_{A'_{1,n}} = \begin{vmatrix} 0 & u_{2,n}(t) & u_{3,n}(t) \\ 0 & u'_{2,n}(t) & u'_{3,n}(t) \\ f_n(t) & u''_{2,n}(t) & u''_{3,n}(t) \end{vmatrix} = \left[u_{2,n}(t)u'_{3,n}(t) - u_{3,n}(t)u'_{2,n}(t)\right] f_n(t) =$$

$$= \left(u_{2n}\left(\sqrt{3} b_n u_{2n} + b_n u_{3n}\right) - u_{3n}\left(b_n u_{2n} - \sqrt{3} b_n u_{3n}\right)\right) f_n =$$

$$= \left(\sqrt{3} b_n u_{2n}^2 + b_n u_{2n} u_{3n} - b_n u_{2n} u_{3n} + \sqrt{3} b_n u_{3n}^2\right) f_n =$$

$$=\sqrt{3}b_n\left(u_{2n}^2+u_{3n}^2\right)=\sqrt{3}b_n e^{2b_n t} f_n=\frac{\sqrt{3}b_n}{u_{1n}} f_n$$

$$\Delta_{A'_{2,n}}=\begin{vmatrix} u_{1,n}(t) & 0 & u_{3,n}(t) \\ u'_{1,n}(t) & 0 & u'_{3,n}(t) \\ u''_{1,n}(t) & f_n(t) & u''_{3,n}(t) \end{vmatrix}=\left[-u_{1,n}(t)u'_{3,n}(t)+u_{3,n}(t)u'_{1,n}(t)\right] f_n(t)=$$

$$=\left(-2b_n u_{1n}u_{3n}-\sqrt{3}b_n u_{1n}u_{2n}-b_n u_{1n}u_{3n}\right) f_n=\left(-3b_n u_{1n}u_{3n}-\sqrt{3}b_n u_{1n}u_{2n}\right) f_n$$

$$\Delta_{A'_{3,n}}=\begin{vmatrix} u_{1,n}(t) & u_{2,n}(t) & 0 \\ u'_{1,n}(t) & u'_{2,n}(t) & 0 \\ u''_{1,n}(t) & u''_{2,n}(t) & f_n(t) \end{vmatrix}=\left[u_{1,n}(t)u'_{2,n}(t)-u_{2,n}(t)u'_{1,n}(t)\right] f_n(t)=$$

$$=\left(b_n u_{1n}u_{2n}-\sqrt{3}b_n u_{1n}u_{3n}+2b_n u_{1n}u_{2n}\right) f_n=\left(3b_n u_{1n}u_{2n}-\sqrt{3}b_n u_{1n}u_{3n}\right) f_n$$

Then we have

$$\begin{cases} A'_{1n}=\dfrac{\Delta_{A'_{1n}}}{\Delta}=\dfrac{1}{12b_n^2 u_{1n}} f_n \\ A'_{2n}=\dfrac{\Delta_{A'_{2n}}}{\Delta}=-\dfrac{1}{12b_n^2}\left(u_{1n}u_{2n}+\sqrt{3}u_{1n}u_{3n}\right) f_n \\ A'_{3n}=\dfrac{\Delta_{A'_{3n}}}{\Delta}=\dfrac{1}{12b_n^2}\left(\sqrt{3}u_{1n}u_{2n}-u_{1n}u_{3n}\right) f_n. \end{cases}$$

Hence, by the aid of integration we obtain

$$A_{1,n}(t)=\frac{1}{12b_n^2}\int_0^t \frac{1}{u_{1,n}(s)} f_n(s)ds \tag{10}$$

$$A_{2,n}(t)=-\frac{1}{12b_n^2}\int_0^t \left(u_{1,n}(s)u_{2,n}+\sqrt{3}u_{1,n}(s)u_{3,n}(s)\right) f_n(s)ds \tag{11}$$

$$A_{3,n}(t)=\frac{1}{12b_n^2}\int_0^t \left(\sqrt{3}u_{1,n}(s)u_{2,n}(s)-u_{1,n}(s)u_{3,n}(s)\right) f_n(s)ds \tag{12}$$

$$A''_{1n}(t)=\frac{1}{12b_n^2}\cdot\frac{f'_n u_{1,n}-f_n u'_{1,n}}{u_{1,n}^2}$$

$$A''_{2,n}=-\frac{1}{12b_n^2}\left[\left(u'_{1,n}u_{2,n}+u_{1,n}u'_{2,n}+\sqrt{3}u'_{1,n}u_{3,n}+\sqrt{3}u_{1,n}u'_{3,n}\right) f_n+\left(u_{1,n}u_{2,n}+\sqrt{3}u_{1,n}u_{3,n}\right) f'_n\right]$$

$$A''_{3,n}=\frac{1}{12b_n^2}\left[\left(\sqrt{3}u'_{1,n}u_{2,n}+\sqrt{3}u_{1,n}u'_{2,n}-u'_{1,n}u_{3,n}-u_{1,n}u'_{3,n}\right) f_n+\left(\sqrt{3}u_{1,n}u_{2,n}-u_{1,n}u_{3,n}\right) f'_n\right]$$

We need the values of the functions $A_{i,n}(t),\ A'_{i,n}(t),\ A''_{i,n}(t)\ \ i=1,2,3$ and $u_{i,n}(t),\ u'_{i,n}(t),\ u''_{i,n}(t)\ \ i=1,2,3$ at zero in the future. Therefore, we evaluate them:

$$\begin{cases} u_{1,n}(0)=1 \\ u_{2,n}(0)=1 \\ u_{3,n}(0)=0 \end{cases} \qquad \begin{cases} u'_{1,n}(0)=-2b_n \\ u'_{2,n}(0)=b_n \\ u'_{3,n}(0)=\sqrt{3}b_n \end{cases} \qquad \begin{cases} u''_{1,n}(0)=4b_n^2 \\ u''_{2,n}(0)=-2b_n^2 \\ u''_{3,n}(0)=2\sqrt{3}b_n^2 \end{cases}$$

$$\begin{cases} A_{1,n}(0)=0 \\ A_{2,n}(0)=0 \\ A_{3,n}(0)=0 \end{cases} \qquad \begin{cases} A'_{1,n}(0)=\dfrac{1}{12b_n^2}f_n(0) \\ A'_{2,n}(0)=-\dfrac{1}{12b_n^2}f_n(0) \\ A'_{3,n}(0)=\dfrac{\sqrt{3}}{12b_n^2}f_n(0) \end{cases} \qquad \begin{cases} A''_{1,n}(0)=\dfrac{1}{12b_n^2}\left(2b_n f_n(0)+f'_n(0)\right) \\ A''_{2,n}(0)=-\dfrac{1}{12b_n^2}\left(2b_n f_n(0)+f'_n(0)\right) \\ A''_{3,n}(0)=\dfrac{\sqrt{3}}{12b_n^2}\left(2\sqrt{3}b_n f_n(0)-\sqrt{3}f'_n(0)\right) \end{cases}$$

Substituting representations (10)-(12) into the formula (9) for a particular solution of the equation (7), we obtain

$$\tilde{u}_n(t)=A_{1,n}(t)u_{1,n}(t)+A_{2,n}(t)u_{2,n}(t)+A_{3,n}(t)u_{3,n}(t) \tag{13}$$

Taking into account (13), we obtain the general solution of the equation (7)

$$u_n(t)=B_{1,n}u_{1,n}(t)+B_{2,n}u_{2,n}(t)+B_{3,n}u_{3,n}(t)+\tilde{u}_n(t), \qquad t\in[0,T], \tag{14}$$

where $B_{i,n}\ (i=1,2,3)$ are yet arbitrary coefficients, which will be determined.

We suppose that the functions $\varphi_i(x)$ are expanding into a Fourier series and using the Fourier coefficients (5), from conditions (3) we obtain

$$u_n(0)=\int_0^1 U(0,x)\vartheta_n(x)\,dx=\int_0^1 \varphi_1(x)\vartheta_n(x)\,dx=\varphi_{1,n}, \tag{15}$$

$$u'_n(0)=\int_0^1 U_t(0,x)\vartheta_n(x)\,dx=\int_0^1 \varphi_2(x)\vartheta_n(x)\,dx=\varphi_{2,n}, \tag{16}$$

$$u''_n(0)=\int_0^1 U_{tt}(0,x)\vartheta_n(x)\,dx=\int_0^1 \varphi_3(x)\vartheta_n(x)\,dx=\varphi_{3,n}. \tag{17}$$

To find the unknown (arbitrary) coefficients $B_{i,n}\,(i=1,2,3)$ in the presentations (14), we use the boundary conditions (15)-(17).

$$U(t,x)=\sum_{n=1}^{\infty}u_n(t)\vartheta_n(x)=\sum_{n=1}^{\infty}\Big[A_{1,n}(t)u_{1,n}(t)+A_{2,n}(t)u_{2,n}(t)+A_{3,n}(t)u_{3,n}(t)+$$

$$+B_{1,n}u_{1,n}(t)+B_{2,n}u_{2,n}(t)+B_{3,n}u_{3,n}(t)\Big]v_n(x),$$

$$U(0,x)=\sum_{n=1}^{\infty}u_n(0)\vartheta_n(x)=\sum_{n=1}^{\infty}\Big[A_{1,n}(0)u_{1,n}(0)+A_{2,n}(0)u_{2,n}(0)+A_{3,n}(0)u_{3,n}(0)+$$

$$+B_{1,n}u_{1,n}(0)+B_{2,n}u_{2,n}(0)+B_{3,n}u_{3,n}(0)\Big]v_n(x)=$$

$$=\sum_{n=1}^{\infty}(0+0+0+B_{1,n}+B_{2,n}+0)v_n(x)=\varphi_1(x),$$

$$B_{1,n}+B_{2,n}=\varphi_{1,n};$$

$$U_t(t,x)=\sum_{n=1}^{\infty}u_n'(t)\vartheta_n(x)=\sum_{n=1}^{\infty}\Big[A_{1,n}'(t)u_{1,n}(t)+A_{1,n}(t)u_{1,n}'(t)+A_{2,n}'(t)u_{2,n}(t)+$$

$$+A_{2,n}(t)u_{2,n}'(t)+A_{3,n}'(t)u_{3,n}(t)+A_{3,n}(t)u_{3,n}'(t)+$$

$$+B_{1,n}u_{1,n}'(t)+B_{2,n}u_{2,n}'(t)+B_{3,n}u_{3,n}'(t)\Big]v_n(x)=\varphi_1(x),$$

$$U_t(0,x)=\sum_{n=1}^{\infty}u_n'(0)\vartheta_n(x)=\sum_{n=1}^{\infty}\Big[A_{1,n}'(0)u_{1,n}(0)+A_{1,n}(0)u_{1,n}'(0)+A_{2,n}'(0)u_{2,n}(0)+$$

$$+A_{2,n}(0)u_{2,n}'(0)+A_{3,n}'(0)u_{3,n}(0)+A_{3,n}(0)u_{3,n}'(0)+$$

$$+B_{1,n}u_{1,n}'(0)+B_{2,n}u_{2,n}'(0)+B_{3,n}u_{3,n}'(0)\Big]v_n(x)=$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{12b_n^2}f_n(0)-\frac{1}{12b_n^2}f_n(0)-2b_nB_{1,n}+b_nB_{2,n}+\sqrt{3}b_nB_{3,n}\right]v_n(x)=$$

$$=\sum_{n=1}^{\infty}\left[-2b_nB_{1,n}+b_nB_{2,n}+\sqrt{3}b_nB_{3,n}\right]v_n(x)=\varphi_2(x),$$

$$-2b_nB_{1,n}+b_nB_{2,n}+\sqrt{3}b_nB_{3,n}=\varphi_{2,n};$$

$$U_{tt}(t,x)=\sum_{n=1}^{\infty}u_n''(t)\vartheta_n(x)=\sum_{n=1}^{\infty}\Big[A_{1,n}''(t)u_{1,n}(t)+2A_{1,n}'(t)u_{1,n}'(t)+A_{1,n}(t)u_{1,n}''(t)+$$

$$+A_{2,n}''(t)u_{2,n}(t)+2A_{2,n}'(t)u_{2,n}'(t)+A_{2,n}(t)u_{2,n}''(t)+A_{3,n}''(t)u_{3,n}(t)+2A_{3,n}'(t)u_{3,n}'(t)+$$

$$+A_{3,n}(t)u'''_{3,n}(t)+B_{1,n}u''_{1,n}(t)+B_{2,n}u''_{1,n}(t)+B_{3,n}u''_{2,n}(t)\Big]v_n(x)=\varphi_3(x),$$

$$U_n(0,x)=\sum_{n=1}^{\infty}u''_n(0)\vartheta_n(x)=\sum_{n=1}^{\infty}\Big[A''_{1,n}(0)u_{1,n}(0)+2A'_{1,n}(0)u'_{1,n}(0)+A_{1,n}(0)u''_{1,n}(0)+$$

$$+A''_{2,n}(0)u_{2,n}(0)+2A'_{2,n}(0)u'_{2,n}(0)+A_{2,n}(0)u''_{2,n}(0)+A''_{3,n}(0)u_{3,n}(0)+2A'_{3,n}(0)u'_{3,n}(0)+$$

$$+A_{3,n}(0)u''_{3,n}(0)+B_{1,n}u''_{1,n}(0)+B_{2,n}u''_{1,n}(0)+B_{3,n}u''_{2,n}(0)\Big]v_n(x)=$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{12b_n^2}(2b_nf_n(0)+f'_n(0))-\frac{4b_n}{12b_n^2}f_n(0)-\frac{1}{12b_n^2}(2b_nf_n(0)+f'_n(0))-\frac{2b_n}{12b_n^2}f_n(0)+\right.$$

$$\left.+\frac{6b_n}{12b_n^2}f_n(0)+4b_n^2B_{1,n}-2b_n^2B_{2,n}+2\sqrt{3}b_n^2B_{3,n}\right]v_n(x)=$$

$$=\sum_{n=1}^{\infty}\Big[4b_n^2B_{1,n}-2b_n^2B_{2,n}+2\sqrt{3}b_n^2B_{3,n}\Big]v_n(x)=\varphi_3(x);$$

$$4b_n^2B_{1,n}-2b_n^2B_{2,n}+2\sqrt{3}b_n^2B_{3,n}=\varphi_{3,n};$$

Using the equations (15)-(17) we obtain system of equation for $B_{i,n}\ \ (i=1;2;3)$

$$\begin{cases}B_{1,n}+B_{2,n}=\varphi_{1,n}\\-2b_nB_{1,n}+b_nB_{2,n}+\sqrt{3}b_nB_{3,n}=\varphi_{2,n}\\4b_n^2B_{1,n}-2b_n^2B_{2,n}+2\sqrt{3}b_n^2B_{3,n}=\varphi_{3,n}\end{cases}$$

Solving the above system with respect to $B_{i,n}\ \ (i=1;2;3)$ yields

$$\begin{cases}B_{1,n}=\dfrac{1}{3}\varphi_{1,n}-\dfrac{1}{6b_n}\varphi_{2,n}+\dfrac{1}{12b_n^2}\varphi_{3,n}\\[2mm]B_{2,n}=\dfrac{2}{3}\varphi_{1,n}+\dfrac{1}{6b_n}\varphi_{2,n}-\dfrac{1}{12b_n^2}\varphi_{3,n}\\[2mm]B_{3,n}=\dfrac{1}{2\sqrt{3}b_n}\varphi_{2,n}+\dfrac{1}{4\sqrt{3}b_n^2}\varphi_{3,n}\end{cases}$$

Substituting the values of $B_{i,n}\ \ (i=1;2;3)$s into (4) and rearranging the sum with respect to $\varphi_{i,n}\ \ (i=1;2;3)$ we obtain the following:

$$\begin{cases} B_{1,n}u_{1,n} = \frac{1}{3}u_{1,n}\varphi_{1,n} - \frac{1}{6b_n}u_{1,n}\varphi_{2,n} + \frac{1}{12b_n^2}u_{1,n}\varphi_{3,n} \\ B_{2,n}u_{2,n} = \frac{2}{3}u_{2,n}\varphi_{1,n} + \frac{1}{6b_n}u_{2,n}\varphi_{2,n} - \frac{1}{12b_n^2}u_{2,n}\varphi_{3,n} \\ B_{3,n}u_{3,n} = \frac{1}{2\sqrt{3}b_n}u_{3,n}\varphi_{2,n} + \frac{1}{4\sqrt{3}b_n^2}u_{3,n}\varphi_{3,n} \end{cases}$$

$$B_{1,n}u_{1,n} + B_{2,n}u_{2,n} + B_{3,n}u_{3,n} = \frac{u_{1,n} + 2u_{2,n}}{3}\varphi_{1,n} +$$

$$- \frac{u_{1,n} - u_{2,n} - \sqrt{3}u_{3,n}}{6b_n}\varphi_{2,n} + \frac{u_{1,n} - u_{2,n} + \sqrt{3}u_{3,n}}{12b_n^2}\varphi_{3,n}. \tag{18}$$

We write the solution of the problem (1)-(3):

$$U(t,x) = \sum_{n=1}^{\infty} u_n(t)\vartheta_n(x) = \sum_{n=1}^{\infty}\Big[A_{1,n}(t)u_{1,n}(t) + A_{2,n}(t)u_{2,n}(t) + A_{3,n}(t)u_{3,n}(t) +$$

$$+ B_{1,n}u_{1,n}(t) + B_{2,n}u_{2,n}(t) + B_{3,n}u_{3,n}(t)\Big] v_n(x) =$$

$$= \sum_{n=1}^{\infty}\Bigg[\frac{u_{1,n}(t)}{12b_n^2}\int_0^t \frac{1}{u_{1,n}(s)} f_n(s)ds - \frac{u_{2,n}(t)}{12b_n^2}\int_0^t (u_{1,n}(s)u_{2,n}(s) + \sqrt{3}u_{1,n}(s)u_{3,n}(s)) f_n(s)ds +$$

$$\frac{u_{3,n}(t)}{12b_n^2}\int_0^t (\sqrt{3}u_{1,n}(s)u_{2,n}(s) - u_{1,n}(s)u_{3,n}(s)) f_n(s)ds +$$

$$+ \frac{u_{1,n} + 2u_{2,n}}{3}\varphi_{1,n} + - \frac{u_{1,n} - u_{2,n} - \sqrt{3}u_{3,n}}{6b_n}\varphi_{2,n} + \frac{u_{1,n} - u_{2,n} + \sqrt{3}u_{3,n}}{12b_n^2}\varphi_{3,n}\Bigg] v_n(x) \tag{19}$$

In this case

$$b_n = \frac{\sqrt[3]{\mu_n}}{2}, \quad u_{1,n}(t) = e^{-\sqrt[3]{\mu_n}t}, \quad u_{2,n}(t) = e^{\frac{\sqrt[3]{\mu_n}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t, \quad u_{2,n}(t) = e^{\frac{\sqrt[3]{\mu_n}t}{2}}\sin\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t$$

We will pu this expressions in (19) and we will use formulae $\sin(\alpha \pm \beta),\ \cos(\alpha \pm \beta)$ we obtain

$$U(t,x) = \sum_{n=1}^{\infty}\Bigg[\frac{e^{-\sqrt[3]{\mu_n}t}}{3\sqrt[3]{\mu_n^2}}\int_0^t e^{\sqrt[3]{\mu_n}s} f_n(s)ds -$$

$$-\frac{2e^{\frac{\sqrt[3]{\mu_n}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t}{3\sqrt[3]{\mu_n^2}}\int_0^t e^{-\frac{\sqrt[3]{\mu_n}}{2}s}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}s+\frac{\pi}{6}\right)f_n(s)ds+$$

$$+\frac{2e^{\frac{\sqrt[3]{\mu_n}t}{2}}\sin\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t}{3\sqrt[3]{\mu_n^2}}\int_0^t e^{-\frac{\sqrt[3]{\mu_n}}{2}s}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}s+\frac{\pi}{6}\right)f_n(s)ds+$$

$$\left.+\psi_{1,n}(t)\varphi_{1,n}+\psi_{2,n}(t)\varphi_{2,n}+\psi_{3,n}(t)\varphi_{3,n}\right]v_n(x) \tag{20}$$

After simplifying we obtain:

$$U(t,x)=\sum_{n=1}^{\infty}\left[\frac{e^{-\sqrt[3]{\mu_n}t}}{3\sqrt[3]{\mu_n^2}}\int_0^t e^{\sqrt[3]{\mu_n}s}f_n(s)ds-\right.$$

$$-\frac{2e^{\frac{\sqrt[3]{\mu_n}t}{2}}}{3\sqrt[3]{\mu_n^2}}\int_0^t e^{-\frac{\sqrt[3]{\mu_n}}{2}s}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}(s-t)+\frac{\pi}{6}\right)f_n(s)ds+$$

$$\left.+\psi_{1,n}(t)\varphi_{1,n}+\psi_{2,n}(t)\varphi_{2,n}+\psi_{3,n}(t)\varphi_{3,n}\right]v_n(x) \tag{21}$$

where,

$$\psi_{1,n}(t)=\frac{e^{-\sqrt[3]{\mu_n}t}+2e^{\frac{\sqrt[3]{\mu_n}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t}{3},$$

$$\psi_{2,n}(t)=\frac{e^{-\sqrt[3]{\mu_n}t}-2e^{\frac{\sqrt[3]{\mu_n}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t+\frac{\pi}{6}\right)}{3\sqrt[3]{\mu_n}},$$

$$\psi_{3,n}(t)=\frac{e^{-\sqrt[3]{\mu_n}t}-2e^{\frac{\sqrt[3]{\mu_n}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}t-\frac{\pi}{6}\right)}{3\sqrt[3]{\mu_n^2}},$$

Introduce the following notation:

$$K_n(t-s)=\frac{1}{3\sqrt[3]{\mu_n^2}}\left[e^{-\sqrt[3]{\mu_n}(t-s)}+2e^{\frac{\sqrt[3]{\mu_n}}{2}(t-s)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n}(t-s)-\frac{\pi}{6}\right)\right] \tag{22}$$

$$P_n(t)=\psi_{1,n}(t)\varphi_{1,n}+\psi_{2,n}(t)\varphi_{2,n}+\psi_{3,n}(t)\varphi_{3,n} \tag{23}$$

Using (22), (23) we write $u_n(t)$ as follows:

$$u_n(t)=\Im(t,u_n(t))=P_n(t)+\int_0^t K_n(t-s)f_n(s)ds \tag{24}$$

Therefore, a unique solvability of the mixed problem (1)-(3) is equivalent to the unique solvability of the integral equation (24), which is a Voltera type integral equation. Once a solution of (24) is obtained, a function defined by

$$U(t,x)=\sum_{n=1}^{\infty}\left[P_n(t)+\int_0^t K_n(t-s)f_n(s)ds\right]v_n(x) \tag{25}$$

defines a solution of the mixed problem (1)-(3), [5-8].

Existencee and uniquness of the solution for problem (1)-(3)

Let us use the following well-known Banach spaces. Consider the space $B_2[0,T]$ of sequences of continuous functions $\left\{\xi_n(t)\right\}_{n=1}^{\infty}$ on the segment $[0,T]$ with norm

$$\left\|\vec{\xi}(t)\right\|_{B_2[0,T]}=\sqrt{\sum_{n=1}^{\infty}\left(\max_{t\in[0,T]}\left|\xi_n(t)\right|\right)^2}<\infty.$$

Coordinate Hilbert space ℓ_2 of number sequences $\left\{\zeta_n\right\}_{n=1}^{\infty}$ with a norm

$$\left\|\vec{\zeta}\right\|_{\ell_2}=\sqrt{\sum_{n=1}^{\infty}\left|\zeta_n\right|^2}<\infty.$$

Space $L_2[0,1]$ of square summab

Under the assumptions $\varphi_j(x)\in C^4[0,1],$ for $j=1,2,3$ we have

$$\varphi_{j,n}\le\left(\frac{1}{\pi}\right)^4\frac{\left|\varphi_{j,n}^{(IV)}\right|}{n^4},\qquad f_n(t)\le\left(\frac{1}{\pi}\right)^2\frac{\left|f_n''(t)\right|}{n^2}, \tag{26}$$

Where $\varphi_{j,n} = \int_0^1 \varphi_j(y)\vartheta_n(y)dy$, $f_n(t) = \int_0^1 f(t,y)\vartheta_n(y)\,dy$ and

$$\varphi_{j,n}^{(IV)} = \int_0^1 \frac{\partial^4 \varphi_j(y)}{\partial y^4}\vartheta_n(y)\,dy, \qquad f_n''(t) = \int_0^1 \frac{\partial^2 f(t,y)}{\partial y^2}\vartheta_n(y)\,dy.$$

Further, the standard estimates, i.e, the Bessel's inequalities hold:

$$\left\|\varphi_j^{(IV)}\right\|_{\ell_2} \le 4\left\|\frac{\partial^4 \varphi_j(x)}{\partial x^4}\right\|_{L_2[0,1]}, \quad \| f''(t)\|_{\ell_2} \le 2\left\|\frac{\partial^2 f(t,x)}{\partial x^2}\right\|_{L_2[0,1]}. \tag{27}$$

Theorem. Assume that the above mentioned smoothness are satisfied. Then (25) has a unique solution $(u_n)_n$ in the space $B_2[0,T]$ and every u_n is the limit of the following sequence:

$$u_n^0(t) = P(t), \quad u_n^k(t) = \Im(u_n^{k-1}),\ k = 1,2,\ldots \tag{28}$$

Proof. Since $\lim\limits_{\lambda_n\to\infty} \frac{1}{\sqrt[3]{\mu_n^2}} = \lim\limits_{\lambda_n\to\infty} \sqrt[3]{\left(\frac{1+\lambda_n}{\lambda_n}\right)^2} = e^{\frac{2}{3}}$ and T is fixed the following estimate holds

$$\max\left\{\max_{0\le t\le T}\left|K_n(t-s)\right|;\max_{j=1,2,3}\max_{0\le t\le T}\left|\psi_{j,n}(t)\right|\right\} \le M_0 < \infty, \quad \text{for some } M_0 > 0. \tag{29}$$

Thus by (23) we have the following estimate

$$|P_n(t)| = |\psi_{1,n}(t)\varphi_{1,n} + \psi_{2,n}(t)\varphi_{2,n} + \psi_{3,n}(t)\varphi_{3,n}| \le M_0 C\frac{1}{n^4} \text{ for some } C > 0 \tag{30}$$

First we show that $u^0 = (u_n^0(t))$ belongs to the space $B_2[0,T]$. By virtue of formulas (26), (27) and the fact that $0 < \mu_n = \frac{\lambda_n}{1+\lambda_n} < 1,$ applying the Cauchy--Shwartz inequality, from approximation (28) we have

$$\left\|u^0(t)\right\|_{B_2[0,t]} \le \sqrt{\sum_{n=1}^{\infty}\left|u_n^0(t)\right|^2} \le \sqrt{\sum_{n=1}^{\infty}|P_n(t)|^2} \le M_0 C\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^8}} = M_1 < +\infty \tag{31}$$

Now we estimate the norm of the difference $u^1(t)-u^0(t)$. For this purpose we estimate $u_n^1(t)-u_n^0(t)$ for every n. By the definition and inequalities (26), (27), (29) and (30) we have

$$\left|u_n^1(t)-u_n^0(t)\right|\leq\frac{1}{1+\lambda_n}\int_0^t K_n(t-s)\left|a(s)\right|\cdot\left|u_n^0(s)\right|ds\leq\frac{1}{n^6}CM_0^2At,$$

where $A=\max_{0\leq t\leq T}\left|a(t)\right|$. Thus, we have

$$\left\|u^1(t)-u^0(t)\right\|_{B_2[0,t]}\leq M_1M_0^2At \tag{32}$$

The next step is to estimate $\left\|u^2(t)-u^1(t)\right\|_{B_2[0,t]}$. Using the definition (28) and inequality (32)

$$\left\|u^2(t)-u^1(t)\right\|_{B_2[0,t]}\leq AM_0C\int_0^t\left\|u^1(s)-u^0(s)\right\|_{B_2[0,s]}ds\leq A^2M_0^3M_1^2\int_0^t sds=$$

$$=\left(AM_0^{3/2}M_1\right)^2\frac{t^2}{2!}. \tag{33}$$

Repeating the estimates inductively as above, for arrbitrary $k\geq 2$ we obtain

$$\left\|u^k(t)-u^{k-1}(t)\right\|_{B_2[0,t]}\leq\left[AM_0^{3/2}M_1\right]^k\frac{t^k}{k!}. \tag{34}$$

The inequalities (30) and (34) implies that the series

$$\left\|u^0(t)\right\|_{B_2[0,T]}+\sum_{k=1}^{\infty}\left\|u^k(t)-u^{k-1}(t)\right\|_{B_2[0,T]}$$

are convergence absolutely and uniformly. Hence implies that the (25) has a unique solution $u(t)\in B_2[0,T]$. The Theorem is proved.

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