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A BOUNDARY-VALUE PROBLEM FOR A COUPLED DEGENERATE HYPERBOLIC EQUATIONS OF THE SECOND KIND

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Abstract: *In the present paper, we formulate and study a boundary value problem for a system of coupled hyperbolic equations of the second kind in a domain consisting of the characteristics of the equations. Using the representation of the R^2-generalized solution of the equation and the properties of fractional differential and integral operators, the problem is equivalently reduced to a Volterra integral equation of the second kind with respect to the trace of the derivative of the solution.*

Keywords: *degenerate hyperbolic equation; boundary-value problem; characteristic domain; generalized solution; Riemann–Liouville fractional operators; Volterra integral equation; Mittag–Leffler function; fractional calculus.*

IKKINCHI TURDAGI BUZILADIGAN GIPERBOLIK TENGLAMALAR UCHUN CHEGARAVIY MASALA

Annotatsiya: *Ushbu maqolada tenglamalar xarakteristikalaridan tashkil topgan sohada ikkinchi turdagi buziladigan giperbolik tenglamalar uchun chegaraviy masala qo'yiladi va tadqiq etiladi. Tenglamaning R^2-umumlashgan yechimi ko'rinishidan hamda kasr tartibli differensial va integral operatorlarning xossalaridan foydalanib, qaralayotgan masala yechim hosilasining iziga nisbatan ikkinchi turdagi Volterra integral tenglamasiga ekvivalent ravishda keltiriladi.*

Kalit so'zlar: *buziladigan giperbolik tenglama; chegaraviy masala; xarakteristik soha; umumlashgan yechim; Riman–Liuvill kasr tartibli operatorlari; Volterra integral tenglamasi; Mittag–Leffler funksiyasi; kasr tartibli hisob.*

КРАЕВАЯ ЗАДАЧА ДЛЯ ВЫРОЖДАЮЩИХСЯ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ ВТОРОГО РОДА

Аннотация: *В настоящей работе поставлена и исследована краевая задача для гиперболических уравнений второго рода в области, ограниченной характеристиками уравнений. Используя представление R^2-обобщённого решения уравнения и свойства дробных дифференциальных и интегральных операторов, рассматриваемая задача эквивалентно сводится к интегральному уравнению Вольтерра второго рода относительно следа производной решения.*

Ключевые слова: *вырождающееся гиперболическое уравнение; краевая задача; характеристическая область; обобщённое решение; условия сопряжения; дробные операторы Римана–Лиувилля; интегральное уравнение Вольтерра; функция Миттаг–Леффлера; дробное исчисление.*

1. Introduction. Degenerate hyperbolic equations form an important class of partial differential equations arising in the study of transonic flows, gas dynamics, models with variable propagation speed, and a number of problems in mathematical physics. In contrast to strictly hyperbolic equations, the characteristic structure of a degenerate equation changes near the line of degeneracy, and this substantially complicates the formulation and analysis of boundary-value problems. Of particular interest are equations for which the degeneracy rate is described by different parameters in different subdomains. Such problems naturally lead to transmission conditions on the degeneracy line and require a careful analysis of the traces of the solution and its derivatives.

Boundary-value problems posed in domains bounded by characteristics occupy a special place in the theory of degenerate hyperbolic equations. Their investigation goes back to classical studies of equations of mixed and degenerate type, where characteristic methods and integral representations of solutions played a decisive role. In particular, the use of generalized solutions makes it possible to treat boundary data with weak singularities and to formulate solvability results under assumptions that are considerably less restrictive than classical smoothness conditions. Related ideas can be traced to the works of Karol [1] and to the systematic treatment of mixed-type equations given by Smirnov [2].

A distinctive feature of the present problem is that two degenerate hyperbolic equations, corresponding to different degeneracy exponents in the upper and lower parts of the domain, are coupled through transmission conditions on the line

$y=0$. The solution therefore has to be reconstructed separately in the subdomains $y>0$ and $y<0$, while the traces arising from both sides must satisfy a common relation on the degeneracy line. This coupling produces a nonlocal relation between the boundary data and the unknown traces and prevents a direct application of the standard characteristic method.

2. Formulation of Problem and main results

We consider the equation

$$|y|^{l}u_{xx}-u_{yy}=0,\ (1.1)$$

where $l=n_1$ for $y>0$ and $l=n_2$ for $y<0$, with $-1<l<0$, in a bounded domain Q enclosed by the characteristics

$$AC: x-\frac{2}{n_1+2}y^{\frac{n_1+2}{2}}=0,\ BC: x+\frac{2}{n_1+2}y^{\frac{n_1+2}{2}}=1,$$

$$AD: x-\frac{2}{n_2+2}(-y)^{\frac{n_2+2}{2}}=0,\ BD: x+\frac{2}{n_2+2}(-y)^{\frac{n_2+2}{2}}=1,$$

corresponding to equation (1.1).

Let us introduce the following notations: $\Omega^{+}=\Omega\cap\{y>0\}$ and $\Omega^{-}=\Omega\cap\{y<0\}$.

Moreover, by I we denote the interval $0<x<1$ on the line $y=0$. Also,

$$u(x,y)=\begin{cases}u^{+}(x,y),\ (x,y)\in\Omega^{+},\\ u^{-}(x,y),\ (x,y)\in\Omega^{-}.\end{cases}$$

It is known that the solution of the Cauchy problem for equation (1.1) in the domain Ω^{+} (respectively, Ω^{-}) has the form [1]

$$u^{\pm}(x,y)=\frac{\Gamma(2+2\beta_i)}{\Gamma^2(1+\beta_i)}\int_0^1\tau_{\pm}(t_i)\zeta^{\beta_i}(1-\zeta)^{\beta_i}d\zeta$$

$$+\frac{\Gamma(2+2\beta_i)}{(n_i+2)\Gamma^2(1+\beta_i)}(\pm y)^{\frac{n_i+2}{2}}\int_0^1\tau'_{\pm}(t_i)\zeta^{\beta_i}(1-\zeta)^{\beta_i}(2\zeta-1)d\zeta$$

$$+\frac{\Gamma(2-2\beta_i)}{\Gamma^2(1-\beta_i)}y\int_0^1\nu_{\pm}(t_i)\zeta^{-\beta_i}(1-\zeta)^{-\beta_i}d\zeta,$$

where

$$\tau_{\pm}(x)=\lim_{y\to\pm0}u^{\pm}(x,y),\ \nu_{\pm}(x)=\lim_{y\to\pm0}u^{\pm}_{y}(x,y),\ (1.3)$$

$$t_i=x+\frac{2}{n_i+2}(\pm y)^{(n_i+2)/2}(2\zeta-1),\ \beta_i=\frac{n_i}{2(n_i+2)},\ i=1,2.$$

We shall assume that the function $u^{+}(x,y)$ (respectively, $u^{-}(x,y)$) is an R_2 -generalized solution of the Cauchy problem for equation (1) [1] if it can be represented in the form (1.2), where

$$\tau_{+}(x)=\tau_{+}(0)+\int_0^x(x-t)^{-2\beta_1}T_{+}(t)dt,$$

$$\tau_{-}(x)=\tau_{-}(1)+\int_x^1(t-x)^{-2\beta_2}T_{-}(t)dt$$

and the functions $T_{+}(t)$, $T_{-}(t)$, as well as $\nu_{+}(t)$, $\nu_{-}(t)$ are continuous and integrable on $(0,1)$.

We will study the following problem for equation (1.1):

Problem 1. Find a solution of equation (1) in the class R_2 satisfying the boundary conditions

$$u^+\big|_{AC}=\psi_+(x),\quad 0\le x\le\frac{1}{2};\ (1.4)$$

$$u^-\big|_{AD}=\psi_-(x),\quad 0\le x\le\frac{1}{2}\ (1.5)$$

and the gluing (transmission) conditions

$$\nu_+(x)=\nu_-(x),\ \ x\in I,\ (1.6)$$

$$\tau_+(x)=\tau_-(x),\ \ x\in\bar{I},\ (1.7)$$

where $\psi_+(x),\psi_-(x)\in C^1(I)\cap C^2(I)$, $\psi_+''(x)$ and $\psi_-''(x)$ may have singularities of order less than one at $x=0$ and $x=1$, respectively.

In the class, on the basis of (1.2), from the boundary (1.3) and (1.4) we obtain [2]R_2

$$\tau_+(x)=\chi_3^+\int_0^x(x-t)^{-2\beta_1}\nu(t)dt+\Phi_1^+(x),\ (1.8)$$

$$\tau_-(x)=\chi_3^-\int_0^x(x-t)^{-2\beta_2}\nu(t)dt+\Phi_1^-(x),\ (1.9)$$

where $\chi_3^+=2\chi_2^+\cos\pi\beta_1$, $\chi_3^-=2\chi_2^-\cos\pi\beta_2$,

$$\chi_2^+=-\left[2(1-2\beta_1)\right]^{2\beta_1}\frac{\Gamma(-2\beta_1)}{\beta_1\Gamma^2(-\beta_1)},\ \chi_2^-=-\left[2(1-2\beta_2)\right]^{2\beta_2}\frac{\Gamma(-2\beta_2)}{\beta_2\Gamma^2(-\beta_2)},$$

$$\Phi_1^+(x)=\frac{2\Gamma(1+\beta_1)}{\Gamma(1+2\beta_1)}D_0^{2\beta_1-1}\left[x^{\beta_1}D_0^{1-\beta_1}\psi_+(x)\right],$$

$$\Phi_1^-(x)=\frac{2\Gamma(1+\beta_2)}{\Gamma(1+2\beta_2)}D_0^{2\beta_2-1}\left[x^{\beta_2}D_0^{1-\beta_2}\psi_-(x)\right].$$

Hence, considering the gluing condition (1.7), from the last equalities, we obtain

$$\chi_3^+\int_0^x(x-t)^{-2\beta_1}\nu(t)dt-\chi_3^-\int_0^x(x-t)^{-2\beta_2}\nu(t)dt=\Phi_2(x),\ (1.10)$$

where $\Phi_2(x)=\Phi_1^-(x)-\Phi_1^+(x)$.

Using the definition of the Riemann-Liouville fractional integral operator, we rewrite (1.10) in the form

$$D_{0x}^{2\beta_1-1}\nu_+(x)=\chi_4\int_0^x(x-t)^{-2\beta_2}\nu_-(t)dt+\Phi_2(x),$$

$$\chi_4=\frac{\chi_3^-}{\chi_3^+\Gamma(1-2\beta_1)},\ \Phi_2(x)=\Phi_1^-(x)-\Phi_1^+(x).$$

Hence, applying differential operator $D_{0x}^{1-2\beta_1}$, we obtain

$$D_{0x}^{1-2\beta_1}D_{0x}^{2\beta_1-1}\nu_+(x)=\chi_4D_{0x}^{1-2\beta_1}\int_0^x(x-t)^{-2\beta_2}\nu_-(t)dt+\Phi_3(x),$$

where

$$\Phi_3(x)=\frac{1}{\chi_3^+\Gamma(1-2\beta_1)}D_{0x}^{1-2\beta_1}\Phi_2(x).$$

Using the representation of the Riemann-Liuvill fractional differential operator, after some computations, the function $\Phi_3(x)$ can be written as

$$\Phi_3(x)=\frac{2\Gamma(1+\beta_2)}{\chi_3\Gamma(1-\beta_1)\Gamma(1+2\beta_2)\Gamma(1+2\beta_1-\beta_2)}x^{\beta_2}\times$$

$$\times\int_0^x\psi_-''(t)(x-t)^{2\beta_1-\beta_2}F\left(-\beta_2,2\beta_1-2\beta_2,2\beta_1-\beta_2+1;\frac{x-t}{x}\right)dt-$$

$$-\frac{x^{\beta_1}}{\chi_3}\Gamma(1-2\beta_1)\Gamma(1+2\beta_1)\int_0^x\psi_-''(t)(x-t)^{\beta_1}dt.$$

Hence, we obtain

$$\nu_+(x)-\frac{\chi_5}{\Gamma(2\beta_1-2\beta_2)}\int_0^x(x-t)^{2\beta_1+2\beta_2-1}\nu_-(t)dt=\Phi_3(x),\ \chi_5=\chi_4\Gamma(1-2\beta_2).$$

We rewrite the last equation in the following form

$$\nu_+(x)-\frac{\lambda}{\Gamma(2\beta_1-2\beta_2)}\int_0^x(x-t)^{2\beta_1-2\beta_2-1}\nu_-(t)dt=\Phi_3(x),\ (1.11)$$

where

$$\lambda=\chi_4\Gamma(1-2\beta_2).$$

It is known that the solution of the following integral equation

$$\varphi(x)-\frac{\lambda}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)dt}{(x-t)^{1-\alpha}}=f(x)$$

has the following form [3]

$$\varphi(x)=\frac{d}{dx}\int_0^x E_\alpha\left[\lambda(x-t)^\alpha\right]f(t)dt.$$

Using this formula, we will find $\nu_+(x)$ in the form

$$\nu_+(x)=\frac{d}{dx}\int_0^x E_{2\beta_1-2\beta_2}\left[\lambda(x-t)^{2\beta_1-2\beta_2}\right]\Phi_3(t)dt.$$

Now, we will find the function $\tau_\pm(x)$ using the formula (1.9):

$$\tau_\pm(x)=$$

$$\Gamma(1-2\beta_1)\chi_3\int_0^x(x-t)^{-2\beta_1}E_{2\beta_1-\beta_2,1-2\beta_1}\left[\lambda(x-t)^{2\beta_1-2\beta_2}\right]\Phi_3(t)dt+\Phi_1^+(x).$$

Substituting the expressions of the functions $\tau_\pm(x)$ and $\nu_\pm(x)$ into (1.2), we have the representation of the solution the considered problem.

Conclusion. In this paper, a boundary-value problem for a coupled system of second-order degenerate hyperbolic equations with different degeneracy parameters in the upper and lower parts of the characteristic domain has been investigated. The coupling of the equations is realized through transmission conditions imposed on the line of degeneracy.

Using the representation formula for the generalized solution of the corresponding Cauchy problem, relations between the boundary data and the traces of the solution and its derivatives on the degeneracy line were obtained. These relations were then rewritten in terms of the Riemann–Liouville fractional integral and differential operators. As a result, the original boundary-value problem was reduced to a Volterra integral equation of the second kind for the unknown trace of the derivative of the solution.

The obtained Volterra equation was solved by using the known representation formula involving the Mittag–Leffler function. The method developed in the paper can also be useful for studying related boundary and transmission problems for degenerate hyperbolic equations with weakly singular kernels, nonlocal conditions, or different orders of degeneracy in adjacent subdomains.

Foydalanilgan adabiyotlar

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