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APPLICATION OF MACHINE LEARNING TO GIBBS MEASURES FOR THE POTTS-ISING MODEL

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Abstract: *This paper investigates tree-indexed Markov chains (Gibbs measures) defined by a Hamiltonian coupling two layers of Potts and Ising models: hidden spins* $s(x) \in \{1,2,3\}$ *and observed spins* $\sigma(x) \in \{\pm 1\}$ *on a Cayley tree. The Hamiltonian incorporates Potts interactions within each layer and component-wise Ising couplings between the layers, extending hidden Markov models to a bilinear Markov random field. In particular, the study focuses on translation-invariant Gibbs measures (TIGMs) associated with this Hamiltonian on Cayley trees.*

Keywords: *Potts-Ising model, Markov chains, Hamiltonian, Cayley tree, translation-invariant Gibbs measures.*

POTTS-ISING MODELIDA GIBBS O‘LCHOVLARI UCHUN MASHINAVIY O‘RGANISH

Annotatsiya: *Ushbu maqolada biz Potts-Ising modelining ikki qatlamini bog‘laydigan Hamiltonian tomonidan aniqlangan daraxt indekslangan Markov zanjirlarini (Gibbs o‘lchovlari) o‘rganamiz: yashirin spinlar* $s(x) \in \{1,2,3\}$ *va kuzatilgan* $\sigma(x) \in \{\pm 1\}$ *spinlari. Hamiltonian har bir qatlam ichidagi Potts o‘zaro ta’sirini va qatlamlar orasidagi komponentlararo Ising aloqalarini o‘z ichiga oladi, yashirin Markov modellarini bilinear Markov tasodifiy maydoniga kengaytiradi. Xususan, biz ushbu Hamiltonianning Keyli daraxtlaridagi translyatsion-invariant Gibbs o‘lchovlarini (TIGO‘) o‘rganamiz.*

Kalit so‘zlar: *Potts-Ising modeli, Markov zanjirlari, Gamiltonian, Keyli daraxti, translyatsion -invariant Gibbs o‘lchovlari.*

ИСПОЛЬЗОВАНИЕ МАШИННОГО ОБУЧЕНИЯ МЕР ГИББСА ДЛЯ МОДЕЛИ ПОТТСА-ИЗИНГА

Аннотация: *В данной работе исследуются марковские цепи, индексируемые деревьями (меры Гиббса), определенные гамильтонианом, связывающим две слоя модели Поттса и Изинга: скрытые спины* $s(x) \in \{1,2,3\}$ *и наблюдаемые спины* $\sigma(x) \in \{\pm 1\}$ *на дереве Кэли. Гамильтониан включает взаимодействия Поттса внутри каждого слоя и покомпонентные связи Изинга между слоями, расширяя скрытые марковские модели до билинейного марковского случайного поля. В частности, исследуются трансляционно-инвариантные меры Гиббса (ТИМГ) данного гамильтониана на деревьях Кэли.*

Ключевые слова: *модель Поттса-Изинга, марковские цепи, гамильтониан, дерево Кэли, трансляционно-инвариантные меры Гиббса.*

1. Introduction

The application of machine learning methods to the study of Gibbs measures represents a promising research direction at the intersection of mathematical statistical mechanics, probability theory, and modern artificial intelligence methods. Gibbs measures are among the fundamental mathematical tools used to describe equilibrium states of complex physical systems and are widely employed in the study of the Ising [1,2], Potts [3,4,5,6], Blume–Capel [7,8,9], hard-core [2,3], and other models of statistical mechanics.

In many models, the determination and classification of Gibbs measures reduce to the study of nonlinear operators, systems of functional equations, and conditions for the existence of fixed points. At the same time, the structure of the set of Gibbs measures can change substantially depending on the parameters of the model, which leads to the emergence of different phases, phase transitions, and the multiplicity of translation-invariant or periodic Gibbs measures.

Machine learning methods make it possible to consider this problem from a new perspective. Instead of directly solving complex nonlinear equations, computational methods can be used for detecting patterns, classifying phase states, identifying regions with multiple Gibbs measures, and predicting critical parameter values. In particular, clustering methods, classification methods, neural networks, and dimensionality reduction methods can be applied to analyze numerical data obtained through iterations of the corresponding nonlinear operators.

Of particular interest is the construction of an algorithm that, for a given set of model parameters, determines the nature of the corresponding state of the system: uniqueness of the Gibbs measure, the existence of multiple Gibbs measures, or the occurrence of a phase transition. Thus, machine learning can serve not only as a tool for numerical analysis but also as an additional method for investigating the qualitative properties of nonlinear operators associated with Gibbs measures.

It is also promising to combine rigorous analytical methods with machine learning methods. Theoretical analysis makes it possible to establish exact conditions for the existence and multiplicity of Gibbs measures, whereas machine learning can be used for the efficient analysis of large datasets, visualization of phase diagrams, and identification of hidden patterns. Such a combined approach can significantly expand the possibilities for investigating complex models of statistical mechanics, particularly in cases where obtaining an explicit analytical solution of the corresponding equations is difficult.

2. Preliminary results

The Cayley tree $\Gamma^k = (V, L)$ with branching factor $k \geq 1$ is a connected infinite graph in which each vertex has exactly $k+1$ neighbors. The graph Γ^k is acyclic, i.e., it does not contain cycles [1,2]. We fix a vertex $x_0 \in V$, which we consider as the root of the tree. A vertex $y \in V$ is called a direct successor of $x \in V$ if x is the penultimate vertex on the unique path from the root x_0 to y, that is $d(x_0, y) = d(x_0, x) + 1$ and $d(x, y) = 1$. The set of direct successors of x is denoted by $S(x)$.

Let

$$W_n = \{x \in V | \ d(x, x_0) = n\}, \ V_n = \{x \in V | \ d(x, x_0) \leq n\}, \ L_n = \{\langle x, y\rangle \in L | \ x, y \in V_n\}.$$

For $x \in W_n$, the set $S(x)$ has the form

$$S(x) = \{y \in W_{n+1} | \ \langle x, y\rangle \in L\}.$$

In this paper, we consider Potts spins $s(x) \in \Phi = \{1, 2, 3\}$ assigned to the vertices of the Cayley tree. A configuration s is a function $x \mapsto s(x)$, and the set of all configurations is denoted by $\Omega = \Phi^V$.

The Hamiltonian, which takes into account the Potts–Ising interactions and the coupling between the hidden and observable states, is given by the following formula (see [2,5]):

$$H(s, \sigma) = -J \sum_{\langle x,y\rangle} \left(\delta_{s(x)s(y)} - \sigma(x)\sigma(y)\right) - \sum_{x \in V} p(\sigma(x) | \ s(x)), \qquad (1)$$

where $J \in \mathbf{R}$ is the interaction parameter and δ is the Kronecker delta. Here, the first term represents the local interactions of the hidden states, whereas the second term represents the probability of observing $\sigma(x)$ given the hidden state $s(x)$.

We define the finite-dimensional distribution of the probability measure μ in the volume V_n as

$$\mu_n(s_n, \sigma_n) = Z_n^{-1} \exp\left(-\beta H_n(s_n, \sigma_n) + \sum_{x \in W_n} h_{s(x), \sigma(x), x} \right), \qquad (2)$$

where $\beta = 1/T$, $T > 0$ - temperature, Z_n - the normalizing factor and

$$h_x = (h_{1,-1,x}, h_{1,1,x}, h_{2,-1,x}, h_{2,1,x}, h_{3,-1,x}, h_{3,1,x}) \in \mathrm{R}^6, \quad x \in V. \qquad (3)$$

The Hamiltonian taking into account the interactions

$$H_n(s_n, \sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} (\delta_{s(x)s(y)} - \sigma(x)\sigma(y)) - \sum_{x \in V_n} p(\sigma(x) | \ s(x)).$$

Definition. The family of distributions (2) is called compatible if, for all $n \geq 1$ and s_{n-1}, σ_{n-1} the following condition is satisfied:

$$\sum_{w_n, \omega_n \in I^{W_n}} \mu_n(s_{n-1} \vee w_n, \sigma_{n-1} \vee \omega_n) = \mu_{n-1}(s_{n-1}, \sigma_{n-1}),$$

where $\vee$ denotes the operation of union of configurations.

According to the Kolmogorov theorem, there exists a unique measure μ on $\Phi^V \times \Phi_1^V$, consistent with μ_n, where $\Phi_1 = \{\pm 1\}$. Such a measure is called a splitting Gibbs measure (SGM) for Hamiltonian (1) and the vector functions (3).

Theorem 1. *The probability distributions* $\mu_n(s_n, \sigma_n)$, $n \in N$, *in (2) are compatible if and only if, for every* $x \in V \setminus \{x^0\}$ *the following equations are satisfied:*

$$z_{1,-1,x} = \prod_{y \in S(x)} \frac{\theta^2 + az_{1,-1,y} + \theta^{-1}bz_{2,-1,y} + \theta cz_{2,1,y} + \theta^{-1}dz_{3,-1,y} + \theta\, fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1} fz_{3,1,y}},$$

$$z_{2,-1,x} = \prod_{y \in S(x)} \frac{\theta + \theta^{-1}az_{1,-1,y} + bz_{2,-1,y} + \theta^2 cz_{2,1,y} + \theta^{-1}dz_{3,-1,y} + \theta\, fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1} fz_{3,1,y}},$$

$$z_{2,1,x} = \prod_{y \in S(x)} \frac{\theta^{-1} + \theta az_{1,-1,y} + \theta^2 bz_{2,-1,y} + cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1} fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1} fz_{3,1,y}}, \quad (4)$$

$$z_{3,-1,x} = \prod_{y \in S(x)} \frac{\theta + \theta^{-1}az_{1,-1,y} + \theta^{-1}bz_{2,-1,y} + \theta cz_{2,1,y} + dz_{3,-1,y} + \theta^2 fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1} fz_{3,1,y}},$$

$$z_{3,1,x} = \prod_{y \in S(x)} \frac{\theta^{-1} + \theta az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta^2 dz_{3,-1,y} + fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1} fz_{3,1,y}},$$

where

$$z_{\varepsilon\tau\alpha,x} = \exp(h_{\varepsilon,\alpha,x} - h_{1,1,x}), \quad \varepsilon = 1,2,3;\ \alpha = \pm 1,\ \theta = \exp(J\beta),$$
$$a = \exp[\beta(p(-1|1) - p(1|1))], \quad b = \exp[\beta(p(-1|2) - p(1|1))],$$
$$c = \exp[\beta(p(1|2) - p(1|1))], \quad d = \exp[\beta(p(-1|3) - p(1|1))],$$
$$f = \exp[\beta(p(1|3) - p(1|1))].$$

3. Translation-Invariant Gibbs Measures.

Let us consider the translation-invariant solutions of system (4), i.e.,

$$z_{\varepsilon,\alpha,x} = z_{\varepsilon,\alpha}, \qquad \forall x \in V.$$

Denote

$$u = az_{1,-1}, \quad v = bz_{2,-1}, \quad w = cz_{2,1}, \quad r = dz_{3,-1}, \quad t = fz_{3,1}.$$

Then system (2) takes the following form:

$$u = a \cdot \left(\frac{\theta^2 + u + \theta^{-1}v + \theta w + \theta^{-1}r + \theta t}{1 + \theta^2 u + \theta v + \theta^{-1}w + \theta r + \theta^{-1}t} \right)^k,$$

$$v = b \cdot \left(\frac{\theta + \theta^{-1}u + v + \theta^2 w + \theta^{-1}r + \theta t}{1 + \theta^2 u + \theta v + \theta^{-1}w + \theta r + \theta^{-1}t} \right)^k,$$

$$w = c \cdot \left(\frac{\theta^{-1} + \theta u + \theta^2 v + w + \theta r + \theta^{-1}t}{1 + \theta^2 u + \theta v + \theta^{-1}w + \theta r + \theta^{-1}t} \right)^k, \quad (5)$$

$$r = d \cdot \left(\frac{\theta + \theta^{-1}u + \theta^{-1}v + \theta w + r + \theta^2 t}{1 + \theta^2 u + \theta v + \theta^{-1}w + \theta r + \theta^{-1}t} \right)^k,$$

$$t = f \cdot \left(\frac{\theta^{-1} + \theta u + \theta v + \theta^{-1}w + \theta^2 r + t}{1 + \theta^2 u + \theta v + \theta^{-1}w + \theta r + \theta^{-1}t} \right)^k.$$

Let us consider the case *a=b=c=d=f=1.* In this case, solving system (5) reduces to finding a fixed point of the equation

$$F(l)=l, \qquad l=(u,v,w,r,t)\in \mathrm{R}_{+}^{5},$$

Where $F:R_{+}^{5}\to R_{+}^{5}$ – is the operator defined by the right-hand sides of the corresponding system (5).

Lemma 1. *The set* $I=\{l=(u,v,w,r,t)\in R_{+}^{5}: u=r=t=1, v=w\}$ *is invariant with respect to the operator* F, *i.e.,* $F(I)\subset I$.

The proof is based on the decomposition of the measure and the application of the compatibility condition.

Restricting (5) to the set I, the equation $l=F(l)$ takes the form

$$v=\left(\frac{2+\theta v}{\theta+1+v}\right)^{k}. \qquad (6)$$

Theorem 2. *Let* $k\geq 2$ *and* $\theta_0=\theta_0(k):=\dfrac{k+2}{k-1}$. *Then there exists* $\theta_c=\theta_c(k)$ *such that* $0<\theta_c<\theta_0$ *and, for the operator* F *on the invariant set* I, *the following statements hold:*

1. If $0<\theta<\theta_c$, *then there exists a unique fixed point,* $\mathbf{1}:=(1,1,1,1,1)$ *i.e.* $u=v=w=r=t=1$;

2. If $\theta=\theta_c$, *then, for the operator* F *on the invariant set* I, *there exist two fixed points* $\mathbf{1}$, $(1,v_c,v_c,1,1)$.

3. If $\theta>\theta_c, \theta\neq\theta_0$, *then there exist three fixed points* $\mathbf{1}, (1,v_i,v_i,1,1),\ i\in\{1,2\}$;

4. If $\theta=\theta_0$ *then there exist two fixed points* $\mathbf{1}, (1,v_0,v_0,1,1)$.

Proof. Let $\sqrt[k]{v}=x$. Then equation (6) becomes

$$x^{k+1}-\theta x^{k}+(\theta+1)x-2=0. \qquad (7)$$

This equation has a solution $x=1$ that is independent of (θ,k). Dividing both sides by $x-1$, we obtain

$$x^{k}+(1-\theta)\left(x^{k-1}+x^{k-2}+\ldots+x^{2}+x\right)+2=0. \qquad (8)$$

Equation (8) again has the solution $x=1$ if and only if $\theta=\theta_0$, where

$$\theta_0:=\frac{k+2}{k-1}.$$

It is well known (see [10], p. 28) that the number of positive roots of polynomial (8) does not exceed the number of sign changes of its coefficients.

Obviously, $1-\theta<0$. Thus, the number of positive roots of polynomial (8) does not exceed 2.

The following lemma provides a complete analysis of equation (8).

Lemma 2. *For every* $k\geq 2$, *there exists exactly one critical value* $\theta_c=\theta_c(k)$ *such that* $0<\theta_c<\theta_0$ *and*

– If $0<\theta<\theta_c$, *then (8) has no positive root.*

– If $\theta=\theta_c$, *then the equation has a unique positive root.*

– If $\theta>\theta_c, \theta\neq\theta_0$, *then it has exactly two positive roots (both different from 1).*

– If $\theta=\theta_0$, *then the equation has two solutions, one of which is* $x=1$.

Proof. Solving (8) with respect to θ, we obtain

$$\theta=\psi_k(x):=1+\frac{2+x^{k}}{x^{k-1}+\ldots+x}.$$

Here, $\psi_k(x)\geq 1,\ x>0$, and the condition $\psi_k'(x)=0$ is equivalent to the following equality:

$$\sum_{j=1}^{k-1}(k-j)x^{k+j-1}-2\sum_{j=1}^{k-1}jx^{k+j-1}=0.$$

According to Descartes' rule of signs (see [10]), this equation has exactly one positive root. For $x = 0$, the function takes the negative value −2, and it takes positive values for sufficiently large x. Consequently, there exists such an $x^* > 0$, that $\psi_k'(x^*) = 0$ and

$$\theta_c = \theta_c(k) = \min_{x>0} \psi_k(x) = \psi_k(x^*).$$

Notice that $x^* \neq 1$, i.e., $x = 1$ does not satisfy (8). Consequently,

$$\theta_0 = \psi_k(1) > \theta_c = \min_{x>0} \psi_k(t).$$

These properties of $\psi_k(x)$ complete the proof. The lemma is proved.

Thus, when $\theta > \theta_c, \theta \neq \theta_0$, according to this lemma, equation (8) has two positive solutions: $x = x_i \neq 1, i = 1,2$ i.e., $v = v_i = x_i^k, i = 1,2$. Theorem 2 is proved.

Taking into account Theorems 1 and 2, we obtain the following.

Theorem 3. *Let* $k \geq 2$. *Then for the Potts–Ising model (1) the following statements hold:*

a) if $0 < \theta < \theta_c$ *then there exists a unique translation-invariant Gibbs measure* μ_0;

b) if $\theta = \theta_c$, *then there exist two translation-invariant Gibbs measures* μ_1, μ_2;

c) if $\theta > \theta_c, \theta \neq \theta_0$, *then there exist three translation-invariant Gibbs measures* μ_0, μ_1, μ_2;

d) if $\theta = \theta_0$, *then there exist two translation-invariant Gibbs measures* μ_0, μ_1,

where the measures μ_0, μ_1, μ_2 *correspond to the solutions* $(1, v_i, v_i, 1, 1), i = 1,2$ *of system (5) on the invariant set* I.

For example, if $k = 2$, then we have $v_i, i = 1,2$,

$$v_1 := v_1(\theta) = \frac{1}{2}\left(\theta^2 - 2\theta - 3 - (\theta - 1)\sqrt{\theta^2 - 2\theta - 7}\right),$$

$$v_2 := v_2(\theta) = \frac{1}{2}\left(\theta^2 - 2\theta - 3 + (\theta - 1)\sqrt{\theta^2 - 2\theta - 7}\right), \theta > 4. \quad (9)$$

4. Interpretation of Results and Applications in Machine Learning

For a given sequence of observed data (configurations) $\sigma = \sigma(x)$ at the vertices x of the tree, one of the main mathematical problems in an HMM based on the Potts–Ising model is to infer the hidden states (spin configurations) from these observations. This can be formalized by calculating the conditional probability $\mu_n(s_n \mid \sigma_n)$ given the conditional (observed) configuration σ_n. We begin by considering the joint distribution $\mu_n(s_n, \sigma_n)$ and the Hamiltonian $H_n(s_n, \sigma_n)$.

By decomposing the Hamiltonian, we separate the terms containing $s(x)$ and $\sigma(x)$. The terms containing $\sigma(x)$ are constants when σ_n is fixed, so they can be taken outside the normalization constant. The remaining terms form an effective Hamiltonian for the s_n-spins, including the interaction between s-spins, the local terms $\beta p(s_n \mid \sigma_n)$ and the boundary fields $h_{s(x),\sigma(x),x}$.

For a given observed configuration σ, the task is to infer the hidden configuration s. The conditional probability is given by the formula

$$\mu(s_n \mid \sigma_n) = \frac{1}{Z(\sigma_n)} \exp\left(\beta J \sum_{\langle x,y \rangle} \delta_{s(x)s(y)} + \sum_{x \in V_n} \beta p(\sigma(x) \mid s(x)) + \sum_{x \in W_n} h_{s(x),\sigma(x),x} \right) =$$

$$= \frac{1}{Z(\sigma_n)} \prod_{\langle x,y \rangle} \theta^{\frac{\delta_{s(x)s(y)}}{2}} \prod_{x \in V_n} exp(\beta p(\sigma(x) \mid s(x))) \prod_{x \in W_n} z_{s(x),\sigma(x),x}, \quad (10)$$

where $Z(\sigma_n)$ is the normalization coefficient depending on σ_n.

Remark 1. *It follows from formula (10) that, in order to determine the conditional probability* $\mu(s_n \mid \sigma_n)$*, it is sufficient to know the conditional probabilities on each edge. Therefore, we investigate these probabilities on each edge. Since the Gibbs measures derived above are translation-invariant, in formula (10) we have* $z_{s(x),\sigma(x),x} = z_{s(x),\sigma(x)}$*, which means that it does not depend on the vertex* x*, but depends on the configuration values at the vertex.*

Recall (see Theorem 1):

$$z_{\varepsilon,\alpha,x} = \exp(h_{\varepsilon,\alpha,x} - h_{1,1,x}), \quad \varepsilon = 1,2,3, \quad \alpha = \pm 1.$$

Without loss of generality, assume that $h_{1,1,x} \equiv 0$.

Then, for each translation-invariant solution, we have

$$h_{\varepsilon,\alpha,x} = \log z_{\varepsilon,\alpha}, \quad \varepsilon = 1,2,3, \quad \alpha = \pm 1. \tag{11}$$

Thus, on Cayley trees, belief propagation (BP) efficiently computes the marginal values $\mu(s_n \mid \sigma_n)$ according to (10). The observed σ acts as fixed boundary conditions, reducing fluctuations in the hidden states.

Case *a=b=c=d=f=1* and **case** I**:** Let us illustrate this for configurations on a fixed edge $l_0 = \langle x, y \rangle$ and for three different Gibbs measures from Result 1: μ0\mu_0, corresponding to $u = v = w = r = t = 1$ and $\mu_i, i = 1,2$, corresponding to the solutions $(1, v_i, v_i, 1, 1)$, with v_i, given by (9).

Measure μ_0**:** In the case of Result 1, we have the condition $a = b = c = d = f = 1$, that is

$$p(-1|1) = p(-1|2) = p(-1|3) = p(1|1) = p(1|2) = p(1|3) \equiv \frac{1}{2}. \tag{12}$$

Since $u = v = w = r = t = 1$ from (11) we obtain

$$h_{\varepsilon,\alpha,x} \equiv 0, \quad \varepsilon = 1,2,3, \quad \alpha = \pm 1..$$

Consequently, for the fixed edge $l_0 = \langle x, y \rangle$ we have

$$\mu_0(s_{l_0} \mid \sigma_{l_0}) = \frac{\theta^{\frac{1}{2}+\delta_{s(x)s(y)}}}{3\theta^{\frac{1}{2}}(\theta+2)}. \tag{13}$$

From this formula, it can be seen that, for the measure μ_0, the conditional probability does not depend on the conditional (observed) configurations. Moreover, the hidden configuration at the endpoints of the edge l_0 has equal values with probability $\theta / 3(\theta+2)$ and different values with probability $1/3(\theta+2)$.

Measure μ_1**:** In this case we have

$$\mu_1(s_{l_0} \mid \sigma_{l_0}) = \frac{\theta^{\frac{1}{2}+\delta_{s(x)s(y)}} z_{s(x),\sigma(x)} z_{s(y),\sigma(y)}}{\sum_{\varepsilon\in\{1,2,3\},\varepsilon_1\in\{1,2,3\}} \theta^{\frac{1}{2}+\delta_{\varepsilon\varepsilon_1}} z_{\varepsilon,\sigma(x)} z_{\varepsilon_1,\sigma(y)}}. \tag{14}$$

For the solution $(1, v_i, v_i, 1, 1)$ this conditional probability also does not depend on the conditional configuration σ_{l_0}. For example,

$$\mu_1((1,1) \mid \sigma_{l_0}) = \frac{\theta}{\theta v_1^2 + 2\theta + 4v_1 + 2}.$$

Measure μ_2**:** For the measure μ_2**, analogously to the case of** μ_1, we obtain

$$\mu_2((1,1) \mid \sigma_{l_0}) = \frac{\theta}{\theta v_2^2 + 2\theta + 4v_2 + 2}. \tag{15}$$

Remark 2. *It follows from formulas (13)–(15) that the system described by Hamiltonian (1) subject to condition (12) has three distinct equilibrium states:*

– For the state corresponding to μ_0*, hidden configurations at the endpoints of each edge of the tree appear with equal values with probability* $\frac{\theta}{3(\theta+2)}$ *and with different values with probability* $\frac{1}{3(\theta+2)}$ *Moreover, as* $\theta \to \infty$*, the hidden configuration may be «all 1, 2, 3» with equal probability* $1/3$ *for the state* μ_0*.*

For the state corresponding to μ_1*, hidden configurations at the endpoints of each edge of the tree are most likely either «all 1» or «all 3», with the highest probability corresponding to* μ_1*.*

For the state corresponding to μ_2*, hidden configurations at the endpoints of each edge of the tree are most likely to be «all 2», with the highest probability corresponding to* μ_2*.*

Foydalanilgan adabiyotlar

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