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CUBIC OPERATORS ON A SET OF IDEMPOTENT MEASURES AND THEIR DYNAMICS

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Abstract: *In this paper described some cubic operators which map the* $(n-1)$*– dimensional simplex of idempotent measures to itself. Such operators are divided to two classes: the first class contains all* $n\times n\times n\times n$ *- matrices with positive entries which for each* $l-$*th cubic matrix exists number* $\exists m_l \le n$*, where all of the elements of the* $l-$*th (*$l \le n$*) cubic matrix are zeroes except elements as* $p_{ijm_l,l}$, $p_{im_l j,l}$ *and* $p_{m_l ij,l}$*; the second class contains all* $n\times n\times n\times n$ *- matrices with positive entries which has at least one cubic matrix. These matrices play a role of the stochastic matrices in the case of idempotent measures. For both classes of cubic operators we find fixed points and their characters.*

Keywords: *cubic operator, simplex, idempotent measure, fixed point, attracting fixed point, repelling fixed point.*

IDEMPOTENT O‘LCHOVLAR TO‘PLAMIDA KUBIK OPERATORLAR VA ULARNING DINAMIKASI

Annotatsiya: *Ushbu ishda idempotent o‘lchovlarning* $(n-1)$*-o‘lchamli simpleksini o‘zini-o‘ziga akslantiruvchi ba’zi kubik operatorlar tavsiflangan. Bunday operatorlar ikki sinfga bo‘linadi: birinchi sinf musbat elementli shunday* $n\times n\times n\times n$ *o‘lchamli barcha matritsalarni o‘z ichiga oladiki, unda har bir* $l-$*chi kubik matritsa uchun shunday* $\exists m_l \le n$ *soni mavjud bo‘lib,* $l-$*chi (*$l \le n$*) kubik matritsaning* $p_{ijm_l,l}$, $p_{im_l j,l}$ *va* $p_{m_l ij,l}$ *indeksli elementlaridan tashqari barcha elementlari nolga teng; ikkinchi sinf kamida bitta kubik matritsaga ega bo‘lgan musbat elementli barcha* $n\times n\times n\times n$ *o‘lchamli matritsalarni o‘z ichiga oladi. Bu matritsalar idempotent o‘lchovlar holida stoxastik matritsalar rolini o‘ynaydi. Kubik operatorlarning ikkala sinfi uchun ham qo‘zg‘almas nuqtalar topilgan va ularning xususiyati (xarakteri) aniqlangan.*

Kalit so‘zlar: *kubik operator, simpleks, idempotent o‘lchov, qo‘zg‘almas nuqta, tortuvchi qo‘zg‘almas nuqta, itaruvchi qo‘zg‘almas nuqta.*

КУБИЧЕСКИЕ ОПЕРАТОРЫ НА МНОЖЕСТВЕ ИДЕМПОТЕНТНЫХ МЕР И ИХ ДИНАМИКА

Аннотация: *В данной работе описаны некоторые кубические операторы, отображающие* $(n-1)$ *-мерный симплекс идемпотентных мер в себя. Такие операторы разделены на два класса: первый класс содержит все матрицы размера* $n\times n\times n\times n$ *с положительными элементами, для которых для каждой*

$l-$й кубической матрицы существует число $\exists m_l \le n$, где все элементы $l-$й ($l \le n$) кубической матрицы являются нулевыми, за исключением элементов с индексами $p_{ijm_l,l}, p_{im_l j,l}$ и $p_{m_l ij,l}$; второй класс содержит все матрицы размера $n\times n\times n\times n$ с положительными элементами, содержащие как минимум одну кубическую матрицу. Эти матрицы играют роль стохастических матриц в случае идемпотентных мер. Для обоих классов кубических операторов найдены неподвижные точки и определен их характер.

Ключевые слова: *кубический оператор, симплекс, идемпотентная мера, неподвижная точка, притягивающая неподвижная точка, отталкивающая неподвижная точка.*

Introduction.

A dynamical system is a system the state of which changes when time is increasing.

Let N_0 be the initial size of the population, N_t be its size after t years. The growth factor *R* is the relative change in the number per year: $R=\left(N_{t+1}-N_t\right)/N_t$. If this value is equal to a constant, then the law governing the population dynamics has the form:

$$N_{t+1}=f\left(N_t\right)=\left(1+R\right)N_t \tag{1}$$

Model (1) is known as the Malthus equation (Thomas Malthus, 1766-1834) in discrete time. Equation (1) is linear and easily solved. After *t* years, the population will be equal to $N_t=\left(1+R\right)^t N_0$. Depending on the sign of the parameter *R*, the number will either increase indefinitely ($R>0$) or decrease ($-1<R<0$) in a geometric progression.

In general, linear models can not be used on longtime intervals, so we should consider a nonlinear model.

Let, $X\subseteq R$. To define a *discrete-time dynamical system* we consider a function $f: X\to X.$ At the moment we do not give conditions of f, the only assumption is that f is from *X* to itself.

For $x\in X$ denote by $f^n\left(x\right)$ the *n*-fold composition of f with itself (i.e. *n* time iteration of f to *x*:

$$f^n\left(x\right)=\underbrace{f(f(f...(f}_{n\ times}(x))...))). \tag{2}$$

Definition1. For arbitrary given $x_0\in X$ and $f: X\to X$ the *discrete-time dynamical system* (also called *forward orbit* or *trajectory* of x_0) is the sequence of points

$$x_0,\ x_1=f\left(x_0\right),\ x_2=f^2\left(x_0\right),\ x_3=f^3\left(x_0\right),... \tag{3}$$

Definition2. A point $x\in X$ is called a *fixed point* for $f: X\to X$ if $f\left(x\right)=x$.

Denote the set of all fixed points by $Fix\left(f\right)$.

Denote the sequence (3), for $x=x_0$, by $\tau\left(x\right)=\left\{x_j=f^j\left(x\right),\ j=0,1,...\right\}.$

Definition 3. The limit set $\omega\left(x_0\right)$ of orbit (3) is defined as

$$\omega(x_0)=\bigcap_{j=1}^{\infty}\overline{\tau\left(f^j(x_0)\right)}, \tag{4}$$

where $\overline{M_j}=\overline{\tau\left(f^j(x_0)\right)}$ is the closure of $M_j=\tau\left(f^j(x_0)\right)$.

The closure of a set *M* of points (in a topological space) consists of all points in M together with all limit points of M. Alternatively, the closure of M is the union of M and its boundary.

Definition 4.

• A fixed point x^* of f is *attracting* or *stable* if there is an open set (neighborhood) U containing x^* such that for all $x\in U$, the orbit $\tau(x)$ with initial point x converges to x^*.

• A fixed point x^* of f is *repelling* if there is an open set (neighborhood) U containing x^* such that for any $x\in U$, $x\neq x^*$, there is some $k\geq 1$ such that $f^k(x)\notin U$.

The main problem: Given a function f and initial point x_0 what ultimately happens with the sequence (3). Does the limit $\lim_{n\to\infty}x_n$ exist? If not what is the set of limit points of the sequence? Is this set finite or infinite?

Literature review

In this paper, we consider the simplex I_n of idempotent measures on $\{1,2,\dots,n\}$, where

$$I_n=\{(x_1,x_2,\dots,x_n)\in R_{\max}^n:\max_{1\leq i\leq n}x_i=0\}=$$

$$=\{(x_1,x_2,\dots,x_n)\in R_{\max}^n:x_1\oplus x_2\oplus\dots\oplus x_n=1\}$$

and consider cubic maps of I_n to itself. The term idempotent measure in our case is used in the sense of idempotent analysis ([1, 2, 3]) rather than in the sense of abstract harmonic analysis as in [4].

In [5] linear maps of the set of idempotent measures are constructed and their dynamical systems are studied. The quadratic operators defined on the set of idempotent measures which we shall consider in this paper are «idempotent» analogies of quadratic stochastic operators of [6].

In [7] we found construction of quadratic operators given by cubic matrices. In [8] described and studied fixed points and their characters of these quadratic operators case n=2. In works [9, 10, 11] described quadratic operators which map I_n to itself and studied fixed points of these operators in case n=3.

In works [12, 13] constructed dynamics of quadratic operators defined on a three dimensional simplex of idempotent measures.

In this paper we will continue researches which studied in works [7, 8, 9, 10, 11, 12, 13].

METHODS

In this section we shall answer the question: which cubic operator maps I_n to tself?

Definition 5. The $P:x\to x'$ mapping is called a cubic operator if the following equality is satisfied:

$$x_l' = \sum_{i,j,k=1}^{n} p_{ijk,l} x_i x_j x_k \tag{5}$$

The following theorem describes such quadratic maps. For the matrix $P = \left(p_{ijk,l}\right)_{i,j,k,l=1}^{n}$ a fixed $l \in \{1, 2, ..., n\}$ the matrix $\left(p_{ijk,l}\right)_{i,j,k=1}^{n}$ is called the *l*-th cubic matrix.

Theorem 1. A cubic operator $P = \left(p_{ijk,l}\right)_{i,j,k,l=1}^{n}$ with $p_{ijk,j} \geq 0$ maps I_n to itself if and only if it satisfies one of following conditions:

i) For each l*-th (*$l \leq n$*) cubic matrix of the operator* $P = \left(p_{ijk,l}\right)_{i,j,k,l=1}^{n}$ *exists number* $\exists m_l \leq n$*, where all of the elements of the* l*-th (*$l \leq n$*) cubic matrix are zeroes except elements as* $p_{m_l jk,l}$*,* $p_{im_l k,l}$ *and* $p_{ijm_l,l}$*. Where* $m_r \neq m_q$ *if* $r \neq q$*,* $i = \overline{1,n}$*,* $j = \overline{1,n}$*,* $k = \overline{1,n}$*,* $l = \overline{1,n}$*;*

ii) Cubic operator $P = \left(p_{ki,j}\right)_{i,j,k=1}^{n}$ *has at least one zero cubic matrix. (*$\exists m \leq n$ *such that the* m*-th cubic matrix consists of only zeroes, e.g.* $p_{ijk,m} = 0$*, where* $i = \overline{1,n}$*,* $j = \overline{1,n}$*,* $k = \overline{1,n}$*).*

Let's denote by $P_{(n;m_1,m_2,...m_n)}$ operator **P** as

$$P_{(n;m_1,m_2,...m_n)} = \left(p_{ijk,l}\right)_{i,j,k,l=1}^{n} \tag{6}$$

where $n-$ order of the cubic operator, $m_l -$ is the number which corresponding to the $l-$th cubic matrix of the cubic operator $P = \left(p_{ijk,l}\right)_{i,j,k,l=1}^{n}$, $m_l \leq n$, $m_r \neq m_q$, $r \neq q$.

In (6) we can see that there are n! cases possible. It will be too difficult to analysis all of these cases. That's why according to the form (6) we can make the set A as:

$$A = \left\{ m_l \,\middle|\, m_l = l,\ l = \overline{1,n} \right\} \tag{7}$$

For $n = 2$ and $n = 3$ from (6) and (7) we'll get following results:

1) if $n = 2$ i.e., operator $P : I_2 \to I_2$ has two cases:

1.1) $|A| = 2$ if $P_{(2;1,2)}$ so $m_1 = 1$, $m_2 = 2$;

1.2) $|A| = 0$ if $P_{(2;2,1)}$ so $m_1 = 2$, $m_2 = 1$.

2) if $n = 3$ i.e., operator $P : I_3 \to I_3$ has three cases:

2.1) $|A| = 3$ if $P_{(3;1,2,3)}$;

2.2) $|A| = 1$ if $P_{(3;1,3,2)}$, $P_{(3;2,1,3)}$ and $P_{(3;3,2,1)}$;

2.3) $|A| = 0$ if $P_{(3;2,3,1)}$ and $P_{(3;3,1,2)}$.

Analogically, for the second condition of the theorem 1 we can denote the set B as:

$$B = \left\{ w_l \,\middle|\, \text{all of the elements of the } l-th \text{ cubic matrix are zeroes}, l = \overline{1,n} \right\} \tag{8}$$

For $n=2$ and $n=3$ from (5) we'll get following results:

1) if $n=2$ i.e., operator $P: I_2 \to I_2$ has two cases:

1.1) $|B|=1$, e.g., $B=\{w_1\}$ and $B=\{w_2\}$;

1.2) $|B|=2$, e.g., $B=\{w_1, w_2\}$.

2) if $n=3$ i.e., operator $P: I_3 \to I_3$ has three cases:

2.1) $|B|=1$, e.g., $B=\{w_1\}$, $B=\{w_2\}$ and $B=\{w_3\}$;

2.2) $|B|=2$, e.g., $B=\{w_1, w_2\}$, $B=\{w_1, w_3\}$ and $B=\{w_2, w_3\}$;

2.3) $|B|=3$, e.g., $B=\{w_1, w_2, w_2\}$.

Results and discussion

Proposition 1. If $n=2$ and condition (i) of the theorem 1 is satisfied, then the set of the solutions of the equation $P(x)=x$ will have the following form:

$$Fix(P)=\begin{cases}\left(-\frac{1}{\sqrt{p_{111,1}}},\ 0\right) \text{ if } |A|=2,\ p_{111,1}\neq 0;\\ \left(0,\ -\frac{1}{\sqrt{p_{222,2}}}\right) \text{ if } |A|=2,\ p_{222,2}\neq 0;\\ (0,\ 0) \text{ for all other cases.}\end{cases} \tag{9}$$

Proof: Assume, P cubic operator is given as: $P=\left(p_{ijk,l}\right)_{i,j,k,l=1}^{2}$. Then by the formula (5) the equation $P(x)=x$ has the form

$$P=\left[\left(\begin{array}{cc|cc} p_{111,1} & p_{121,1} & p_{112,1} & p_{122,1}\\ p_{211,1} & p_{221,1} & p_{212,1} & p_{222,1}\end{array}\right)\left(\begin{array}{cc|cc} p_{111,2} & p_{121,2} & p_{112,2} & p_{122,2}\\ p_{211,2} & p_{221,2} & p_{212,2} & p_{222,2}\end{array}\right)\right]$$

$$\begin{cases} x_1=p_{111,1}x_1^3+\left(p_{121,1}+p_{112,1}+p_{211,1}\right)x_1^2x_2+\left(p_{122,1}+p_{212,1}+p_{221,1}\right)x_1x_2^2+p_{222,1}x_2^3\\ x_2=p_{111,2}x_1^3+\left(p_{121,2}+p_{112,2}+p_{211,2}\right)x_1^2x_2+\left(p_{122,2}+p_{212,2}+p_{221,2}\right)x_1x_2^2+p_{222,2}x_2^3\end{cases} \tag{10}$$

Let P satisfies condition (i) of theorem 1 then we have two cases: $P_{(2;1,2)}$ and $P_{(2;2,1)}$ as:

$$P_{(2;1,2)}=\left[\left(\begin{array}{cc|cc} p_{111,1} & p_{121,1} & p_{112,1} & p_{122,1}\\ p_{211,1} & p_{221,1} & p_{212,1} & 0\end{array}\right)\left(\begin{array}{cc|cc} 0 & p_{121,2} & p_{112,2} & p_{122,2}\\ p_{211,2} & p_{221,2} & p_{212,2} & p_{222,2}\end{array}\right)\right]$$

$$P_{(2;2,1)}=\left[\left(\begin{array}{cc|cc} 0 & p_{121,1} & p_{112,1} & p_{122,1}\\ p_{211,1} & p_{221,1} & p_{212,1} & p_{222,1}\end{array}\right)\left(\begin{array}{cc|cc} p_{111,2} & p_{121,2} & p_{112,2} & p_{122,2}\\ p_{211,2} & p_{221,2} & p_{212,2} & 0\end{array}\right)\right]$$

1a) Let P satisfies condition (i) of theorem 1 and $m_l = 1$ e.g., $|A| = 2$ and $P_{(2;1,2)}$. Then by (10) we'll find following system:

$$P_{(2;1,2)} : I_2 \to I_2 \Rightarrow \begin{cases} x_1 = p_{111,1}x_1^3 + (p_{121,1} + p_{112,1} + p_{211,1})x_1^2 x_2 + (p_{122,1} + p_{212,1} + p_{221,1})x_1 x_2^2 \\ x_2 = p_{222,2}x_2^3 + (p_{121,2} + p_{112,2} + p_{211,2})x_1^2 x_2 + (p_{122,2} + p_{212,2} + p_{221,2})x_1 x_2^2 \end{cases}$$

Following three cases are possible:

1a.1) Let $x_1 \le 0$, $x_2 = 0$ then we get: $\begin{cases} x_1 = p_{111,1}x_1^3 \\ x_2 = 0 \end{cases}$. The solutions are $x_2 = -\frac{1}{\sqrt{p_{111,1}}}$ $(p_{111,1} \neq 0)$ and $x_1 = 0$, hence the fixed points are $(0,0)$ and $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$.

1a.2) Assume $x_1 = 0$, $x_2 \le 0$ then we get $\begin{cases} x_1 = 0 \\ x_2 = p_{222,2}x_2^3 \end{cases}$. The solutions are is $x_2 = 0$ and $x_2 = -\frac{1}{\sqrt{p_{222,2}}}$ $(p_{222,2} \neq 0)$, hence the fixed points are $(0,0)$ and $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$.

1a.3) Let $x_1 = x_2 = 0$ then we get fixed point $(0,0)$.

1b) Let P satisfies condition (i) of theorem 1 and $m_l \neq l$ e.g., $|A| = 0$ and $P_{(2;2,1)}$. Then by (8) we'll find following system:

$$P_{(2;2,1)} : I_2 \to I_2 \Rightarrow \begin{cases} x_1 = (p_{121,1} + p_{112,1} + p_{211,1})x_1^2 x_2 + (p_{122,1} + p_{212,1} + p_{221,1})x_1 x_2^2 + p_{222,1}x_2^3 \\ x_2 = p_{111,2}x_1^3 + (p_{121,2} + p_{112,2} + p_{211,2})x_1^2 x_2 + (p_{122,2} + p_{212,2} + p_{221,2})x_1 x_2^2 \end{cases}$$

There is at least one zero element in $x = (x_1, x_2)$. So $x_1 = 0$ or $x_2 = 0$ then we'll get that $P_{(2;2,1)}$ has only one fixed point as: $(0,0)$.

Preposition 1 is proved.

Preposition 2. If $n = 2$ and condition (ii) of the theorem 1 is satisfied, then the set of the solutions of the equation $P(x) = x$ will have the following form:

$$Fix(P) = \begin{cases} \left(-\frac{1}{\sqrt{p_{111,1}}},\ 0\right) \text{ if } B = \{w_2\},\ p_{111,1} \neq 0; \\ \left(0,\ -\frac{1}{\sqrt{p_{222,2}}}\right) \text{ if } B = \{w_1\},\ p_{222,2} \neq 0; \\ (0,\ 0) \text{ for all other cases.} \end{cases} \tag{11}$$

Proof: Assume, P is given as a cubic matrix: $P = (p_{ijk,l})_{i,j,k,l=1}^{2}$. Then by the formula (1) the equation $P(x) = x$ has the form

$$\begin{cases} x_1 = p_{111,1}x_1^3 + \left(p_{121,1} + p_{112,1} + p_{211,1}\right)x_1^2x_2 + \left(p_{122,1} + p_{212,1} + p_{221,1}\right)x_1x_2^2 + p_{222,1}x_2^3 \\ x_2 = p_{111,2}x_1^3 + \left(p_{121,2} + p_{112,2} + p_{211,2}\right)x_1^2x_2 + \left(p_{122,2} + p_{212,2} + p_{221,2}\right)x_1x_2^2 + p_{222,2}x_2^3 \end{cases} \quad (12)$$

Let P satisfies condition (ii) of theorem 1 then two cases are possible:

2a) Let P satisfies condition (ii) of theorem 1 and $|B| = 1$ e.g., $B = \{w_1\}$ and $B = \{w_2\}$.

$$B = \{w_1\} \Rightarrow P = \left[\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{cc|cc} p_{111,2} & p_{121,2} & p_{112,2} & p_{122,2} \\ p_{211,2} & p_{221,2} & p_{212,2} & p_{222,2} \end{array}\right)\right]$$

$$B = \{w_2\} \Rightarrow P = \left[\left(\begin{array}{cc|cc} p_{111,1} & p_{121,1} & p_{112,1} & p_{122,1} \\ p_{211,1} & p_{221,1} & p_{212,1} & p_{222,1} \end{array}\right)\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\right]$$

Let's check the quadratic operator which $B = \{w_1\}$. Then from (12) we'll find following system:

$$P : I_2 \to I_2 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = p_{222,2}x_2^3 \end{cases}$$

Following three cases are possible:

2a.1) Let $x_1 \le 0$, $x_2 = 0$ then we'll find solution as: $x_1 = 0$, $x_2 = 0$, hence the fixed point is $(0,0)$.

2a.2) Assume $x_1 = 0$, $x_2 \le 0$ then we get $x_2 = p_{222,2}x_2^3$. The solutions are is $x_2 = 0$ and $x_2 = -\frac{1}{\sqrt{p_{222,2}}}$ $\left(p_{222,2} \neq 0\right)$, hence the fixed points are $(0,0)$ and $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$.

2a.3) Let $x_1 = x_2 = 0$ then we get fixed point $(0,0)$.

Analogically we can show that points $(0,0)$ and $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ are fixed points of the case $B = \{w_2\}$.

2b) Let P satisfies condition (ii) of theorem 1 and $|B| = 2$, e.g., $B = \{w_1, w_2\}$.

$$B = \{w_1, w_2\} \Rightarrow P = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then from (12) we'll find following system:

$$P : I_2 \to I_2 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

In this case there is only one fixed point as: $(0,0)$.

Preposition 2 is proved.

From prepositions 1 and 2 we shall get following theorem:

Theorem 2. If $n = 2$ and theorem 1 is satisfied, then $Fix(P)$ the set of all fixed point of cubic operator $P : I_2 \to I_2$ will have the following form:

$$Fix(P)=\begin{cases}\left(-\frac{1}{\sqrt{p_{111,1}}},\ 0\right) \text{ if } \left(|A|=2 \text{ or } B=\{w_2\}\right) \text{ and } p_{111,1}\neq 0;\\ \left(0,\ \frac{1}{p_{22,2}}\right) \text{ if } \left(|A|=2 \text{ or } B=\{w_1\}\right) \text{ and } p_{222,2}\neq 0;\\ (0,\ 0) \text{ for all other cases.}\end{cases} \tag{13}$$

When we map the non-fixed points by using quadratic maps they will change their coordinates. In this paper our aim is to find the trajectory of non-fixed points if we use quadratic maps to them so many times. In other words, we should find the solution of the following limit:

$$\lim_{n\to\infty} P^n(x)=\lim_{n\to\infty} P(P(P(...(P(x))...))) \tag{14}$$

Let, n=2. For getting the trajectories of quadratic maps of I_2 to itself let's use the results $\left(-\frac{1}{\sqrt{p_{111,1}}},0\right)$, $\left(0,-\frac{1}{\sqrt{p_{222,2}}}\right)$ and $(0,0)$. These results are found by using cubic operator $P_{(2;1,2)}$.

Definition 6. [see 10]. Suppose x_0 is fixed point for $F(x)$. Then x_0 is an *attracting fixed point* if $|F'(x_0)|<1$. The fixed point x_0 is a *repelling fixed point* if $|F'(x_0)|>1$. Finally, if $|F'(x_0)|=1$ the fixed point x_0 is called neutral or indifferent.

Theorem 3. For the cubic operator $P_{(2;1,2)}$:

a) the fixed point $(0,0)$ *is an attracting fixed point*

b) fixed points $\left(-\frac{1}{\sqrt{p_{111,1}}},0\right)$ *and* $\left(0,-\frac{1}{\sqrt{p_{222,2}}}\right)$ *are repelling fixed points.*

Proof. We have following system for $P_{(2;1,2)}$:

$$P(x)=x\Rightarrow\begin{cases}x_1=p_{111,1}x_1^3+\left(p_{121,1}+p_{112,1}+p_{211,1}\right)x_1^2x_2+\left(p_{122,1}+p_{212,1}+p_{221,1}\right)x_1x_2^2+p_{222,1}x_2^3\\ x_2=p_{111,2}x_1^3+\left(p_{121,2}+p_{112,2}+p_{211,2}\right)x_1^2x_2+\left(p_{122,2}+p_{212,2}+p_{221,2}\right)x_1x_2^2+p_{222,2}x_2^3\end{cases}$$

We know that at least one element of $x=(x_1,x_2)$ *is* zero. These arise following cases:

Case 1. Let $x_1\le 0$, $x_2=0$. Then we get one-dimensional function as $P(x_1)=p_{111,1}x_1^3$. Solving the equation $P(x)=x$ we obtain $x_1=0$ and $x_1=-\frac{1}{\sqrt{p_{111,1}}}$ $\left(p_{111,1}>0\right)$, hence fixed points are $\left(-\frac{1}{\sqrt{p_{111,1}}},0\right)$ and $(0,0)$. Derivative of the function $P'(x_1)=3p_{111,1}x_1^2$.

$$\left|P'(x_1)\right|=\left|3\cdot p_{111,1}\cdot x_1^2\right|=\left|3\cdot p_{111,1}\cdot\left(-\frac{1}{\sqrt{p_{111,1}}}\right)^2\right|=3>1$$

1.1) Since $x_1=-\frac{1}{\sqrt{p_{111,1}}}$ the fixed point $\left(p_{111,1}>0\right)$ is repelling for the function $P(x_1)=p_{111,1}x_1^3$.

1.2) The fixed point $x_1 = 0$ is an attracting fixed point for $P(x_1) = p_{111,1}x_1^3$ since $|P'(x_1)| = |3 \cdot p_{111,1} \cdot x_1^2| = |3 \cdot p_{111,1} \cdot 0^2| = 0 < 1$.

Case 2. Let $x_1 = 0$, $x_2 \le 0$. Then we get one-dimensional function as $P(x_2) = p_{222,2}x_2^3$. Solving the equation $P(x) = x$ we obtain $x_2 = 0$ and $x_2 = -\frac{1}{\sqrt{p_{222,2}}}$ $\left(p_{222,2} > 0\right)$, hence fixed points are $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ and $(0,0)$. Derivative of the function $P'(x_2) = 3p_{222,2}x_2^2$.

$$|P'(x_2)| = \left|3 \cdot p_{222,2} \cdot x_2^2\right| = \left|3 \cdot p_{222,2} \cdot \left(-\frac{1}{\sqrt{p_{222,2}}}\right)^2\right| = 3 > 1$$

1.1) Since $x_2 = -\frac{1}{\sqrt{p_{222,2}}}$ the fixed point $\left(p_{222,2} > 0\right)$ is repelling for the function $P(x_2) = p_{222,2}x_2^3$.

1.2) The fixed point $x_2 = 0$ is an attracting fixed point for $P(x_2) = p_{222,2}x_2^3$ since $|P'(x_2)| = |3 \cdot p_{222,2} \cdot x_2^2| = |3 \cdot p_{222,2} \cdot 0^2| = 0 < 1$.

Let us check the validity of the obtained results from Cases 1 and 2 by another method which used in [8]. In this method we should check the points, which are situated on two different sides of the fixed point $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$. These points we can denote as $\left(-\frac{k}{\sqrt{p_{111,1}}}, 0\right)$. Moreover, if $k > 1$ we will get point which situated between points $(-\infty, 0)$ and $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$; if $0 < k < 1$ we will get point which situated between points $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ and $(0,0)$. Then we solve the expression (14) step by step:

Step 1. Let $n = 1$, $x_1 < 0$ and $x_2 = 0$ then for $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ we get:

$$P_{(2;1,2)}(x) = P_{(2;1,2)}\begin{pmatrix} -\frac{k}{\sqrt{p_{111,1}}} \\ 0 \end{pmatrix} = \left[\left(\begin{array}{cc|cc} p_{111,1} & p_{121,1} & p_{112,1} & p_{122,1} \\ p_{211,1} & p_{221,1} & p_{212,1} & 0 \end{array}\right)\left(\begin{array}{cc|cc} 0 & p_{121,2} & p_{112,2} & p_{122,2} \\ p_{211,2} & p_{221,2} & p_{212,2} & p_{222,2} \end{array}\right)\right] \cdot \begin{pmatrix} -\frac{k}{\sqrt{p_{111,1}}} \\ 0 \end{pmatrix} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$$

$$\begin{cases} x_1^{(1)} = p_{111,1} \cdot \left(-\frac{k}{\sqrt{p_{111,1}}}\right)^3 + \left(p_{112,1} + p_{121,1} + p_{211,1}\right) \cdot \left(-\frac{k}{\sqrt{p_{111,1}}}\right)^2 \cdot 0 + \left(p_{122,1} + p_{221,1} + p_{212,1}\right) \cdot \left(-\frac{k}{\sqrt{p_{111,1}}}\right) \cdot 0 \\ x_2^{(1)} = p_{222,2} \cdot 0^2 + \left(p_{112,2} + p_{121,2} + p_{211,2}\right) \cdot \left(-\frac{k}{\sqrt{p_{111,1}}}\right)^2 \cdot 0 + \left(p_{122,2} + p_{221,2} + p_{212,2}\right) \cdot \left(-\frac{k}{\sqrt{p_{111,1}}}\right) \cdot 0 \end{cases}$$

$$P_{(2;1,2)}(x) = \begin{pmatrix} -\frac{k^3}{\sqrt{p_{111,1}}} \\ 0 \end{pmatrix}$$

Step 2. After this we should find the solution of expression (11) for $n = 2$.

$$P^2_{(2;1,2)}(x)=P_{(2;1,2)}\left(P_{(2;1,2)}(x)\right)=P_{(2;1,2)}\begin{pmatrix}-\dfrac{k^3}{\sqrt{p_{111,1}}}\\0\end{pmatrix}=\begin{pmatrix}x_1^{(2)}\\x_2^{(2)}\end{pmatrix}$$

$$\begin{cases}x_1^{(1)}=p_{111,1}\cdot\left(-\dfrac{k^3}{\sqrt{p_{111,1}}}\right)^3+\left(p_{112,1}+p_{121,1}+p_{211,1}\right)\cdot\left(-\dfrac{k^3}{\sqrt{p_{111,1}}}\right)^2\cdot 0+\left(p_{122,1}+p_{221,1}+p_{212,1}\right)\cdot\left(-\dfrac{k^3}{\sqrt{p_{111,1}}}\right)\cdot 0\\ x_2^{(1)}=p_{222,2}\cdot 0^2+\left(p_{112,2}+p_{121,2}+p_{211,2}\right)\cdot\left(-\dfrac{k^3}{\sqrt{p_{111,1}}}\right)^2\cdot 0+\left(p_{122,2}+p_{221,2}+p_{212,2}\right)\cdot\left(-\dfrac{k^3}{\sqrt{p_{111,1}}}\right)\cdot 0\end{cases}$$

$$P^2_{(2;1,2)}(x)=P_{(2;1,2)}\left(P_{(2;1,2)}(x)\right)=\begin{pmatrix}-\dfrac{k^9}{\sqrt{p_{111,1}}}\\0\end{pmatrix}=\begin{pmatrix}-\dfrac{k^{3^2}}{\sqrt{p_{111,1}}}\\0\end{pmatrix}$$

Step 3. Then, we will find the solution of expression (14) for case $n=3$

$$P^3_{(2;1,2)}(x)=P_{(2;1,2)}\left(P_{(2;1,2)}\left(P_{(2;1,2)}(x)\right)\right)=P_{(2;1,2)}\begin{pmatrix}\dfrac{k^9}{p_{11,1}}\\0\end{pmatrix}=\begin{pmatrix}x_1^{(3)}\\x_2^{(3)}\end{pmatrix}$$

$$\begin{cases}x_1^{(3)}=p_{111,1}\cdot\left(-\dfrac{k^9}{\sqrt{p_{111,1}}}\right)^3\\ x_2^{(3)}=p_{222,2}\cdot 0^2\end{cases}$$

$$P^3(x)=P\left(P\left(P(x)\right)\right)=\begin{pmatrix}-\dfrac{k^{3^3}}{\sqrt{p_{111,1}}}\\0\end{pmatrix}$$

Repeating operations we get solution of expression (14) for all natural values of n as:

$$\lim_{n\to\infty}P^n_{(2;1,2)}(x)=\lim_{n\to\infty}P_{(2;1,2)}\left(P_{(2;1,2)}\left(P_{(2;1,2)}\left(\ldots\left(P_{(2;1,2)}(x)\right)\ldots\right)\right)\right)=\left(-\frac{k^{3^n}}{\sqrt{p_{111,1}}},0\right) \quad (15)$$

From (15) we get conclusion, for $k>1$, i.e. if the point is on the left from the fixed point it will be move to the fixed point $(-\infty,0)$; for $k<1$, i.e., if the point is on the right from the fixed point it will be move to the fixed point $(0,0)$.

There are 3 cases possible:

1) for $k>1$ we will get the point as: $\lim_{n\to\infty}P^n_{(2;1,2)}(x)=\lim_{n\to\infty}\left(-\dfrac{k^{3^n}}{\sqrt{p_{111,1}}},0\right)=(-\infty,0)$;

2) for $k<1$ we will get the point as: $\lim_{n\to\infty}P^n_{(2;1,2)}(x)=\lim_{n\to\infty}\left(-\dfrac{k^{3^n}}{\sqrt{p_{111,1}}},0\right)=(0,0)$;

3) for $k=1$ we will get the point as: $\lim_{n\to\infty}P^n_{(2;1,2)}(x)=\lim_{n\to\infty}\left(-\dfrac{k^{3^n}}{\sqrt{p_{111,1}}},0\right)=\left(-\dfrac{1}{\sqrt{p_{111,1}}},0\right)$.

Similar considerations apply to the case $x_1 = 0$, $x_2 < 0$ then for $x_0 = \left(0, -\frac{k}{\sqrt{p_{222,2}}}\right)$ we will get following result for all natural values of n expression (14) will be defined as:

$$\lim_{n\to\infty} P^n_{(2;1,2)}(x) = \lim_{n\to\infty} P_{(2;1,2)}\left(P_{(2;1,2)}\left(P_{(2;1,2)}\left(\ldots\left(P_{(2;1,2)}(x)\right)\ldots\right)\right)\right) = \left(0, -\frac{k^{3^n}}{\sqrt{p_{222,2}}}\right) \quad (16)$$

From (16) we do conclusion, for $k > 1$, i.e. if the point x_0 is situated between points $(0, -\infty)$ and $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ it will be move to the point $(0, -\infty)$; for $k < 1$, i.e. if the point x_0 is between fixed points $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ and $(0,0)$, it will be move to the fixed point $(0,0)$.

There are 3 cases possible:

1) for $k > 1$ we will get the point as: $\lim_{n\to\infty} P^n_{(2;1,2)}(x) = \lim_{n\to\infty}\left(0, -\frac{k^{3^n}}{\sqrt{p_{222,2}}}\right) = (0, -\infty)$;

2) for $k < 1$ we will get the point as: $\lim_{n\to\infty} P^n_{(2;1,2)}(x) = \lim_{n\to\infty}\left(0, -\frac{k^{3^n}}{\sqrt{p_{222,2}}}\right) = (0,0)$;

3) for $k = 1$ we will get the point as: $\lim_{n\to\infty} P^n_{(2;1,2)}(x) = \lim_{n\to\infty}\left(0, -\frac{k^{3^n}}{\sqrt{p_{222,2}}}\right) = \left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$.

In other cases if we map the quadratic operator only one time all of the points in I_2 become fixed point as $(0,0)$. That's why, we don't need to research them.

ConclusionS

There are three markedly different types of fixed points: *attracting, repelling* and *neutral* fixed points.

In figure 1 we can see, any $x = (x_1, 0)$ which coordinates between points $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ and $(0,0)$, no matter how close to $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ lead to an orbit that tends «far» from $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ and close $(0,0)$; all point as $x = (0, x_2)$ which coordinates between points $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ and $(0,0)$, lead to an orbit that tends «far» from $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ and close $(0,0)$. Other point which coordinates between $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$

and $(-\infty, 0)$ or $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ and $(0, -\infty)$ are lead to an orbit that tends «far» from $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ or $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$. So, fixed points $\left(-\frac{1}{\sqrt{p_{111,1}}}, 0\right)$ and $\left(0, -\frac{1}{\sqrt{p_{222,2}}}\right)$ are *repelling fixed points*. Finally, fixed point $(0, 0)$ is an *attracting fixed point*.

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