NAMANGAN DAVLAT UNIVERSITETI
ILMIY AXBOROTNOMASI, [2026-10]
ISSN:2181-1458
ISSN:2181-0427

01.00.00-FIZIKA-MATEMATIKA
journal.namdu.uz
PHYSICS AND MATHEMATICS

UDK 517.95

OLTINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMA UCHUN TO‘G‘RI VA TESKARI MASALALAR

Abdukodirov Abdurashid Tolibjonovich
Farg‘ona davlat universiteti
ORCID: 0009-0007-6646-5993
E-mail: abdurashid1976@mail.ru
Tulqinboyev Tulqinjon Azizjon o‘g‘li
Farg‘ona davlat universiteti
ORCID: 0009-0006-5265-1114
E-mail: tulqinjon98@mail.ru

Annotatsiya: *Maqolada oltinchi tartibli xususiy hosilali differensial tenglama uchun Koshi masalasi va teskari masala tadqiq qilingan. Asosiy e’tibor berilgan boshlang‘ich shartlarda yechimning mavjudligi va yagonaligi masalalariga qaratilgan. Koshi masalasining qo‘yilishi, boshlang‘ich ma’lumotlarga qo‘yiladigan shartlar va yechimni qurish usuli batafsil bayon etilgan. Ketma-ket almashtirishlar yordamida oltinchi tartibli tenglama ikkinchi tartibli giperbolik tenglamalar sistemasiga keltiriladi. Hosil bo‘lgan Koshi masalalari esa Dalamber tipidagi formulalar va Dyuamel prinsipi yordamida yechiladi. Natijada boshlang‘ich shartlarni qanoatlantiruvchi yagona yechimning oshkor formulasi olingan. Bundan tashqari, Koshi masalasining yechimi asosida qo‘shimcha shart (ortiqcha aniqlovchi shart) bo‘yicha tenglamaning o‘ng tomonini aniqlash haqidagi teskari masala qo‘yilgan va uning bir qiymatli yechiluvchanligi isbotlangan. Olingan natijalar yuqori tartibli xususiy hosilali differensial tenglamalar nazariyasiga hissa qo‘shadi va to‘g‘ri hamda teskari masalalarni o‘rganishda qo‘llanilishi mumkin.*

Kalit so‘zlar: *Koshi masalasi, teskari masala, oltinchi tartibli differensial tenglama, xususiy hosila, yechimning mavjudlik va yagonaligi, Dalamber formulasi.*

ПРЯМАЯ И ОБРАТНАЯ ЗАДАЧИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ШЕСТОГО ПОРЯДКА

Аннотация: *В статье исследуются задача Коши и обратная задача для уравнения в частных производных шестого порядка. Основное внимание уделено вопросам существования и единственности решения при заданных начальных условиях. Подробно изложены постановка задачи Коши, условия, налагаемые на начальные данные, и метод построения решения. С помощью последовательных замен уравнение шестого порядка сводится к системе гиперболических уравнений второго порядка, а полученные задачи Коши решаются с помощью формул типа Даламбера и принципа Дюамеля. В результате получена явная формула единственного решения, удовлетворяющего начальным условиям. Кроме того, на основе решения задачи Коши сформулирована обратная задача определения правой части уравнения по дополнительному условию (условию переопределения) и доказана её однозначная разрешимость. Полученные результаты вносят вклад в теорию уравнений в частных производных высокого порядка и могут быть применены при исследовании прямых и обратных задач.*

Ключевые слова: *задача Коши, обратная задача, дифференциальное уравнение шестого порядка, частные производные, существование и единственность решения, формула Даламбера.*

DIRECT AND INVERSE PROBLEMS FOR A SIXTH-ORDER PARTIAL DIFFERENTIAL EQUATION

Abstract: *This article investigates the Cauchy problem and an inverse problem for a sixth-order partial differential equation. The main attention is paid to the existence and uniqueness of the solution under given initial conditions. The statement of the Cauchy problem, the conditions imposed on the initial data, and the method of constructing the solution are presented in detail. Using successive substitutions, the sixth-order equation is reduced to a system of second-order hyperbolic equations, and the resulting Cauchy problems are solved by means of d’Alembert-type formulas and Duhamel’s principle. As a result, an explicit formula for the unique solution satisfying the initial conditions is obtained. Furthermore, based on the solution of the Cauchy problem, an inverse problem of determining the right-hand side of the equation from an additional (overdetermination) condition is formulated, and its unique*

solvability is proved. The results contribute to the theory of higher-order partial differential equations and can be applied to the study of direct and inverse problems.

Keywords: *Cauchy problem, inverse problem, sixth-order differential equation, partial derivatives, existence and uniqueness of the solution, d'Alembert formula.*

Kirish

Zamonaviy matematika va uning amaliy yo‘nalishlarida xususiy hosilali differensial tenglamalar muhim o‘rin egallaydi. Ayniqsa, yuqori tartibli tenglamalar murakkab jarayonlarni ifodalashda keng qo‘llaniladi. Shunday tenglamalardan biri oltinchi tartibli xususiy hosilali differensial tenglamalardir. Bunday tenglamalar fizika, mexanika va muhandislik sohalarida uchraydigan murakkab tizimlarni tavsiflashda ishlatiladi.

Koshi masalasi esa differensial tenglamani yechishda boshlang‘ich shartlar yordamida aniq yechimni topishga xizmat qiladi. Ya’ni, tenglama bilan birga berilgan boshlang‘ich qiymatlar asosida yagona yechimni aniqlash masalasi Koshi masalasi deb ataladi. Ushbu masala nazariy jihatdan ham, amaliy jihatdan ham katta ahamiyatga ega.

Mazkur maqolada haqiqiy va har xil xarakteristikalarga ega bo‘lgan oltinchi tartibli xususiy hosilali differensial tenglama uchun Koshi masalasining qo‘yilishi, uning asosiy xossalari hamda yechimning mavjudligi va yagonaligi sodda va tushunarli tarzda isbotlangan. Koshi masalasi yechimidan foydalanib oltinchi tartibli xususiy hosilali differensial tenglama uchun teskari masalaning qo‘yilishi, yechimning mavjudligi va yagonaligi o‘rganilgan.

Adabiyotlar tahlili

Koshi masalasining qo‘yilishi va uning yechimining mavjudligi hamda yagonaligi masalalari birinchi navbatda umumiy differensial tenglamalar nazariyasida ishlab chiqilgan [1,2]. Giperbolik tipdagi tenglamalar uchun Koshi masalasi [3] ishda o‘rganilgan. Keyinchalik bu natijalar xususiy hosilali tenglamalarga ham tatbiq etilgan.

Yuqori tartibli tenglamalar uchun esa masala biroz murakkablashadi [4]. Chunki bu holatda boshlang‘ich shartlar soni ko‘payadi va yechimni aniqlash uchun qo‘shimcha shartlar talab etiladi. Shu sababli ko‘plab tadqiqotlarda bunday tenglamalarni soddalashtirish, kanonik ko‘rinishga keltirish hamda maxsus usullar yordamida yechish yo‘llari ishlab chiqilgan [5].

Oltinchi tartibli chiziqli xususiy hosilali differensial tenglamalarni to‘la klassifikatsiya qilish va mos kanonik ko‘rinishga keltirish masalalari [6], [7] ishlarda o‘rganilgan. Jumladan, haqiqiy va har xil xarakteristikalarga ega bo‘lgan oltinchi tartibli xususiy hosilali differensial tenglamalarni quyidagi

$$\left[\frac{\partial}{\partial y}-a\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}-b\frac{\partial}{\partial x}\right]\left[\frac{\partial^2}{\partial y^2}-c^2\frac{\partial^2}{\partial x^2}\right]\left[\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}\right]=0 \qquad (1)$$

ko‘rinishda ifodalash mumkinligi isbotlangan [6].

Natijalar va ularning tahlili

Haqiqiy va har xil xarakteristikalarga ega bo‘lgan (1) oltinchi tartibli xususiy hosilali differensial tenglamani $\Omega=\{(x,y): x\in R, y>0\}$ sohada qaraymiz, bu yerda $u(x,y)$ – noma’lum funksiya, a,b,c – berilgan o‘zgarmas haqiqiy sonlar bo‘lib, $a\neq b$, $a\neq\pm c$, $b\neq\pm c$, $a\neq\pm 1$, $b\neq\pm 1$, $c\neq\pm 1$ va $c\neq 0$.

(1) tenglama uchun Koshi masalasi $a=1, b=-1, c=1$ bo‘lgan holda [8] ishda o‘rganilgan.

K_1 **masala.** (1) tenglamani va

$$\left.\frac{\partial^j u(x,y)}{\partial y^j}\right|_{y=0}=\varphi_j(x),\ x\in R \qquad (2)$$

boshlang‘ich shartlarni qanoatlantiruvchi $u(x,y)\in C^5(\bar{\Omega})\cap C^6(\Omega)$ funksiya topilsin, bu yerda $\varphi_j(x)$ – berilgan funksiyalar, $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$.

Bu masalani yechish uchun

$$V(x,y)=\left[\frac{\partial^2}{\partial y^2}-c^2\frac{\partial^2}{\partial x^2}\right]\left[\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}\right] \qquad (3)$$

belgilashni kiritaylik. Bu belgilashga asosan (1) tenglamani

$$\left[\frac{\partial}{\partial y}-a\frac{\partial}{\partial x}\right]\left[\frac{\partial V}{\partial y}-b\frac{\partial V}{\partial x}\right]=0 \qquad (4)$$

ko‘rinishda yozib olamiz. (2) boshlang‘ich shartlardan va (3) belgilashdan foydalanib,

$$V(x,0)=\varphi_4(x)-\left(1+c^2\right)\varphi_2^{(2)}(x)+c^2\varphi_0^{(4)}(x) \quad (5)$$

$$V_y(x,0)=\varphi_5(x)-\left(1+c^2\right)\varphi_3^{(2)}(x)+c^2\varphi_1^{(4)}(x) \quad (6)$$

tengliklarni hosil qilamiz.

(4), (5), (6) ikkinchi tartibli giperbolik tipdagi tenglama uchun Koshi masalasi bo‘lib, uning yechimini xarakteristikalar usulidan foydalanib, quyidagi

$$V(x,y)=\frac{a}{a-b}c^2\varphi_0^{(4)}(x+by)-\frac{b}{a-b}c^2\varphi_0^{(4)}(x+ay)+\frac{c^2}{a-b}\left[\varphi_1^{(3)}(x+ay)-\varphi_1^{(3)}(x+by)\right]-$$
$$+\frac{b}{a-b}\left(1+c^2\right)\varphi_2^{(2)}(x+ay)-\frac{a}{a-b}\left(1+c^2\right)\varphi_2^{(2)}(x+by)-\frac{\left(1+c^2\right)}{a-b}\left[\varphi_3'(x+ay)-\varphi_3'(x+by)\right]-$$
$$-\frac{b}{a-b}\varphi_4(x+ay)+\frac{a}{a-b}\varphi_4(x+by)+\frac{1}{a-b}\int_{x+by}^{x+ay}\varphi_5(z)\,dz \quad (7)$$

ko‘rinishda topamiz.

Isbotlash mumkinki, agar $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ bo‘lsa, u holda (4) - (6) masalaning yechimi mavjud, yagona va turg‘un bo‘ladi.

$V(x,y)$ funksiya berilganlar orqali (7) ko‘rinishda topilganidan so‘ng, (3) ni quyidagi

$$\left[\frac{\partial^2}{\partial y^2}-c^2\frac{\partial^2}{\partial x^2}\right]\left[\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}\right]=V(x,y) \quad (8)$$

ko‘rinishdagi bir jinsli bo‘lmagan tenglama sifatida qarashimiz mumkin.

Agar ushbu

$$W(x,y)=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2} \quad (9)$$

belgilashni kiritsak, u holda (8) tenglamani

$$\frac{\partial^2 W}{\partial y^2}-c^2\frac{\partial^2 W}{\partial x^2}=V(x,y) \quad (10)$$

ko‘rinishda yozib olamiz. (9) belgilashdan va (2) boshlang‘ich shartlardan foydalanib esa,

$$W(x,0)=\varphi_2(x)-\varphi_0^{(2)}(x) \quad (11)$$

$$W_y(x,0)=\varphi_3(x)-\varphi_1^{(2)}(x) \quad (12)$$

boshlang‘ich shartlarni hosil qilamiz.

Natijada (10) bir jinsli bo‘lmagan tor tebranish tenglamasi uchun (11), (12) boshlang‘ich shartli Koshi masalasiga ega bo‘lamiz. Ushbu masalaning yechimini Dalamber formulasidan foydalanib [1], quyidagi

$$W(x,y)=-\frac{1}{2}\left[\varphi_0^{(2)}(x+cy)+\varphi_0^{(2)}(x-cy)\right]-\frac{1}{2c}\left[\varphi_1'(x+cy)-\varphi_1'(x-cy)\right]+$$
$$+\frac{1}{2}\left[\varphi_2(x+cy)+\varphi_2(x-cy)\right]+\frac{1}{2c}\int_{x-cy}^{x+cy}\varphi_3(z)\,dz+\frac{1}{2c}\int_0^y ds\int_{x-c(y-s)}^{x+c(y-s)}V(z,s)\,dz \quad (13)$$

ko‘rinishda topamiz.

$W(x,y)$ funksiya berilganlar orqali (13) ko‘rinishda aniqlanganidan so‘ng, (9) belgilash va (2) boshlang‘ich shartdan foydalanib,

$$\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}=W(x,y) \quad (14)$$

$$u(x,0)=\varphi_0(x) \quad (15)$$

$$u_y(x,0)=\varphi_1(x) \quad (16)$$

masalani hosil qilamiz. Ushbu masalaning yechimini ham Dalamber formulasidan foydalanib [1]

$$u(x,y)=\frac{1}{2}\left[\varphi_0(x+y)+\varphi_0(x-y)\right]+\frac{1}{2}\int_{x-y}^{x+y}\varphi_1(z)\,dz+\frac{1}{2}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}W(\xi,\tau)\,d\xi \quad (17)$$

ko‘rinishda topamiz.

(13) ni (17) ga qo‘yib ba’zi hisoblashlarni amalga oshirib, uni

$$u(x,y)=\frac{c^2}{2\left(c^2-1\right)}\left[\varphi_0(x+y)+\varphi_0(x-y)\right]-\frac{1}{2\left(c^2-1\right)}\left[\varphi_0(x+cy)+\varphi_0(x-cy)\right]+$$
$$+\frac{c^2}{2\left(c^2-1\right)}\int_{x-y}^{x+y}\varphi_1(z)\,dz-\frac{1}{2c\left(c^2-1\right)}\int_{x-cy}^{x+cy}\varphi_1(z)\,dz$$
$$+\frac{1}{4}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}\left[\varphi_2(\xi+c\tau)+\varphi_2(\xi-c\tau)\right]d\xi+\frac{1}{4c}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_{\xi-c\tau}^{\xi+c\tau}\varphi_3(z)\,dz+$$
$$+\frac{1}{4c}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}V(z,s)\,dz \qquad (18)$$

ko‘rinishda topamiz.

(7) ni (18) ga qo‘yib ba’zi hisoblashlarni va soddalashtirishlarni amalga oshirsak K_1 masalaning yechimini quyidagi:

$$u(x,y)=\frac{ab}{2\left(c^2-1\right)}\left[\frac{c^2\varphi_0(x+y)}{(a-1)(b-1)}+\frac{c^2\varphi_0(x-y)}{(a+1)(b+1)}-\frac{\varphi_0(x+cy)}{(a-c)(b-c)}-\frac{\varphi_0(x-cy)}{(a+c)(b+c)}\right]+$$
$$+\frac{ac^2}{(a-b)\left(b^2-1\right)\left(b^2-c^2\right)}\varphi_0(x+by)-\frac{bc^2}{(a-b)\left(a^2-1\right)\left(a^2-c^2\right)}\varphi_0(x+ay)+$$
$$+\frac{c^2}{2\left(c^2-1\right)}\int_{x-y}^{x+y}\varphi_1(z)\,dz-\frac{1}{2c\left(c^2-1\right)}\int_{x-cy}^{x+cy}\varphi_1(z)\,dz+$$
$$+\frac{c}{4(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_1^{(3)}(z+as)-\varphi_1^{(3)}(z+bs)\right]dz+$$
$$+\frac{\left(1+c^2\right)}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[b\varphi_2^{(2)}(z+as)-a\varphi_2^{(2)}(z+bs)\right]dz+$$
$$+\frac{1}{4}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}\left[\varphi_2(\xi+c\tau)+\varphi_2(\xi-c\tau)\right]d\xi+\frac{1}{4c}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_{\xi-c\tau}^{\xi+c\tau}\varphi_3(z)\,dz-$$
$$-\frac{\left(1+c^2\right)}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_3'(z+as)-\varphi_3'(z+bs)\right]dz+$$
$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[a\varphi_4(z+bs)-b\varphi_4(z+as)\right]dz+$$
$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}dz\int_{z+bs}^{z+as}\varphi_5(t)\,dt \qquad (19)$$

ko‘rinishda aniqlaymiz.

K_1 masalaning (19) ko‘rinishdagi yechimining mavjud va yagona ekanligi yuqoridagi (4), (10) va (14) tenglamalar uchun o‘rganilgan boshlang‘ich shartli masalalarga mos yechimlarning mavjud va yagona ekanligidan kelib chiqadi.

Natijada, quyidagi teorema isbotlandi:

1-teorema. Agar $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ bo‘lsa, u holda K_1 masalaning yechimi mavjud va yagona bo‘lib, u (19) tenglik bilan aniqlanadi.

Endi (1) tenglamaga mos ushbu

$$\left[\frac{\partial}{\partial y}-a\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}-b\frac{\partial}{\partial x}\right]\left[\frac{\partial^2}{\partial y^2}-c^2\frac{\partial^2}{\partial x^2}\right]\left[\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}\right]=f(x,y) \qquad (20)$$

bir jinsli bo‘lmagan oltinchi tartibli xususiy hosilali differensial tenglamani $\Omega=\{(x,y): x\in R, y>0\}$ sohada qaraymiz, bu yerda $u(x,y)$ – noma’lum funksiya, $f(x,y)$ - berilgan funksiya, $f(x,y)\in C^4(\overline{\Omega})$ va a,b,c – berilgan o‘zgarmas haqiqiy sonlar bo‘lib, $a\neq b$, $a\neq\pm c$, $b\neq\pm c$, $a\neq\pm 1$, $b\neq\pm 1$, $c\neq\pm 1$ va $c\neq 0$.

K_2 **masala.** (20) tenglamani va

$$\left.\frac{\partial^j u(x,y)}{\partial y^j}\right|_{y=0}=0,\ j=\overline{0,5} \qquad (21)$$

bir jinsli boshlang‘ich shartlarni qanoatlantiruvchi $u(x,y)\in C^5(\overline{\Omega})\cap C^6(\Omega)$ funksiya topilsin.

K_2 masalani yechish uchun, avval, (20) tenglamada quyidagi

$$\left[\frac{\partial}{\partial y}-a\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}-b\frac{\partial}{\partial x}\right]\left[\frac{\partial^2 u}{\partial y^2}-c^2\frac{\partial^2 u}{\partial x^2}\right]=V(x,y) \qquad (22)$$

almashtirishni bajaramiz. U holda (20) quyidagi

$$\frac{\partial^2 V}{\partial y^2}-\frac{\partial^2 V}{\partial x^2}=f(x,y) \quad (23)$$

bir jinsli bo‘lmagan tor tebranish tenglamasi ko‘rinishiga ega bo‘ladi.

Osongina ko‘rsatish mumkinki, (21) bir jinsli boshlang‘ich shartlarga hamda (22) almashtirishga asosan

$$V(x,0)=0,\ V_y(x,0)=0 \quad (24)$$

ekanligi kelib chiqadi.

Natijada, $V(x,y)$ funksiyani bir jinsli bo‘lmagan tor tebranish tenglamasi uchun bir jinsli boshlang‘ich shartli Koshi masalasi yechimi sifatida quyidagi ko‘rinishda topamiz:

$$V(x,y)=\frac{1}{2}\int_0^y d\eta\int_{x-(y-\eta)}^{x+(y-\eta)} f(\xi,\eta)\,d\xi. \qquad (25)$$

Buni e’tiborga olsak, (22) ning o‘ng tomoni ma’lum bo‘lgan to‘rtinchi tartibli tenglama sifatida qarashimiz mumkin. Agar (22) tenglamada quyidagi

$$\left[\frac{\partial}{\partial y}-a\frac{\partial}{\partial x}\right]\left[\frac{\partial u}{\partial y}-b\frac{\partial u}{\partial x}\right]=V_1(x,y) \qquad (26)$$

almashtirish bajarsak, u quyidagi ko‘rinishga ega bo‘ladi:

$$\frac{\partial^2 V_1}{\partial y^2}-c^2\frac{\partial^2 V_1}{\partial x^2}=V(x,y). \qquad (27)$$

Bevosita tekshirish mumkinki, (21) bir jinsli boshlang‘ich shartlarga asosan

$$V_1(x,0)=0,\ V_{1y}(x,0)=0 \quad (28)$$

ekanligi kelib chiqadi.

Natijada, $V_1(x,y)$ funksiyani ham bir jinsli bo‘lmagan tor tebranish tenglamasi uchun bir jinsli boshlang‘ich shartli Koshi masalasi yechimi sifatida quyidagi ko‘rinishda topamiz:

$$V_1(x,y)=\frac{1}{2c}\int_0^y dt\int_{x-c(y-t)}^{x+c(y-t)} V(s,t)\,ds. \qquad (29)$$

Buni e’tiborga olsak, (26) ni o‘ng tomoni ma’lum bo‘lgan ikkinchi tartibli tenglama sifatida qarashimiz mumkin. (26) tenglamada quyidagi almashtirishni bajaramiz:

$$s=2x+(a+b)y,\qquad t=(a-b)y, \qquad (30)$$

bu yerda, umumiylikni chegaralamagan holda $a>b$ deb hisoblashimiz mumkin.

(30) va $u(x,y)=U(s,t)$ belgilashga asosan, xususiy hosilalarni olib (26) ga qo‘yib ba’zi hisoblashlarni va soddalashtirishlarni amalga oshirib, uni quyidagicha yozib olamiz:

$$\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial s^2}=W(s,t) \qquad (31)$$

bu yerda $W(s,t)=\frac{1}{(a-b)^2}V_1\left(\frac{1}{2}\left(s-\frac{a+b}{a-b}t\right),\frac{t}{a-b}\right)$.

Endi (21) boshlang‘ich shartlarning dastlabki ikkitasini, (30) almashtirishga asosan yangi o‘zgaruvchilarga o‘tkizib quyidagi

$$U(s,0)=0,\ U_t(s,0)=0 \quad (32)$$

boshlang‘ich shartlarni keltirib chiqaramiz.

Natijada, $U(s,t)$ funksiyani bir jinsli bo‘lmagan tor tebranish tenglamasi uchun bir jinsli boshlang‘ich shartli Koshi masalasi yechimi sifatida quyidagi ko‘rinishda topamiz:

$$U(s,t)=\frac{1}{2}\int_0^t d\eta_1\int_{s-(t-\eta_1)}^{s+(t-\eta_1)}W(\xi_1,\eta_1)\,d\xi_1. \qquad (33)$$

Topilgan yechim ko‘rinishlaridan foydalanib, ba’zi soddalashtirishlarni amalga oshirib quyidagi yechimga ega bo‘lamiz:

$$u(x,y)=\frac{1}{2(a-b)^2}\int_0^{(a-b)y}d\eta_1\int_{2(x+by)+\eta_1}^{2(x+ay)-\eta_1}V_1\left(\frac{1}{2}\left(\xi_1-\frac{a+b}{a-b}\eta_1\right),\frac{\eta_1}{a-b}\right)d\xi_1 \qquad (34)$$

Topilgan (34) yechim ko‘rinishidan foydalanib, integral chegaralarini almashtirib ba’zi soddalashtirishlarni amalga oshirib quyidagi yechimga ega bo‘lamiz:

$$u(x,y)=\frac{1}{(a-b)}\int_0^y dt_1\int_{x+by-bt_1}^{x+ay-at_1}V_1(s_1,t_1)\,ds_1 \qquad (35)$$

Topilgan (35) yechim ko‘rinishidan foydalanib, (29) va (25) larni inobatga olib quyidagi yechimga ega bo‘lamiz:

$$u(x,y)=\frac{1}{4c(a-b)}\int_0^y dt_1\int_{x+by-bt_1}^{x+ay-at_1}ds_1\int_0^{t_1}dt\int_{s_1-c(t_1-t)}^{s_1+c(t_1-t)}ds\int_0^t d\eta\int_{s-(t-\eta)}^{s+(t-\eta)}f(\xi,\eta)\,d\xi. \qquad (36)$$

Bulardan xulosa qilib aytish mumkinki, (20) bir jinsli bo‘lmagan tenglamaning (2) bir jinsli bo‘lmagan boshlang‘ich shartlarni qanoatlantiruvchi yechimi (19) va (36) larning yig‘indisidan iborat bo‘ladi:

$$\begin{aligned}u(x,y)=&\frac{ab}{2(c^2-1)}\left[\frac{c^2\varphi_0(x+y)}{(a-1)(b-1)}+\frac{c^2\varphi_0(x-y)}{(a+1)(b+1)}-\frac{\varphi_0(x+cy)}{(a-c)(b-c)}-\frac{\varphi_0(x-cy)}{(a+c)(b+c)}\right]+\\&+\frac{ac^2}{(a-b)(b^2-1)(b^2-c^2)}\varphi_0(x+by)-\frac{bc^2}{(a-b)(a^2-1)(a^2-c^2)}\varphi_0(x+ay)+\\&+\frac{c^2}{2(c^2-1)}\int_{x-y}^{x+y}\varphi_1(z)\,dz-\frac{1}{2c(c^2-1)}\int_{x-cy}^{x+cy}\varphi_1(z)\,dz+\\&+\frac{c}{4(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_1^{(3)}(z+as)-\varphi_1^{(3)}(z+bs)\right]dz+\\&+\frac{(1+c^2)}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[b\varphi_2^{(2)}(z+as)-a\varphi_2^{(2)}(z+bs)\right]dz+\\&+\frac{1}{4}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}\left[\varphi_2(\xi+c\tau)+\varphi_2(\xi-c\tau)\right]d\xi+\frac{1}{4c}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_{\xi-c\tau}^{\xi+c\tau}\varphi_3(z)\,dz-\\&-\frac{(1+c^2)}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_3'(z+as)-\varphi_3'(z+bs)\right]dz+\end{aligned}$$

$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[a\varphi_4(z+bs)-b\varphi_4(z+as)\right]dz+$$

$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)} dz\int_{z+bs}^{z+as}\varphi_5(t)\,dt+$$

$$+\frac{1}{4c(a-b)}\int_0^y dt_1\int_{x+by-bt_1}^{x+ay-at_1} ds_1\int_0^{t_1} dt\int_{s_1-c(t_1-t)}^{s_1+c(t_1-t)} ds\int_0^t d\eta\int_{s-(t-\eta)}^{s+(t-\eta)} f(\xi,\eta)\,d\xi\,. \qquad (37)$$

Endi ushbu

$$\left[\frac{\partial}{\partial y}-a\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}-b\frac{\partial}{\partial x}\right]\left[\frac{\partial^2}{\partial y^2}-c^2\frac{\partial^2}{\partial x^2}\right]\left[\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}\right]=f(y) \qquad (38)$$

oltinchi tartibli xususiy hosilali differensial tenglamani $\Omega=\{(x,y): x\in R, y>0\}$ sohada qaraylik, bu yerda $u(x,y)$ va $f(y)$ – noma'lum funksiyalar, a,b,c – berilgan o'zgarmas haqiqiy sonlar bo'lib, $a\neq b$, $a\neq\pm c$, $b\neq\pm c$, $a\neq\pm 1$, $b\neq\pm 1$, $c\neq\pm 1$ va $c\neq 0$.

Teskari masala. (38) tenglamani, (2) boshlang'ich va

$$u(y,y)=\omega(y) \qquad (39)$$

qo'shimcha shartlarni qanoatlantiruvchi $u(x,y)\in C^5(\bar{\Omega})\cap C^6(\Omega)$ va $f(y)\in C[0,+\infty)$ funksiyalar juftligi topilsin, bu yerda $\omega(y)$ va $\varphi_j(x)$ – berilgan funksiyalar, $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ hamda $\omega^{(k)}(0)=\sum_{j=0}^{k}C_k^j\,\varphi_j^{(k-j)}(0)$, $k=\overline{0,5}$.

Teskari masalani tadqiq etishda $f(x,y)=f(y)$ bo'lgandagi Koshi masalasining (37) ko'rinishda topilgan yechimidan foydalanamiz. Shundan kelib chiqib, (37) yechimni $f(x,y)=f(y)$ bo'lganda qayta yozib

$$u(x,y)=\frac{ab}{2(c^2-1)}\left[\frac{c^2\varphi_0(x+y)}{(a-1)(b-1)}+\frac{c^2\varphi_0(x-y)}{(a+1)(b+1)}-\frac{\varphi_0(x+cy)}{(a-c)(b-c)}-\frac{\varphi_0(x-cy)}{(a+c)(b+c)}\right]+$$

$$+\frac{ac^2}{(a-b)(b^2-1)(b^2-c^2)}\varphi_0(x+by)-\frac{bc^2}{(a-b)(a^2-1)(a^2-c^2)}\varphi_0(x+ay)+$$

$$+\frac{c^2}{2(c^2-1)}\int_{x-y}^{x+y}\varphi_1(z)\,dz-\frac{1}{2c(c^2-1)}\int_{x-cy}^{x+cy}\varphi_1(z)\,dz+$$

$$+\frac{c}{4(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_1^{(3)}(z+as)-\varphi_1^{(3)}(z+bs)\right]dz+$$

$$+\frac{(1+c^2)}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[b\varphi_2^{(2)}(z+as)-a\varphi_2^{(2)}(z+bs)\right]dz+$$

$$+\frac{1}{4}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)}\left[\varphi_2(\xi+c\tau)+\varphi_2(\xi-c\tau)\right]d\xi+\frac{1}{4c}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_{\xi-c\tau}^{\xi+c\tau}\varphi_3(z)\,dz-$$

$$-\frac{(1+c^2)}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_3'(z+as)-\varphi_3'(z+bs)\right]dz+$$

$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^\tau ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[a\varphi_4(z+bs)-b\varphi_4(z+as)\right]dz+$$

$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{x-(y-\tau)}^{x+(y-\tau)} d\xi\int_0^{\tau} ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)} dz\int_{z+bs}^{z+as}\varphi_5(t)\,dt+$$

$$+\frac{1}{4c(a-b)}\int_0^y dt_1\int_{x+by-bt_1}^{x+ay-at_1} ds_1\int_0^{t_1} dt\int_{s_1-c(t_1-t)}^{s_1+c(t_1-t)} ds\int_0^t d\eta\int_{s-(t-\eta)}^{s+(t-\eta)} f(\eta)\,d\xi\,. \qquad (40)$$

ga ega bo‘lamiz. Demak, agar $f(y)\in C[0,+\infty)$ noma’lum funksiyani berilganlar orqali aniqlasak, yuqoridagi teskari masalaning yechimi (40) ko‘rinishda aniqlanadi.

Shuning uchun, $f(y)\in C[0,+\infty)$ noma’lum funksiyani topish uchun (40) ni (39) qo‘shimcha shartga qo‘yib, ba’zi bir soddalashtirishlarni amalga oshirgach quyidagiga ega bo‘lamiz:

$$\int_0^y dt_1\int_{y+by-bt_1}^{y+ay-at_1} ds_1\int_0^{t_1} dt\int_{s_1-c(t_1-t)}^{s_1+c(t_1-t)} ds\int_0^t (t-\eta)\,f(\eta)\,d\eta = 2c(a-b)\left[\omega(y)-R(y)\right], \qquad (41)$$

bu yerda, $R(y)=\dfrac{ab}{2(c^2-1)}\left[\dfrac{c^2\varphi_0(2y)}{(a-1)(b-1)}+\dfrac{c^2\varphi_0(0)}{(a+1)(b+1)}-\dfrac{\varphi_0(y(1+c))}{(a-c)(b-c)}-\dfrac{\varphi_0(y(1-c))}{(a+c)(b+c)}\right]+$

$$+\frac{ac^2}{(a-b)(b^2-1)(b^2-c^2)}\varphi_0(y(1+b))-\frac{bc^2}{(a-b)(a^2-1)(a^2-c^2)}\varphi_0(y(1+a))+$$

$$+\frac{c^2}{2(c^2-1)}\int_0^{2y}\varphi_1(z)\,dz-\frac{1}{2c(c^2-1)}\int_{y(1-c)}^{y(1+c)}\varphi_1(z)\,dz+$$

$$+\frac{c}{4(a-b)}\int_0^y d\tau\int_{\tau}^{2y-\tau} d\xi\int_0^{\tau} ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_1^{(3)}(z+as)-\varphi_1^{(3)}(z+bs)\right]dz+$$

$$+\frac{(1+c^2)}{4c(a-b)}\int_0^y d\tau\int_{\tau}^{2y-\tau} d\xi\int_0^{\tau} ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[b\varphi_2^{(2)}(z+as)-a\varphi_2^{(2)}(z+bs)\right]dz+$$

$$+\frac{1}{4}\int_0^y d\tau\int_{\tau}^{2y-\tau}\left[\varphi_2(\xi+c\tau)+\varphi_2(\xi-c\tau)\right]d\xi+\frac{1}{4c}\int_0^y d\tau\int_{\tau}^{2y-\tau} d\xi\int_{\xi-c\tau}^{\xi+c\tau}\varphi_3(z)\,dz-$$

$$-\frac{(1+c^2)}{4c(a-b)}\int_0^y d\tau\int_{\tau}^{2y-\tau} d\xi\int_0^{\tau} ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[\varphi_3'(z+as)-\varphi_3'(z+bs)\right]dz+$$

$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{\tau}^{2y-\tau} d\xi\int_0^{\tau} ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)}\left[a\varphi_4(z+bs)-b\varphi_4(z+as)\right]dz+$$

$$+\frac{1}{4c(a-b)}\int_0^y d\tau\int_{\tau}^{2y-\tau} d\xi\int_0^{\tau} ds\int_{\xi-c(\tau-s)}^{\xi+c(\tau-s)} dz\int_{z+bs}^{z+as}\varphi_5(t)\,dt\,.$$

Osongina ko‘rsatish mumkinki, $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ bo‘lganda yuqoridagi $R(y)$ funksiya $C^6[0,+\infty)$ sinfga tegishli bo‘ladi.

(41) tenglikning chap tomonini, undagi oxirgi integral S o‘zgaruvchiga bog‘liq bo‘lmaganligi uchun oxirgi ikki integralda integrallash tartibini o‘zgartirib quyidagicha yozib olamiz:

$$I=\int_0^y dt_1\int_{y+by-bt_1}^{y+ay-at_1} ds_1\int_0^{t_1} dt\int_{s_1-c(t_1-t)}^{s_1+c(t_1-t)} ds\int_0^t (t-\eta)\,f(\eta)\,d\eta=\int_0^y dt_1\int_{y+by-bt_1}^{y+ay-at_1} ds_1\int_0^{t_1} dt\int_0^t (t-\eta)\,f(\eta)\,d\eta\int_{s_1-c(t_1-t)}^{s_1+c(t_1-t)} ds\,. \qquad (42)$$

Endi (42) tenglikdagi oxirgi integralni hisoblasak hamda keyin qoladigan oxirgi ikki integralda ham integrallash tartibini o‘zgartirsak

$$I=\frac{c}{3}\int_0^y dt_1\int_{y+by-bt_1}^{y+ay-at_1} ds_1\int_0^{t_1}(t_1-\eta)^3\,f(\eta)\,d\eta \qquad (43)$$

ga ega bo‘lamiz. (43) tenglikning o‘ng tomonidagi oxirgi integral S_1 o‘zgaruvchiga bog‘liq bo‘lmaganligi uchun oxirgi ikki integralda integrallash tartibini o‘zgartirib

$$I=\frac{c}{3}\int_0^y dt_1\int_0^{t_1}(t_1-\eta)^3\,f(\eta)\,d\eta\int_{y+by-bt_1}^{y+ay-at_1} ds_1=\frac{c}{3}\int_0^y dt_1\int_0^{t_1}(a-b)(y-t_1)(t_1-\eta)^3\,f(\eta)\,d\eta$$

ni hosil qilamiz. Agar oxirgi integral ifodada ham yana integrallash tartibini o‘zgartirsak

$$I=\frac{c(a-b)}{3}\int_0^y dt_1\int_0^{t_1}(y-t_1)(t_1-\eta)^3\,f(\eta)\,d\eta=\frac{c(a-b)}{3}\int_0^y f(\eta)\,d\eta\int_\eta^y(y-t_1)(t_1-\eta)^3\,dt_1$$

munosabat o‘rinli bo‘ladi. Agar yuqoridagi tenglikning o‘ng tomonidagi oxirgi integralda $z=t_1-\eta$ deb almashtirish bajarsak, u holda quyidagiga ega bo‘lamiz:

$$\int_\eta^y(y-t_1)(t_1-\eta)^3\,dt_1=\int_0^{y-\eta}(y-\eta-z)z^3dz=\left[(y-\eta)\frac{z^4}{4}-\frac{z^5}{5}\right]_{z=0}^{z=y-\eta}=\frac{(y-\eta)^5}{20}.$$

Yuqoridagilarni e’tiborga olsak, u holda, (43) ni

$$\frac{c(a-b)}{60}\int_0^y(y-\eta)^5\,f(\eta)\,d\eta=2c(a-b)\left[\omega(y)-R(y)\right]$$

yoki

$$\frac{1}{120}\int_0^y(y-\eta)^5\,f(\eta)\,d\eta=\omega(y)-R(y) \qquad (44)$$

ko‘rinishda yozib olishimiz mumkin.

(44) birinchi tur Volterra integral tenglamasi bo‘lib, uning chap tomoni va 5-tartibgacha hosilalari $y=0$ da har doim nolga teng. Shuning uchun (44) yechimga ega bo‘lishi uchun o‘ng tomon ham shunday bo‘lishi shart. Bu esa masala shartida berilgan ulash shartlaridan kelib chiqadi.

Yuqoridagini inobatga olib, (44) tenglikning har ikki tomonini, parametr bo‘yicha hosila olish qoidalaridan foydalanib, y o‘zgaruvchi bo‘yicha ketma-ket 6 marta differensiallasak, $f(y)$ ni quyidagi ko‘rinishda topamiz:

$$f(y)=\omega^{(6)}(y)-R^{(6)}(y). \qquad (45)$$

Agar $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ bo‘lganda $R(y)$ funksiya $C^6[0,+\infty)$ sinfga tegishli ekanidan hamda $\omega(y)\in C^6[0,+\infty)$ ekanidan (45) ga asosan $f(y)\in C[0,+\infty)$ ekanligi kelib chiqadi. Shunday qilib, quyidagi teorema isbotlandi.

2-teorema. Agar $\omega(y)\in C^6[0,+\infty)$, $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ bo‘lib, quyidagi ulash shartlari bajarilsa:

$$\omega^{(k)}(0)=\sum_{j=0}^{k}C_k^j\,\varphi_j^{(k-j)}(0),\quad k=\overline{0,5},$$

u holda teskari masala yagona yechimga ega va u (40), (45) formulalar bilan aniqlanadi.

Xulosa

Ushbu maqolada haqiqiy va har xil xarakteristikalarga ega bo‘lgan oltinchi tartibli (1) xususiy hosilali differensial tenglama uchun to‘g‘ri va teskari masalalar o‘rganildi.

Ketma-ket (3), (9), (22) va (26) almashtirishlar yordamida oltinchi tartibli tenglama ikkinchi tartibli giperbolik tenglamalar ketma-ketligiga keltirildi. Hosil bo‘lgan Koshi masalalari Dalamber formulasi va Dyuamel prinsipi yordamida yechildi. Natijada bir jinsli tenglama uchun bir jinsli bo‘lmagan boshlang‘ich shartli K_1 masalaning yechimi (19) oshkor formula ko‘rinishida topildi. $\varphi_j(x)\in C^{6-j}(R)$, $j=\overline{0,5}$ bo‘lganda bu yechimning mavjudligi va yagonaligi isbotlandi (1-teorema).

Bir jinsli bo‘lmagan (20) tenglama uchun bir jinsli boshlang‘ich shartli K_2 masalaning yechimi olti karrali integral ko‘rinishida (36) qurildi. Superpozitsiya prinsipiga ko‘ra umumiy Koshi masalasining yechimi (37) formula bilan ifodalandi.

Koshi masalasi yechimiga asoslanib, tenglamaning noma'lum o'ng tomoni $f(y)$ ni $u(y,y)=\omega(y)$ qo'shimcha shart bo'yicha aniqlash haqidagi teskari masala qo'yildi. Bu masala $(y-\eta)^5$ yadroli birinchi tur Volterra integral tenglamasiga keltirildi va uning yechimi (45) oshkor formula bilan topildi. $\omega(y)\in C^6[0,+\infty)$ bo'lganda va ulash shartlari bajarilganda teskari masala yagona yechimga ega ekanligi isbotlandi (2-teorema).

Olingan natijalar $a=1$, $b=-1$, $c=1$ bo'lgan holda [8] ishda o'rganilgan masalani umumlashtiradi. Ularni karrali xarakteristikali hollarni ($a=b$ yoki $c^2=1$), shuningdek o'ng tomoni $f(x,y)=g(x)h(y)$ ko'rinishidagi teskari masalalarni o'rganishda ham qo'llash mumkin.

Foydalanilgan adabiyotlar

1.Tikhonov A.N., Samarskii A.A. Equations of Mathematical Physics. – New York: Macmillan, **1963**. – 765 p.

2.Koshlyakov N.S., Smirnov M.M., Gliner E.B. Differential Equations of Mathematical Physics. – Amsterdam: North-Holland Publishing Company, 1964. –701 p.

3.Tersenov S.A. On the Cauchy problem, with data given on a curve of degenerate type, for a hyperbolic equation. Differential Equations. 1966, Vol.2, No 1. pp. 125-130.

4. Baikuziev K.B. Boundary value problems for higher-order equations. – Tashkent, 1972.

5. Karimov Sh. T. The Cauchy problem for the degenerated partial differential equation of the high even order. Siberian Electronic Mathematical Reports,

6. 15 (2018), 853–862.

6. Abdukodirov A.T, Tulkinboev T.A. Classification and reduction to canonical form of linear differential equations partial of the sixth-order with non-multiple characteristics. Bulletin of the Karaganda University. Mathematics Series, 2025, 3(119), pp. 4-21.

7. Abdukodirov A.T, Tulkinboev T.A. Classification and reduction to canonical form of sixth-order linear partial differential equations with multiple and complex characteristics. Bull. Inst. Math., 2024, Vol.7, No 6, pp. 1-13.

8. Уринов А.К., Каримов Ш.Т. Операторы Эрдейи-Кобера и их приложения дифференциальным уравнениям в частных производных. Фергана: изд. «Фарғона», 2021. - 202 с.