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EXTREMALITY OF TRANSLATION-INVARIANT GIBBS MEASURES FOR THE HC-BLUME-CAPEL MODEL IN THE «CYCLE» CASE ON THE CAYLEY TREE

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Abstract: *In this paper, we consider the translation-invariant Gibbs measures for the HC-Blume–Capel model in the case of a «cycle» on a second-order Cayley tree. It is proved that for* $\theta > 0$ *there exists a unique translation-invariant Gibbs measure in the «cycle» case for the model. In addition, the problem of the (non-)extremality of this measure is studied.*

Keywords: *Cayley tree, configuration, HC-Blume–Capel model, Gibbs measures, extreme measure.*

KELI DARAXTIDA «SIKL» HOLATIDAGI HC BLYUM-KAPEL MODELI UCHUN TRANSLATSION-INVARIANT GIBBS O‘LCHOVINING EKSTREMALLIGI

Annotatsiya: *Ushbu maqolada ikkinchi tartibli Keli daraxtida «sikl» holatidagi HC-Blyum–Kapel modeli uchun translatsion-invariant Gibbs o‘lchovlari qaraladi.* $\theta > 0$ *bo‘lganda modelning «sikl» holatida yagona translatsion-invariant Gibbs o‘lchovi mavjudligi isbotlanadi. Bundan tashqari, ushbu o‘lchovning ekstremalligi (noekstremalligi) masalasi o‘rganiladi.*

Kalit so‘zlar: *Keli daraxti, konfiguratsiya, HC-Blyum–Kapel modeli, Gibbs o‘lchovlari, ekstremal o‘lchov.*

ЭКСТРЕМАЛЬНОСТЬ ТРАНСЛЯЦИОННО-ИНВАРИАНТНЫХ МЕР ГИББСА ДЛЯ НС-МОДЕЛИ БЛЮМА–КАПЕЛЯ В СЛУЧАЕ «ЦИКЛА» НА ДЕРЕВЕ КЭЛИ

Аннотация: *В данной статье рассматриваются трансляционно-инвариантные меры Гиббса для НС-модели Блюма–Капеля в случае «цикла» на дереве Кэли второго порядка. Доказано, что при* $\theta > 0$ *в случае «цикла» для данной модели существует единственная трансляционно-инвариантная мера Гиббса. Кроме того, исследуется вопрос об экстремальности (неэкстремальности) данной меры.*

Ключевые слова: *дерево Кэли, конфигурация, НС-модель Блюма–Капеля, меры Гиббса, экстремальная мера.*

Introduction

In the theory of Gibbs measures, the main problem is a complete description of all limit Gibbs measures for a given Hamiltonian. The Gibbs measure is a fundamental law that determines the probability of a microscopic state of a given physical system and it plays an important role in determining the existence of a phase transition of a particular physical system, because each limiting Gibbs measure is associated with one phase of the physical system and occurs phase transition when the Gibbs measure is not unique [1–5].

In addition, it is known that the set of all limit Gibbs measures forms a non-empty convex compact subset in the set of all probability measures. Recall (see [8]) that a Gibbs measure μ *(as an element of a convex set) is called extremal if* $\mu \neq s\nu + (1-s)\nu'$ for different Gibbs measure ν, ν' and $0 < s < 1$. Each point of this convex set is uniquely decomposed in terms of its extremal points. In this regard, of particular interest is the description of all extreme points of this convex set, i.e. extremal Gibbs measures (see [6–9]).

In [10], Mazel and Suhov introduced and studied the *HC*-model (hard disk, hard core) on the $d-$ dimensional lattice Z^d. Quite a lot of papers are devoted to the study of Gibbs limit measures for HC models on the Cayley tree (see [11–17]). HC-model is of interest from the point of view of statistical mechanics, combinatorics and theory of neural networks. In

numerous works (see, for example, [16, 17]) periodic Gibbs measures on the Cayley tree are studied for various models of statistical mechanics. These measures are mostly translation-invariant or periodic with period two.

The papers [18, 19] consider the problem of extremality Gibbs measures for SOS and Potts-SOS models on the Cayley tree.

The Blume-Capel model is a two-dimensional spin system where the spin can take three values: $-1,0,1$. It was originally introduced to study the $He^3 - He^4$ phase transition (see [20–22]). Concerning other results according to the Blume-Capel model see for example [23–27]. In particular, in [23], translation-invariant Gibbs measures (TIGM) was considered for the Blume-Capel model on a Cayley tree of order $k \geq 2$. In addition, the (non) extreme problem for the only Gibbs measure is studied. In [24], TIGM was considered for the Blume-Capel model in the case of a «wand» on a Cayley tree of order $k \geq 2$.. The exact critical value $\theta_{cr} = 1$ is found such that for $\theta \geq \theta_{cr}$ there is only one TIGM, and for $0 < \theta < \theta_{cr}$ there are exactly three TIGMs in the case of a «wand» for the model under consideration. In addition, the problem of (non)extremality of these measures is studied. In the paper [26] the extreme periodic Gibbs measures for the HC-Blume-Capel model in the case of a «wand» on the Cayley tree are studied.

In this paper, we study translation-invariant Gibbs measures for the *HC*-Blume-Capel model in the case of a «cycle» on a second-order Cayley tree. It is proved that for the model under consideration. In addition, the problem of (non)extremality of this measure is studied.

Preliminaries and system of functional equations

The Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e. graph without cycles, from each vertex which comes out $k+1$ edges, where V is the set of vertices Γ^k, $L-$ is its set of edges. Let $i-$ be the incidence function assigning each edge $l \in L$ *to its endpoints* $x, y \in V$. *If* $i(l) = \{x, y\}$, then the vertices x and y are called *nearest neighbors* and are denoted by $\langle x, y \rangle$. Let $d(x, y), x, y \in V$ *be the distance between the vertices* $x, y,$ i.e. the number of edges of the shortest path connecting x and y. Consider a model where spin takes values from the set $\Phi = \{-1, 0, 1\}$. Then *configuration* σ on V is defined as the function $x \in V \to \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. Let $A \subset V$. We denote by Ω_A *the space of configurations defined on the set* A.

Consider a graph with three vertices $-1,0,1$ (on the set of values $\sigma(x)$), which has the following form (see [9–11]):

cycle: $\{0,-1\},\{0,1\},\{-1,1\}$

The Hamiltonian of the Blume-Capel model is defined as follows:

$$H(\sigma) = J \sum_{\langle x,y\rangle, x,y \in V} (\sigma(x) - \sigma(y))^2, \quad (1)$$

where $J \in R$.

Let $x^o \in V$ *– be a fixed point. We will write* $x \prec y$ *if the path from* x^0 to y *passes through* x.

We denote:

$$W_n = \{x \in V : d(x^0, x) = n\}, \quad V_n = \{x \in V : d(x^0, x) \leq n\}.$$

Point y is called the «direct descendant» of point x *if* $x \prec y$ and $d(x, y) = 1$.

Let us denote by $S(x)$ – the set «*direct descendants*» of the point $x \in V$.

Let $O = \{cycle\}, G \in O$. A configuration σ *is called a* $G-$*admissible configuration* on the Cayley tree (in V_n or W_n), if $\{\sigma(x), \sigma(y)\}-$ is an edge G *for any nearest pair of neighbors* x, y from V *(from* V_n*)*. We denote the set of $G-$ admissible configurations by $\Omega^G (\Omega^G_{V_n})$.

Let $h : x \to h_x = (h_{-1,x}, h_{0,x}, h_{1,x}) \in R^3$ is a real vector-valued function with respect to the variable $x \in V \setminus \{x^0\}$. Consider the probability distribution μ^n *on* $\Omega^G_{V_n}$:

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} \tilde{h}_{\sigma(x),x}\}, \quad (2)$$

where $\sigma_n \in \Omega^G_{V_n}, Z_n = \sum_{\bar{\sigma}_n \in \Omega^G_{V_n}} \exp\{-\beta H(\bar{\sigma}_n) + \sum_{x \in W_n} h_{\bar{\sigma}(x),x}\}$ *and* $h_{\bar{\sigma},x} \in R.$

The probability distribution $\mu^{(n)}$, $(\forall n \geq 1)$ is said to be consistent if

$$\sum \mu^{(n)}(\sigma_{n-1},\sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}) \quad (3)$$

for all $n \geq 1$ *and* $\sigma_{n-1} \in \Omega^G_{V_{n-1}}$.

In this case, there is a unique measure μ on $\Omega^G_{V_n}$ such that

$$\mu(\{\sigma|_{V_n=\sigma_n}\}) = \mu^{(n)}(\sigma_n)$$

for all $n \geq 1$ and $\sigma_n \in \Omega^G_{V_n}$.

The following statement describes a condition on that ensures consistency of $\mu^{(n)}(\sigma_n)$ for the Blume-Capel model.

Theorem 1. *[24] Let* $k \geq 2$. *Sequence of probability distributions* $\mu^{(n)}(\sigma_n),\ n = 1,2,\ldots$ *in (2) is consistent if and only if for any* $x \in V$ *the following functional equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \ (4)$$

where $\theta = \exp\{-J\beta\}, \beta = 1/T,$ *vector* $h_x = (\tilde{h}_{1,x} - \tilde{h}_{0,x}, \tilde{h}_{-1,x} - \tilde{h}_{0,x})$ *and function* $F(\cdot,\theta): R^2 \to R^2,$ *where* $F(h,\theta) = (F_1(h,\theta), F_{-1}(h,\theta))$ *defined as:*

$$F_i(h,\theta) = \ln \frac{\theta^{(-1-i)^2} e^{h_{-1}} + \theta + \theta^{(-1+i)^2} e^{h_1}}{\theta e^{h_{-1}} + 1 + \theta e^{h_1}},$$
$$h = (h_1, h_{-1}).$$

Translation-Invariant Gibbs Measures

In the case of a»cycle» from the functional equation (4) we obtain (see [19]):

$$\begin{cases} z_{1,x} = \prod_{y \in S(x)} \dfrac{1+\theta^3 z_{-1,y}}{z_{-1,y} + z_{1,y}} \\ z_{1,x} = \prod_{y \in S(x)} \dfrac{1+\theta^3 z_{1,y}}{z_{-1,y} + z_{1,y}} \end{cases} \quad (5)$$

where $z_{i,x} = \exp(\tilde{h}_{i,x} - \tilde{h}_{0,x}),\ i = -1,1.$

Translation-invariant Gibbs measures correspond to solutions of (4) with $z_{i,x} = z_i$ for all $x \in V$ and $i = -1,1$. For convenience, let's denote $z_1 = z_1,\ z_{-1} = z_2$. Then the system of equations (5) in the case of»cycle» will look like:

$$\begin{cases} z_1 = \left(\dfrac{1+\theta^3 z_2}{z_1 + z_2}\right)^k, \\ z_2 = \left(\dfrac{1+\theta^3 z_1}{z_1 + z_2}\right)^k, \end{cases} \quad (6)$$

Let $z_1 = z_2 = z$. Then from (6) we get

$$z = \left(\frac{1+\theta^3 z}{2z}\right)^k \quad (7)$$

The following lemma is true.

Lemma 1. Let $k \geq 2$. *Then for any* $\theta > 0$ *the equation (7) has a unique solution. Proof.* We write the equation (7) in the form

$$z = f(z) \ (8)$$

where

$$f(z) = \left(\frac{1+\theta^3 z}{2z}\right)^k.$$

Note that the derivative of the function $f(z)$

$$f'(z) = -\frac{k}{2z^2}\left(\frac{1+\theta^3 z}{2z}\right)^{k-1} < 0,$$

i.e. the function f(z) is decreasing for $z > 0$. Hence, the equation (7) has a unique solution $z^* = z^*(k,\theta)$ *for any* $\theta > 0$.

Further, subtracting the second equation from the first equation of system (6), we obtain

$$\left(z_1 - z_2\right)\left[1+\theta^3 \frac{\left(1+\theta^3 z\right)^{k-1} + \ldots + \left(1+\theta^3 z\right)^{k-1}}{\left(z_1+z_2\right)^k}\right] = 0. \quad (9)$$

Hence, $z_1 = z_2$ and

$$1+\theta^3 \frac{\left(1+\theta^3 z\right)^{k-1} + \ldots + \left(1+\theta^3 z\right)^{k-1}}{\left(z_1+z_2\right)^k} \neq 0.$$

Then the following theorem is true.

Theorem 2. *Let* $k \geq 2$. *Then for the HC-Blume-Capel model in the»cycle» case for* $\theta > 0$ *there is only one translation-invariant Gibbs measure* μ_0.

Conditions for the non-extremality of the TI measure

To study the (non)-extremality, we use the methods from [22]- [24] for translationinvariant Gibbs measures and from [25] for periodic (non-translation-invariant) Gibbs measures with period two.

To do this, consider Markov chains with states $\{-1,0,1\}$ and the matrix P_μ of probabilistic transitions $P_{\sigma(x)\sigma(y)}$ defined by the given translation-invariant Gibbs measure μ, i.e., $P_{\sigma(x)\sigma(y)}$ is shift probability from $\sigma(x)$ state to $\sigma(y)$ state.

A sufficient condition for the non-extremality of the Gibbs measure corresponding to the matrix P_μ is that $k\lambda_2^2 > 1$, where λ_2 *is the second modulo the maximum eigenvalue of the matrix* P_μ (Kesten-Stigum condition).

To check this condition, we need to know the explicit form of the solution to system (11). Exact solutions are known to us only for $k = 2$.

It is clear that for $k = 2$ the system of equations (11) for $\theta > 0$ has a unique solution $\left(z^*, z^*, z^*, z^*\right)$, where z^* is the only solution of the equation

$$z = \left(\frac{1+\theta^3 z}{2z}\right)^2 \quad (10)$$

Let us find conditions for the measure to be non-extremality corresponding to this solution. For this, consider

$$P_{\sigma(x)\sigma(y)} = \frac{a_{\sigma(x)\sigma(y)} \exp\left\{-J\beta\left(\sigma(x)-\sigma(y)\right)^2 + h_{\sigma(y)}\right\}}{\sum\limits_{\tilde{\sigma}(y)\in\{-1,0,1\}} a_{\sigma(x)\sigma(y)} \exp\left\{-J\beta\left(\sigma(x)-\sigma(y)\right)^2 + h_{\sigma(y)}\right\}}.$$

In this case $G = \{cycle\}$

$$a_{-1.-1} = 0, a_{-1,0} = 1, a_{-1,1} = 1,$$
$$a_{0,-1} = 1, a_{0,0} = 0, a_{0,1} = 1,$$
$$a_{1,-1} = 1, a_{1,0} = 1, a_{1,1} = 0.$$

Thence, we get

$$P_{-1,-1}=0,\quad P_{-1,0}=\frac{\theta^3 z}{1+\theta^3 z},\quad P_{-1,1}=\frac{\theta^3 z}{1+\theta^3 z},$$
$$P_{0,-1}=\frac{1}{2},\quad P_{0,0}=0,\quad P_{0,1}=\frac{1}{2},$$
$$P_{1,-1}=\frac{\theta^3 z}{1+\theta^3 z},\ P_{1,0}=\frac{\theta^3 z}{1+\theta^3 z},\ P_{1,1}=0.$$

Hence,

$$\mathrm{P}=\begin{pmatrix} 0 & \frac{1}{1+\theta^3 z} & \frac{\theta^3 z}{1+\theta^3 z} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\theta^3 z}{1+\theta^3 z} & \frac{1}{1+\theta^3 z} & 0 \end{pmatrix} \quad (11)$$

It is clear that one of the eigenvalues of this matrices $s_3=1$. We find s_1 and s_2:

$$s_1=-\frac{1}{1+\theta^3 z},\ s_2=-\frac{\theta^3 z}{1+\theta^3 z},$$

where z is the solution of equation (9). We find $\max\{|s_1|,|s_2|\}$:

$$|s_1|-|s_2|=\frac{1}{1+\theta^3 z}-\frac{\theta^3 z}{1+\theta^3 z}=\frac{1-\theta^3 z}{1+\theta^3 z}.$$

Using the Cardano formula, we solve the equation (9). It has one real positive solution

$$z=z(\theta)=\frac{1}{12}\sqrt[3]{36\theta^9+216+\theta^{18}+24\sqrt{3\theta^9+81}}+$$
$$+\frac{2\theta^3+\frac{1}{12}\theta^{12}}{\sqrt[3]{36\theta^9+216+\theta^{18}+24\sqrt{3\theta^9+81}}}+\frac{1}{12\theta^6}. \quad (12)$$

The expression $1-\theta^3 z$ can be rewritten as

$$1-\theta^3 z=(1-\theta)A(\theta),\ (13)$$

where $A(\theta)>0$ and z has the form (11). This can also be seen from Figure 1.

Then if $0<\theta<1$, then $1-\theta^3 z>0$. From here we get

$$\max\{|s_1|,|s_2|\}=|s_1|.$$

If $\theta>1$, then $1-\theta^3 z<0$. Then $\max\{|s_1|,|s_2|\}=|s_2|$, i.e.

$$\max\{|s_1|,|s_2|\}=\begin{cases}|s_1|, & if\quad 0<\theta<1,\\ |s_2|, & if\quad \theta>1.\end{cases}$$

Hence it follows that $0<\theta<1$, $|s_2|<|s_1|<|s_3|=1$.

Let $0<\theta<1$. In this case, we check the Kesten-Stigum condition for the measure $\mu_0: 2s_1^2>1$ to be non-extremality. To determine the non-extremality interval of this measure, we check the condition

$$2s_1^2-1=2\cdot\left(\frac{1}{1+\theta^3 z}\right)^2-1>0,$$

where z has the form (11). This inequality is equivalent to $\theta^3 z-\left(\sqrt{2}-1\right)<0$. *Using the Maple program, we can see that the last inequality is true for* $\theta\in(\theta_1;1)$, where $\theta_1\approx 0.8051112259$, i.e. under this condition, the measure μ_0 is not extremal (Figure 2).

Let now $\theta > 1$. In this case $|s_2| < |s_1| < s_3 = 1$. Then the Kesten-Stigum condition for the measure μ_0 to be non extremal: $2s_2^2 > 1$.

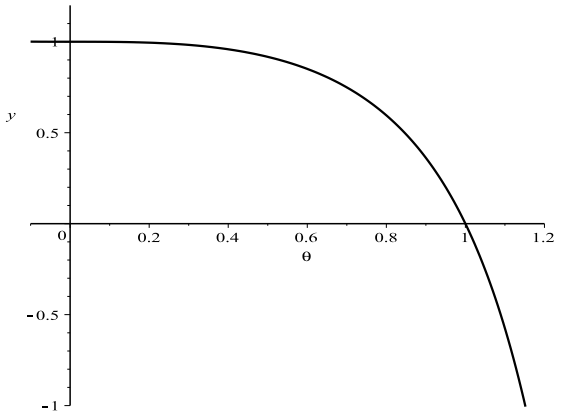


Figure 1. Graph of the function $1-\theta^3 z$

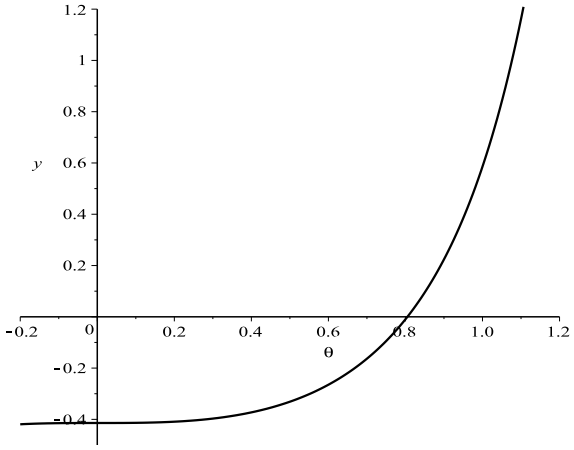


Figure 2. Graph of the function $\theta^3 z-\left(\sqrt{2}-1\right)$

To determine the non-extremality interval of the measure μ_0 in the case $\theta > 1$, we check the condition

$$2s_1^2 - 1 = 2\cdot\left(\frac{\theta^3 z}{1+\theta^3 z}\right)^2 - 1 > 0$$

where z has the form (11). This inequality is equivalent $\left(\sqrt{2}-1\right)\theta^3 z - 1$. Using the Maple program, you can see that the last inequality is true for $\theta \in \left(\theta_2; +\infty\right)$ where $\theta_2 \approx 1.191182787$, i.e. under this condition, the measure μ_0 is not extremal (Figure 3).

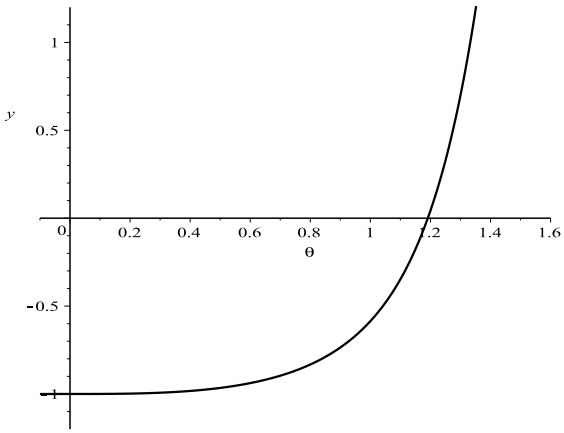


Figure 3. Graph of the function $\left(\sqrt{2}-1\right)\theta^3 z - 1$

Thus, the following theorem is true.

Theorem 5. Let $k=2$. *Then for the HC-Blume-Capel model in the case»cycle» the measure μ0 at* $\theta\in(0.8051112259;1)\cup(1.191182787;+\infty)$ *is not extremal.*

To study the extremality, the methods from [23] are applicable. Let us carry out the necessary definitions from [23]. If we remove an arbitrary edge $\langle x^0,x^1\rangle=l\in L$ from the Cayley tree Γ^k, then it splits into two components $\Gamma^k_{x^0}$ and $\Gamma^k_{x^1}$, each of which is called a semi-infinite tree or Cayley's half-tree.

Consider a finite complete subtree $\Im$ that contains all initial points of the semitree $\Gamma^k_{x^0}$. The boundary $\partial\Im$ of a subtree $\Im$ consists of the nearest neighbors of its vertices, which lie in $\Gamma^k_{x^0}\setminus\Im$. We identify the subtree $\Im$ with the set of its vertices. Denote by $E(A)$ the set of all edges A and ∂A.

The paper [23] introduces two key quantities: κ and γ, which play an important role in studying the extremality of the TI of Gibbs measures. These quantities are properties of the set of Gibbs measures $\{\mu^\tau_\Im\}$, where the boundary condition τ is fixed and $\Im$ is an arbitrary, initial, complete, final subtree of $\Gamma^k_{x^0}$. Given initial subtree $\Gamma^k_{x^0}$ and vertex $x\in\Im$ we will write $\Im_x$ for the (maximum) subtree $\Im$ with initial point in x. When x is not the initial point of $\Im$, *we denote by* $\{\mu^s_\Im\}$ the Gibbs measure in which the»ancestor» x has spin s *and configuration on the lower bound of* $\Im_x$, (i.e. on $\partial\Im\setminus\{"ancestor"\ x\}$) is given by Γ.

For two measures on Ω, by μ_1 and μ_2 we denote the distance in the norm

$$\|\mu_1-\mu_2\|_x=\sum_{i\in\{-1,0,1\}}\left|\mu_1(\sigma(x)=i)-\mu_2(\sigma(x)=i)\right|.$$

Let $\eta^{x,s}$ be the configuration η with spin in x set to s.

Following [23], we define

$$\kappa\equiv\kappa(\mu)=\sup_{x\in\Gamma^k}\max_{x,s,s'}\left\|\mu^s_{\Im_x}-\mu^{s'}_{\Im_x}\right\|_x,$$

$$\gamma\equiv\gamma(\mu)=\sup_{A\subset\Gamma^k}\max\left\|\mu^{\eta^{y,s}}_A-\mu^{\eta^{y,s'}}_A\right\|_x,$$

where the maximum is taken over all boundary conditions η, all $y\in\partial A$, all neighbors $x\in A$ of y, and all spins $s,s'\in\{-1,0,1\}$.

Let us find the extremality condition for the measure μ_0. It is clear that $1-\theta^3z>0$ for $0<\theta<1$ and $1-\theta^3z<0$ for $\theta>1\Rightarrow$ (see Fig.1).

Note that κ has a particularly simple formula

$$\kappa=\frac{1}{2}\max\sum_{l\in\{-1,0,1\}}\left|P_{il}-P_{jl}\right|.$$

Hence it is clear that $\left|P_{il}-P_{jl}\right|=0$ for $i=j$. *Using [23], for* $i\neq j$ we compute

$$\sum_{l\in\{-1,0,1\}}\left|P_{il}-P_{jl}\right|=\begin{cases}\dfrac{2}{1+\theta^3z}, & 0<\theta<1, \quad ij=0,\\ 1, & \theta>1, \qquad ij=0,\\ \dfrac{2\theta^3z}{1+\theta^3z}, & \theta>0, \qquad ij=-1.\end{cases}$$

Hence, for solutions (z^*,z^*,z^*,z^*) for $0<\theta<1$ we have

$$\kappa=\frac{1}{1+\theta^3z},$$

and at $\theta>1$

$$\kappa=\frac{\theta^3z}{1+\theta^3z}.$$

Now, like in [23], p. 15, we will look for an estimate for γ, in the following form:

$$\gamma=\max\left\{\left\|\mu^{\eta^{y,-1}}_A-\mu^{\eta^{y,0}}_A\right\|_x,\left\|\mu^{\eta^{y,-1}}_A-\mu^{\eta^{y,1}}_A\right\|_x,\left\|\mu^{\eta^{y,0}}_A-\mu^{\eta^{y,1}}_A\right\|_x\right\},$$

where

$$\left\|\mu_A^{\eta^{y,1}}-\mu_A^{\eta^{y,0}}\right\|_x=\frac{1}{2}\sum_{s\in\{-1,0,1\}}\left|\mu_A^{\eta^{y,1}}\left(\sigma(x)=s\right)-\mu_A^{\eta^{y,0}}\left(\sigma(x)=s\right)\right|=$$

$$=\frac{1}{2}\left(\left|P_{1,-1}-P_{0,1}\right|+\left|P_{1,0}-P_{0,0}\right|+\left|P_{1,1}-P_{0,1}\right|\right)=$$

$$=\frac{1}{2}\frac{\left|1-\theta^3 z\right|+3+\theta^3 z}{2\left(1+\theta^3 z\right)}=\begin{cases}\dfrac{1}{1+\theta^3 z}, & 0<\theta<1,\\ \dfrac{1}{2}, & \theta>1,\end{cases}$$

$$\left\|\mu_A^{\eta^{y,1}}-\mu_A^{\eta^{y,-1}}\right\|_x=\frac{1}{2}\sum_{l\in\{-1,0,1\}}\left|P_{1,l}-P_{-1,l}\right|=\frac{1}{2}\frac{2\theta^3 z}{1+\theta^3 z}=\frac{\theta^3 z}{1+\theta^3 z},\theta>0,$$

$$\left\|\mu_A^{\eta^{y,0}}-\mu_A^{\eta^{y,-1}}\right\|_x=\frac{1}{2}\sum_{l\in\{-1,0,1\}}\left|P_{0,l}-P_{-1,l}\right|=$$

$$=\frac{1}{2}\frac{\left|1-\theta^3 z\right|+3+\theta^3 z}{2\left(1+\theta^3 z\right)}=\begin{cases}\dfrac{1}{1+\theta^3 z}, & 0<\theta<1,\\ \dfrac{1}{2}, & \theta>1,\end{cases}$$

Therefore, for $0<\theta<1$ we get

$$\gamma=\max\left\{\frac{1}{1+\theta^3 z},\frac{\theta^3 z}{1+\theta^3 z}\right\}=\frac{1}{1+\theta^3 z},$$

at $\theta>1$

$$\gamma=\max\left\{\frac{1}{1+\theta^3 z},\frac{\theta^3 z}{1+\theta^3 z}\right\}=\frac{1}{1+\theta^3 z}.$$

Now, for the measure μ_0, we check the extremality condition: $2\kappa\gamma<1$. For $0<\theta<1$ this condition has the form

$$2\kappa\gamma-1=2\cdot\left(\frac{1}{1+\theta^3 z}\right)^2-1<0,$$

and at $\theta>1$

$$2\kappa\gamma-1=2\cdot\left(\frac{\theta^3 z}{1+\theta^3 z}\right)^2-1<0$$

From figure 2 and figure 3 we get that the last inequalities are true for where $\theta\in(0,\theta_1)\cup(1,\theta_2)$, where $\theta_1\approx 0.8051112259$, $\theta_2\approx 1.91182787$. So, the following theorem is true.

Theorem 3. *Let* $k=2$. *Then for the HC-Blum-Capel model in the»cycle» case the measure* μ_0 *at* $\theta\in(0;0.8051112259)\cup(1;1.191182787)$ *is extremal.*

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