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PARABOLIK TENGLAMA UCHUN CHEKLI AYIRMALI SXEMALAR

Inomiddinov Sardorbek Nizomiddinovich
Namangan davlat universiteti katta o‘qituvchisi
ORCID: 0000-0003-4806-1286
e-mail: sn.inomiddinov@gmail.com

Annotatsiya: *Ushbu maqolada parabolik turdagi differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalarni chekli ayirmali sxemalar yordamida taqribiy yechish usullari o‘rganilgan. Xususan, oshkor, sof oshkormas va Krank–Nikolson sxemalarini qurish, ularning approksimatsiya aniqligi hamda turg‘unlik xossalarini tahlil qilish masalalari ko‘rib chiqilgan. Sxemalarning dispersion munosabatlari asosida turg‘unlik shartlari aniqlangan. Oshkor sxemaning shartli turg‘unligi, oshkormas va Krank–Nikolson sxemalarining esa absolyut turg‘unligi asoslab berilgan. Shuningdek, mazkur sxemalarni hisoblash algoritmlari, uch diagonalli algebraik tenglamalar sistemasini progonka usuli bilan yechish hamda hisoblashlarni Mathcad kabi matematik dasturlar yordamida amalga oshirish imkoniyatlari yoritilgan. Olingan natijalar parabolik tenglamalarni sonli yechishda mos chekli ayirmali sxemani tanlash va hisoblash jarayonining samaradorligini baholashga xizmat qiladi.*

Kalit so‘zlar: *parabolik differensial tenglama, boshlang‘ich-chegaraviy masala, chekli ayirmali sxema, oshkor sxema, oshkormas sxema, Krank–Nikolson sxemasi, turg‘unlik, progonka usuli.*

КОНЕЧНО-РАЗНОСТНЫЕ СХЕМЫ ДЛЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

Аннотация: *В статье рассматриваются конечно-разностные методы приближённого решения начально-краевых задач для параболических дифференциальных уравнений. Цель исследования состоит в систематическом изложении явной, полностью неявной и схемы Кранка–Николсона, а также в сопоставлении их аппроксимационных свойств, устойчивости и вычислительных алгоритмов. Методология включает построение пространственно-временной сетки, замену производных конечными разностями, оценку локальной погрешности аппроксимации и анализ устойчивости с помощью гармонических решений. Показано, что явная схема проста в реализации, однако её устойчивость ограничивается условием на шаги сетки. Полностью неявная схема и схема Кранка–Николсона для классической диффузионной задачи обладают безусловной устойчивостью, но на каждом временном слое требуют решения системы линейных алгебраических уравнений с трёхдиагональной матрицей. Для её эффективного решения может применяться метод прогонки. Численные результаты отдельного эксперимента должны быть рассчитаны автором и включены в окончательную версию статьи.*

Ключевые слова: *параболическое дифференциальное уравнение, начально-краевая задача, конечно-разностная схема, явная схема, неявная схема, схема Кранка–Николсона, устойчивость, метод прогонки.*

FINITE-DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

Abstract: *This paper considers finite-difference approaches for the approximate solution of initial-boundary value problems for parabolic differential equations. The aim is to present the explicit, fully implicit, and Crank–Nicolson schemes within a consistent methodological framework and to compare their approximation properties, stability, and computational algorithms. The methodology includes constructing a space-time grid, replacing derivatives with finite differences, estimating local truncation error, and examining stability through harmonic solutions. The explicit scheme is straightforward to implement, but its stability is restricted by a condition on the grid steps. For the classical diffusion problem, the fully implicit and Crank–Nicolson schemes are unconditionally stable; each time layer requires solving a tridiagonal system of linear algebraic equations, which can be solved efficiently by the Thomas algorithm. The paper summarizes the principal advantages and limitations of the schemes and proposes criteria for a reproducible computational experiment. Numerical results from an actual experiment should be calculated by the author and added to the final manuscript.*

Keywords: *parabolic differential equation, initial-boundary value problem, finite-difference scheme, explicit scheme, implicit scheme, Crank–Nicolson scheme, stability, Thomas algorithm.*

Kirish

Parabolik turdagi xususiy hosilali differensial tenglamalar issiqlik tarqalishi, diffuziya, namlikning g‘ovak muhitda ko‘chishi va vaqtga bog‘liq boshqa jarayonlarning matematik modellarida keng qo‘llanadi. Bunday tenglamalarda yechim vaqt bo‘yicha evolyutsiyalanadi, fazoviy o‘zgaruvchilar bo‘yicha esa hosilalar jarayonning tarqalish xususiyatlarini ifodalaydi. Ko‘plab amaliy masalalarda analitik yechimni yopiq ko‘rinishda topish qiyin yoki imkonsiz bo‘lgani sababli sonli usullarga murojaat qilinadi.

Chekli ayirmali usul differensial masalani to‘r tugunlarida aniqlangan algebraik munosabatlar sistemasi bilan almashtiradi. Ushbu yondashuv algoritmni aniq belgilash va uni kompyuterda amalga oshirish imkonini beradi. Biroq to‘rni tanlashda faqat tugunlar sonini oshirish yetarli emas: sxemaning approksimatsiya tartibi, turg‘unligi, hisoblash xarajati va yechimning fizik ma’noga mosligi ham hisobga olinishi lozim.

Ushbu ishning maqsadi diffuziya tenglamasi uchun uchta klassik chekli ayirmali sxemani ko‘rib chiqish va ularni metodik jihatdan taqqoslashdir. Tadqiqot obyekti boshlang‘ich-chegaraviy shartlar bilan berilgan parabolik tenglama, tadqiqot predmeti esa oshkor, sof oshkormas va Krank–Nikolson sxemalarining qurilishi, approksimatsiyasi, turg‘unligi hamda yechish algoritmlaridir. Vazifalar to‘r qurish, sxemalarni hosil qilish, xatolik va turg‘unlikni tahlil qilish, algebraik sistemalarni yechish usulini bayon etish va amaliy tavsiyalar ishlab chiqishni qamrab oladi.

Maqola nazariy-metodik tahlilga asoslangan. Dastlabki hujjatda keltirilmagan hisoblash natijalari yoki o‘lchovlar haqiqiy tajriba sifatida taqdim etilmaydi. Nashrga topshirishdan oldin aniq test masalasi bo‘yicha hisoblash tajribasi o‘tkazish, xatoliklarni jadvalda ko‘rsatish va natijalarni takrorlash uchun zarur parametrlarni berish tavsiya etiladi.

Adabiyotlar tahlili

Chekli ayirmali usullar hisoblash matematikasining klassik yo‘nalishlaridan biri bo‘lib, ularning nazariy asoslari differensial operatorlarni to‘r operatorlari bilan almashtirish, yaqinlashish, turg‘unlik va korrektlik tushunchalari bilan bog‘liq. Hisoblash usullariga oid o‘quv adabiyotlarida to‘r funksiyalari, ayirmali operatorlar, approksimatsiya xatoligi va algebraik sistemalarni yechish algoritmlari izchil bayon etiladi [1, 2]. Ushbu tushunchalar parabolik tenglamalar uchun sonli sxemalarni tahlil qilishda tayanch vazifasini bajaradi.

Oshkor sxemada keyingi vaqt qatlamidagi qiymatlar avvalgi qatlamdagi ma’lumotlar orqali bevosita aniqlanadi. Shu sababli algoritm sodda, biroq klassik diffuziya tenglamasida turg‘unlik vaqt va fazo qadamlarining o‘zaro nisbatiga bog‘liq. Fazoviy to‘rni mayda qilish vaqt qadamini ham kamaytirishni talab qilishi mumkin, bu esa umumiy hisoblashlar sonini oshiradi.

Sof oshkormas sxemada vaqt bo‘yicha hosila yangi qatlamdagi qiymatlar orqali ifodalanadi. Natijada har bir vaqt qadamida chiziqli algebraik tenglamalar sistemasini yechish zarur bo‘ladi. Bir o‘lchovli, bir jinsli diffuziya tenglamasi uchun sistema odatda uch diagonalli tuzilishga ega. Bunday tuzilma progonka algoritmidan foydalanish imkonini beradi.

Krank–Nikolson sxemasi vaqt bo‘yicha ikki qatlamdagi fazoviy operator qiymatlarining o‘rtachasini qo‘llaydi. Klassik sharoitlarda u vaqt bo‘yicha ikkinchi tartibli approksimatsiyaga ega bo‘lib, oshkor sxemaga xos qat’iy turg‘unlik cheklovini bartaraf etadi. Shu bilan birga, sxema oshkormas bo‘lgani uchun algebraik sistemani yechishni talab qiladi. Turg‘unlikning o‘zi har qanday holatda yuqori aniqlik yoki soxta tebranishlarning yo‘qligini kafolatlamaydi.

Manbalarda [1, 2] sonli usullarni qurish va hisoblash algoritmlariga e’tibor berilgan. Mazkur maqola sxemalarni approksimatsiya, turg‘unlik va algoritmik xarajat nuqtayi nazaridan birgalikda taqqoslaydi. Dastlabki maqoladagi differensial o‘yinlar va integral tenglamalar haqidagi ishlar [3–5] parabolik tenglamalar uchun ayirmali sxemalarning bevosita nazariy manbalari bo‘lmagani sababli, ular ushbu tahlilning asosiy dalili sifatida ishlatilmaydi.

Tadqiqot metodologiyasi

PDT uchun boshlang‘ich-chegara masala:, $L\bar{u} = \bar{u}_t - \bar{u}_{xx} = f(x,t)$

$\bar{u} = \bar{u}(x,t)$ – aniq yechim,

$$l\bar{u} = \bar{u}(x,0) = g_0(x),\ 0 \le x \le 1, \quad l_1\bar{u} = \bar{u}(0,t) = g_1(t),$$
$$l_2\bar{u} = \bar{u}(1,t) = g_2(t),\ 0 \le t \le T.$$

PDT uchun chekli ayirmali sxemalar:

1) Oshkor CHAS. $\left(\Lambda u_h = \Lambda_{xx} u_k^j = \frac{u_{k-1}^j - 2u_k^j + u_{k+1}^j}{h^2}\right)$

$$L_h^{(1)} u_h = \frac{u_k^{j+1} - u_k^j}{\tau} - \Lambda_{xx} u_k^j = f_k^j; \quad u_k^j = u(x_k, t_j) \approx \bar{u}(x_k, t_j) = \bar{u}_k^j$$
$$u_k^0 = g_0(x_k),\ k = 0, \dots, m; \quad u_0^{j+1} = g_1^{j+1},\ u_m^{j+1} = g_2^{j+1}, \quad j = 0, \dots, n.$$

2) Oshkormas CHAS

$$L_h^{(2)}u_h = \frac{u_k^{j+1} - u_k^j}{\tau} - \Lambda_{xx}u_k^{j+1} = f_k^{j+1};$$

$$u_k^0 = g_0(x_k),\ k = 0, \dots, m; \quad u_0^{j+1} = g_1^{j+1},\ u_m^{j+1} = g_2^{j+1}, \quad j = 0, \dots, n$$

3) Krank-Nikolsonning yuqori tartibli approksimasiyali sxemasi

$$L_h^{(3)}u_h = \frac{u_k^{j+1} - u_k^j}{\tau} - 0.5\Lambda\left(u_k^j + u_k^{j+1}\right) = 0.5\left(f_k^j + f_k^{j+1}\right) = 0.5f_k^{j+0.5}$$

Algoritmlar:

$$u_k^{j+1} = u_k^j + \tau\Lambda_{xx}u_k^j + \tau f_k^j;$$

$$(E - \tau\Lambda_{xx})u_k^{j+1} = u_k^j + \tau f_k^{j+1} \Leftrightarrow Au^{j+1} = u^j + \tau f^{j+1};$$

$$(E - 0.5\tau\Lambda_{xx})u_k^{j+1} = (E + 0.5\tau\Lambda_{xx})u_k^j + \tau f_k^{j+\frac{1}{2}} \Leftrightarrow Au^{j+1} = Bu^j + \tau f^{j+\frac{1}{2}}.$$

Approksimasiyani tekshirish:

$$R_h^{(1)} = L_h^{(1)}u_h - f_h = \left\{Lu\left(x_k, t_j\right) - f\left(x_k, t_j\right)\right\} = O(\tau + h^2),$$

$$R_h^{(2)} = L_h^{(2)}u_h - f_h = \left\{Lu\left(x_k, t_j\right) - f\left(x_k, t_j\right)\right\} = O(\tau + h^2),$$

$$R_h^{(3)} = L_h^{(3)}u_h - f_h = \left\{Lu\left(x_k, t_j\right) - f\left(x_k, t_j\right)\right\} = O(\tau^2 + h^4).$$

Turg‘unlikni tekshirish: a) oshkor sxemaning turg‘unlik sharti;$\frac{\tau}{h^2} \le \frac{1}{2}$

b) oshkormas sxema shartsiz (absolyut) turg‘un.

PDT uchun boshlang‘ich chegara masala.

$D = \{0 \le x \le 1,\ 0 \le t \le T\}$ sohada diffuziya tenglamasining boshlang‘ich-chegara shartlarga

$$Lu = u_t - u_{xx} = f(x,t), \qquad (1)$$

$$lu = g, \quad lu = [u(x,0), u(0,t), u(1,t)], \quad g = [u_0(x), g_1(t), g_2(t)], \qquad (2)$$

bo‘ysunadigan aniq yechimini topish talab etiladi. Faraz qilamizki, (1)-(2) masala yechimi yagona va mavjud, bu yechim sohada, uzluksiz hosilalarga ega.$\bar{u} = \bar{u}(x,t)D\bar{u}_{tt}\bar{u}_{xxxx}$

Yechimni CHAS usuli bilan taqribiy topamiz. Buning uchun sohada to‘g‘ri chiziqlar$D$

$$x_k = kh,\ k = 0..m,\ h = \frac{1}{m}, \quad t_j = j\tau,\ j = 0..n,\ \tau = \frac{T}{n}$$

yordamida to‘r hosil qilamiz.$D_h = D_{h\tau} = \left\{\left(x_k, t_j\right) = (kh, j\tau),\ k = 0..m,\ j = 0..n\right\}$

Asosiy belgilash quyidagidan iborat:.$u_k^j = u\left(x_k, t_j\right) \approx \bar{u}_k^j = \bar{u}\left(x_k, t_j\right)$

To‘rning nuqtalarida (1), (2) tenglamalarni yozamiz:

$$u_t\left(x_k, t_j\right) = u_{xx}\left(x_k, t_j\right) + f\left(x_k, t_j\right), \quad k = 1..m-1,\ j = 0..n-1. \qquad (3)$$

$$u(x_k, 0) = g_0(x_k),\ k = 0..m, \quad u\left(0, t_j\right) = g_1\left(t_j\right),\ u\left(1, t_j\right) = g_2\left(t_j\right),\ j = 0..n. \qquad (4)$$

Qisqalik uchun hosilaning chekli ayirmali approksimasiyasini kiritamiz:$u_{xx}(x,t)$

$$\Lambda u_h = \Lambda_{xx}u\left(x_k, t_j\right) = \frac{u\left(x_{k-1}, t_j\right) - 2u\left(x_k, t_j\right) + u\left(x_{k+1}, t_j\right)}{h^2} = u_{xx}\left(x_k, t_j\right) + O(h^2). \qquad (5)$$

Λ_{xx} operator ixtiyoriy to‘r funksiyaga ham shu kabi ta’sir etadi:$u_h = \left\{u_k^j\right\}$

$$\Lambda u_h = \Lambda u_k^j = \frac{u_{k-1}^j - 2u_k^j + u_{k+1}^j}{h^2}. \qquad (6)$$

(3) tengliklarga kiruvchi hosilalarni quyidagicha almashtirish mumkin:

$$\bar{u}_t\left(x_k, t_j\right) = \frac{\bar{u}\left(x_k, t_{j+1}\right) - \bar{u}\left(x_k, t_j\right)}{\tau} + O(\tau) = \frac{\bar{u}_k^{j+1} - \bar{u}_k^j}{\tau} + O(\tau), \qquad (7)$$

$$u_{xx}(x_k,t_j)=\Lambda u_h+O(h^2)=\frac{u(x_{k-1},t_j)-2u(x_k,t_j)+u(x_{k+1},t_j)}{h^2}+O(h^2), \qquad (8)$$

$$u_{xx}(x_k,t_j)=\Lambda u(x_k,t_{j+1})+O(h^2)$$
$$=\frac{u(x_{k-1},t_{j+1})-2u(x_k,t_{j+1})+u(x_{k+1},t_{j+1})}{h^2}+O(h^2). \qquad (9)$$

Hosilani chekli ayirmalar bilan approksimasiya qilishda ishlatiladigan to‘rning nuqtalar to‘plamiga *shablon* deyiladi. Endi (3) larga hosilalarning (7)-(9) qiymatlarini qo‘yamiz. Boshlang‘ich-chegara shartlarda hosila yo‘q, ular aniq approksimasiya qilinadi. (3) ga avval (8), keyin (9) ni qo‘yib ikki xil tengliklar hosil qilamiz:

$$L_h^{(1)}\left(\bar{u}(x_k,t_j)\right)=\frac{\bar{u}(x_k,t_{j+1})-\bar{u}(x_k,t_j)}{\tau}-\Lambda\bar{u}(x_k,t_j)=f(x_k,t_j)+O(\tau+h^2). \qquad (10)$$

$$L_h^{(2)}\left(\bar{u}(x_k,t_j)\right)=\frac{\bar{u}(x_k,t_{j+1})-\bar{u}(x_k,t_j)}{\tau}-\Lambda\bar{u}(x_k,t_{j+1})$$
$$=f(x_k,t_{j+1})+O(\tau+h^2). \qquad (11)$$

Cheksiz kichik miqdorlarni tashlab yuborib va, deb ikki xil chekli ayirmali sxema hosil qilamiz:$u_k^j=u(x_k,t_j)\approx\bar{u}(x_k,t_j)f_k^j=f(x_k,t_j)$

$$L_h^{(1)}u_h=\frac{u_k^{j+1}-u_k^j}{\tau}-\Lambda u_k^j=f_k^j;\quad k=1,\dots,m-1;\ j=0,\dots,n-1. \qquad (12)$$

$$L_h^{(2)}u_h=\frac{u_k^{j+1}-u_k^j}{\tau}-\Lambda u_k^{j+1}=f_k^{j+1},\quad k=1,\dots,m-1;\ j=0,\dots,n-1. \qquad (13)$$

(12) sxemaning shabloni teskari (T):. $(x_k,t_{j+1}),(x_{k-1},t_j),(x_k,t_j),(x_{k+1},t_j)$

(13) sxemaniki esa (T):. (12) oshkor sxema deyiladi, (13) sof oshkormas sxema deyiladi. Ularni qo‘shib va ikkiga bo‘lib Krank-Nikolson sxemasini olamiz:$(x_k,t_j),(x_{k-1},t_{j+1}),(x_k,t_{j+1}),(x_{k+1},t_{j+1})$

$$L_h^{(3)}u_h=\frac{u_k^{j+1}-u_k^j}{\tau}-0.5\Lambda\left(u_k^j+u_k^{j+1}\right)=0.5\left(f_k^j+f_k^{j+1}\right),\quad k=1,\dots,m-1;$$
$$j=0,\dots,n-1 \qquad (14)$$

Natijalar va ularning tahlili

Nazariy tahlil uchta sxemaning hisoblash xususiyatlari farqlanishini ko‘rsatadi. Oshkor sxemada har bir yangi qatlam bevosita hisoblanadi; bu algoritmni sodda qiladi, biroq parametriga turg‘unlik sharti qo‘yadi. Sof oshkormas sxemada yangi qatlamdagi qiymatlar birgalikda aniqlanadi. Natijada har bir vaqt qatlamida chiziqli algebraik sistema paydo bo‘ladi, ammo klassik diffuziya tenglamasi uchun turg‘unlik vaqt qadamining fazo qadamiga nisbatan qat’iy cheklanishini talab qilmaydi.$r=\frac{a\tau}{h^2}$

Krank–Nikolson sxemasi vaqt bo‘yicha yuqoriroq aniqlikka ega bo‘lishi bilan ajralib turadi. Shu sababli silliq yechimlarda aniqlik va hisoblash xarajatini muvozanatlash mumkin. Biroq bu sxema ham oshkormas sistema yechimini talab etadi. Algoritm tanlashda turg‘unlik sharti bilan bir qatorda talab qilinayotgan aniqlik, hisoblash resurslari va yechimning xususiyatlari ham hisobga olinishi kerak.

Endi sxemalarni boshlang‘ich-chegara shartlar bilan birga yozamiz:

Oshkor CHAS	$\frac{u_k^{j+1}-u_k^j}{\tau}-\frac{u_{k-1}^j-2u_k^j+u_{k+1}^j}{h^2}=f_k^j,\ k=1..m-1;\ j=0..n-1$
Sof oshkormas CHAS	$\frac{u_k^{j+1}-u_k^j}{\tau}-\frac{u_{k-1}^{j+1}-2u_k^{j+1}+u_{k+1}^{j+1}}{h^2}=f_k^{j+1},\ k=1..m-1;\ j=0..n-1$
Krank-Nikolson sxemasi	$\frac{u_k^{j+1}-u_k^j}{\tau}-0.5\Lambda\left(u_k^j+u_k^{j+1}\right)=0.5\left(f_k^j+f_k^{j+1}\right),\ k=1,\dots,m-1;\ j=0,\dots,n-1$

BCH shartlar	$u_k^0 = g_0(x_k),\ k = 0..m;\ u_0^{j+1} = g_1^{j+1},\ u_m^{j+1} = g_2^{j+1},\ j = 0, \dots, n-1$

Approksimasiyani tekshirish

Boshlang‘ich-chegara shartlar aniq approksimasiya qilingani tufayli faqat differensial tenglamaning approksimasiyasini tekshirish qoladi. O‘ng tomon ham aniq approksimasiya qilinayapti, shuning uchun va aniq yechim jadvali ni kiritib (10), (11) dan quyidagini topamiz:$f_h = f\left(x_k, t_j\right)\bar{u}_k^j = \bar{u}\left(x_k, t_j\right)$

$$R_h^{(1)} = L_h^{(1)}\bar{u}_h - f_h = \left\{L\bar{u}\left(x_k, t_j\right) - f\left(x_k, t_j\right)\right\} = \frac{\bar{u}_k^{j+1} - \bar{u}_k^j}{\tau} - \frac{\bar{u}_{k-1}^j - 2\bar{u}_k^j + \bar{u}_{k+1}^j}{h^2} - f_k^j$$
$$= O(\tau + h^2),$$
$$R_h^{(2)} = L_h^{(2)}\bar{u}_h - f_h = \left\{L\bar{u}\left(x_k, t_{j+1}\right) - f\left(x_k, t_{j+1}\right)\right\}$$
$$= \frac{\bar{u}_k^{j+1} - \bar{u}_k^j}{\tau} - \frac{\bar{u}_{k-1}^{j+1} - 2\bar{u}_k^{j+1} + \bar{u}_{k+1}^{j+1}}{h^2} - f_k^{j+1} = O(\tau + h^2).$$

Ikkala sxema ham differensial tenglamani bilan approksimasiya qilyapti.$O(\tau + h^2)$

Krank-Nikolson sxemasi aniqligi ga teng.$R_h^{(3)} = O(\tau^2 + h^4)$

Demak,.$L_h^{(l)}\bar{u}_h - f_h \to Lu - f = 0,\ l = 1..3$

Turg‘unlikni tekshirish

Sxemalarning xususiy yechimlarini garmonika, ko‘rinishida izlaymiz. Bu xususiy yechimni (12) va (13) ga qo‘yib quyidagi dispersion munosabatlarni (xarakteristik tenglamalarni) topamiz (deb olamiz):$u_k^j = \lambda^j e^{i\alpha k}\lambda = \lambda(\alpha) = ?\, f_k^j = 0$

$$\frac{\lambda_1^{j+1}e^{i\alpha k} - \lambda_1^j e^{i\alpha k}}{\tau} = \lambda_1^j\,\frac{e^{i\alpha(k-1)} - 2e^{i\alpha k} + e^{i\alpha(k+1)}}{h^2},$$
$$\frac{\lambda_2^{j+1}e^{i\alpha k} - \lambda_2^j e^{i\alpha k}}{\tau} = \lambda_2^{j+1}\,\frac{e^{i\alpha(k-1)} - 2e^{i\alpha k} + e^{i\alpha(k+1)}}{h^2}.$$

Avvalo,

$$e^{i\alpha(-1)} - 2 + e^{i\alpha(1)} = 2\cos(\alpha) - 2 = 2(\cos(\alpha) - 1) = -4\sin^2\left(\frac{\alpha}{2}\right).$$

Yuqoridagi tengliklarni bir xil ko‘paytuvchilar ga qisqartirib$\left(\lambda^j e^{i\alpha k}\right)$

$$\frac{\lambda_1 - 1}{\tau} = \frac{e^{i\alpha(-1)} - 2 + e^{i\alpha(1)}}{h^2} \Rightarrow \lambda_1 = 1 - 4r\sin^2\left(\frac{\alpha}{2}\right),\quad r = \frac{\tau}{h^2}, \qquad (15)$$
$$\frac{\lambda_2 - 1}{\tau} = \lambda_2\,\frac{e^{i\alpha(-1)} - 2 + e^{i\alpha(1)}}{h^2} \Rightarrow \lambda_2 = \frac{1}{1 + 4r\sin^2\left(\frac{\alpha}{2}\right)},\quad r = \frac{\tau}{h^2}, \qquad (16)$$

dispersion munosabatlarni topamiz. Bu munosabatlar garmonika ayirmali sxemaning qachon yechimi bo‘lishini bildiradi.$u_k^j = \lambda^j e^{i\alpha k}$

$\left|u_k^j\right| = |\lambda|^j$ ekanidan, agar bo‘lsa va turg‘unlik yo‘q. Agar bo‘lsa va turg‘unlik bo‘lishi mumkin. (16) dan doimo va oshkormas sxema doimo turg‘un ekan. Oshkor sxemada bo‘lishi uchun$|\lambda| > 1\left|u_k^j\right| \to \infty|\lambda| \le 1\left|u_k^j\right| \to 0\lambda_2 < 1|\lambda_1| \le 1$

bo‘lishi kerak.$\frac{\tau}{h^2} \le \frac{1}{2}$

Demak, oshkor sxema (12) shartli turg‘un, ya’ni bo‘lsa turg‘un va oshkormas sxemalar (13), (14) absolyut turg‘un ekan.$\frac{\tau}{h^2} \le \frac{1}{2}$

Algoritmlar

a) Oshkor (12) sxemani yechish masalasini ko‘raylik. Undan ni topamiz:u_k^{j+1}

$$u_k^{j+1} = u_k^j + r\left(u_{k-1}^j - 2u_k^j + u_{k+1}^j\right) + \tau f_k^j = r\left(u_{k-1}^j + u_{k+1}^j\right) + (1-2r)u_k^j + \tau f_k^j, \quad r = \frac{\tau}{h^2},\ j = 0..n-1; \qquad (17)$$

$$u_k^0 = g_{0k},\ k = 0,1..m; \quad u_0^j = g_1^j,\ u_m^j = g_2^j, \quad j = 0,1..n.$$

Bu yerdan ko‘rinadiki, chekli ayirmali sxemada faqat ikki qatlam nuqtalari ishtirok etyapti (,), undan tashqari yuqori qatlam () da faqat bitta noma’lum qiymat bor (). Demak, (17) asosida qatlamda berilgan boshlang‘ich va chegara shart qiymatlaridan foydalanib qatlamdagi, so‘ng keyingi qatlamdagi qiymatlarni ham rekurrent formulalar (17) yordamida topishimiz mumkin ekan.$t = t_j t = t_{j+1} t = t_{j+1} u_k^{j+1} t = t_0 t = t_1$

b) Endi sof oshkormas sxemani yechish masalasini ko‘raylik. Har qatlamda uchta noma’lum bor:. Ularni topish uchun ushbu uch diagonalli sistemani olamiz:$t = t_{j+1} u_{k-1}^{j+1}, u_k^{j+1}, u_{k+1}^{j+1}$

$$u_0^{j+1} = g_1^{j+1}, \quad j = 0..n-1,$$
$$-ru_{k-1}^{j+1} + (1+2r)u_k^{j+1} - ru_{k+1}^{j+1} = u_k^j + \tau f_k^{j+1}, \quad k = 1..m-1;\ j = 1..n-1, \quad u_k^0 = g_{0k}, \qquad (18)$$
$$u_m^{j+1} = g_2^{j+1}, \quad j = 0,1..n-1; \quad r = \frac{\tau}{h^2}$$

Bu yerda ham ikkita qatlam ishtirok etyapti. Yuqori qatlamda uch noma’lum, quyi qatlamda bitta qiymat ishtirok etyapti. Shuning uchun ham bu sxemani sof oshkormas sxema deyiladi. Yechishni qatlamdan boshlanadi. Bu qatlamda (18) ga progonka usuli qo‘llaniladi. Birinchi qatlamda noma’lumlar topilgach, ikkinchi qatlamdagi noma’lumlar yana progonka usuli bilan topiladi va hokazo.$t = t_{j+1} t = t_1$

Sof oshkormas sxemani quyidagicha yozish mumkin:

$$Au^{\langle j+1\rangle} = u^{\langle j\rangle} + \tau f^{\langle j+1\rangle}, \quad j = 0..n-1, \qquad (19)$$

bu yerda, — vektorlar va matrisa elementlari quyidagicha hisoblanadi:$u^{\langle j\rangle} = \left[u_0^j, u_1^j, \dots, u_m^j\right]^T f^{\langle j\rangle} = \left[f_0^j, f_1^j, \dots, f_m^j\right]^T A = \left[a_{ij}\right]$

$0)\ a_{00} = 1,\ i = 1..m,\ a_{0i} = 0;$

$i)\ i = 1..m-1,\ a_{ii-1} = a_{ii+1} = -r,\ a_{ii} = 1+2r,\ a_{ij} = 0,\ j \neq i-1, i, i+1;$

$m)\ a_{mi} = 0,\ i = 0..m-1,\ a_{mm} = 1,$

c) Krank-Nikolson sxemasi ham oshkormas sxema va quyidagicha yoziladi:

$$u_0^{j+1} = g_1^{j+1}, \quad j = 0..n-1,$$
$$-u_{k-1}^{j+1} + 2(\mu+1)u_k^{j+1} - u_{k+1}^{j+1} = u_{k-1}^j + 2(\mu-1)u_k^j + u_{k+1}^j + 2h^2 f_k^{j+0.5}, \quad k = 1,\dots,m-1;\ j = 1,\dots,n-1,$$
$$u_k^0 = g_{0k}, \quad u_m^{j+1} = g_2^{j+1},\ j = 0..n-1; \quad r = \frac{\tau}{h^2},\ \mu = \frac{1}{r}.$$

Krank-Nikolson sxemasini vektor-matrisa ko‘rinishida yozish mumkin:

$$Au^{\langle j+1\rangle} = Bu^{\langle j\rangle} + 2h^2 f^{\langle j+0.5\rangle}, \quad j = 0..n-1, \quad f_{i,j} = f\left(x_i, t_j + \frac{\tau}{2}\right), \qquad (20)$$

bu yerda, — vektorlar va, matrisalar quyidagicha aniqlanadi:$u^{\langle j\rangle} = \left[u_0^j, u_1^j, \dots, u_m^j\right]^T f^{\langle j\rangle} = \left[f_0^j, f_1^j, \dots, f_m^j\right]^T A = \left[a_{ij}\right] B = \left[b_{ij}\right]$

0-satr. $a_{00} = 1,\ i = 1..m,\ a_{0i} = 0;\ b_{0i} = 0,\ i = 0..m;$

i-satr. $i = 1..m-1,\ a_{ii-1} = a_{ii+1} = -1,\ a_{ii} = 2(1+\mu),$

$a_{ij} = 0,\ j \neq i-1, i, i+1;\ \mu = \frac{1}{r};$

$$b_{ii-1} = b_{ii+1} = 1,\ b_{ii} = 2(\mu-1),\ b_{ij} = 0,\ j \neq i-1, i, i+1;$$

m-satr. $a_{mi} = 0,\ i = 0..m-1,\ a_{mm} = 1;\ b_{mi} = 0,\ i = 0..m;$

Demak, algoritmik tomondan oshkor sxema qulay, lekin shartli turg‘un, sof oshkormas va Krank-Nikolson sxemalari absolyut turg‘un, va har bir qatlamda uch diagonalli algebraik tenglamalar sistemasini progonka usuli bilan yechish kerak. Shartli turg‘unlik to‘r qadamlariga jiddiy shart qo‘yadi, bu esa ko‘p hisoblashlarni bajarishga olib keladi.

(17)-(20) chekli ayirmali sxemalar kompyuterda programma tuzib yoki matematik tizimlar, masalan, Mathcad dasturida yechish mumkin.

Xulosa

Parabolik differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalarni chekli ayirmali sxemalar yordamida taqribiy yechish yondashuvlari ko‘rib chiqildi. Oshkor, sof oshkormas va Krank–Nikolson sxemalarining formulalari, approksimatsiya xususiyatlari, turg‘unlik shartlari va hisoblash algoritmlari yagona metodik doirada bayon qilindi.

Oshkor sxema algoritmik jihatdan sodda va har bir yangi qatlamni bevosita hisoblash imkonini beradi. Shu bilan birga, klassik diffuziya tenglamasida uning turg‘unligi

sharti bilan cheklanadi. Sof oshkormas sxema va Krank–Nikolson sxemasi har bir qatlamda uch diagonalli sistema yechishni talab qiladi, biroq klassik diffuziya masalasi uchun absolyut turg‘unlik xususiyatiga ega. Krank–Nikolson sxemasi silliq yechimlarda vaqt bo‘yicha ikkinchi tartibli approksimatsiyani ta’minlashi bilan ajralib turadi.$r = \frac{a\tau}{h^2} \leq \frac{1}{2}$

Sof oshkormas va Krank–Nikolson sxemalarida hosil bo‘ladigan uch diagonalli sistemalarni progonka usuli bilan yechish hisoblash xarajatini kamaytiradi. Sxema tanlashda turg‘unlik bilan bir qatorda aniqlik, hisoblash resurslari, boshlang‘ich ma’lumotlarning silliqligi va masalaning fizik xususiyatlari hisobga olinishi zarur.

Keyingi bosqich sifatida aniq yechimga ega test masalasi bo‘yicha hisoblash tajribasini o‘tkazish, bir nechta to‘r zichliklarida maksimal xatolikni hisoblash, natijalarni jadval va grafiklarda berish hamda sxemalarning amaliy aniqlik tartibini tasdiqlash tavsiya etiladi. Ushbu sonli natijalar kiritilgach, maqolaning tajribaviy asoslanganligi va ilmiy yangilik da’vosi aniqroq baholanishi mumkin.

Mualliflar hissasi, manfaatlar to‘qnashuvi va moliyalashtirish

Mualliflar hissasi: maqola yakka muallif tomonidan tayyorlangan. Manfaatlar to‘qnashuvi: muallif manfaatlar to‘qnashuvi yo‘qligini bildiradi. Moliyalashtirish: tadqiqot tashqi moliyaviy yordamsiz amalga oshirilgan.

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