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NONLOCAL INTEGRAL PROBLEMS FOR HIGH-ORDER MIXED-DERIVATIVE EQUATIONS

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Abstract: *We investigate a high-order mixed-derivative equation subject to initial, boundary and integral conditions. An auxiliary polynomial realizing the prescribed boundary and integral condition is constructed through a positive-definite moment matrix. The associated finite-rank moment-preserving projection is characterized. For* $m=1$*, explicit finite trigonometric expansions yield existence and uniqueness within a specified modal class and demonstrate high-frequency instability. For* $m\geq 2$*, we construct an explicit basis of a* $(m-1)$*-dimensional polynomial space and prove existence and uniqueness for compatible condition within the corresponding solution class. The analysis does not establish well-posedness for given function in unrestricted function spaces.*

Keywords: *mixed-derivative equation; nonlocal integral condition; auxiliary polynomial; moment matrix; finite-rank projection; existence and uniqueness; instability.*

YUQORI TARTIBLI ARALASH HOSILALI TENGLAMALAR UCHUN NOLOKAL INTEGRAL MASALALAR

Annotatsiya: *Ushbu maqolada boshlang‘ich, chegaraviy va vaznli integral shartlari bilan berilgan yuqori* $2m+1$ *tartibli aralash hosilali differensial tenglama tadqiq etiladi. Musbat aniqlangan momentlar matritsasi yordamida berilgan chegaraviy va integral shartlarini qanoatlantiruvchi yordamchi ko‘phad quriladi. Integral shartlarni saqlovchi mos chekli rangli proyeksiya operatorining xossalari o‘rganiladi. Birinchi qaralayotgan holat* $m=1$ *uchun chekli trigonometrik yoyilmalar yordamida belgilangan yordamchi funksiyalar sinfida yechimning mavjudligi va yagonaligi isbotlanadi hamda yuqori chastotali g‘alayonlarga nisbatan noturg‘unlik aniqlanadi. Ikkinchi holat* $m\geq 2$ *uchun chekli* m *- o‘lchamli ko‘phadlar fazosining aniq bazisi quriladi va* $(m-1)$ *moslik shartlarini qanoatlantiruvchi berilganlar uchun tegishli yechimlar sinfida yechimning mavjudligi hamda yagonaligi isbotlanadi. Olingan natijalar ixtiyoriy berilganlar uchun cheklanmagan funksional fazolarda masalaning korrektligini to‘liq ta’minlamaydi.*

Kalit so‘zlar: *aralash hosilali differensial tenglama; nolokal integral shart; yordamchi ko‘phad; momentlar matritsasi; chekli rangli proyeksiya operatori; mavjudlik va yagonalik; noturg‘unlik.*

НЕЛОКАЛЬНЫЕ ИНТЕГРАЛЬНЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЙ СМЕШАННОГО ТИПА ВЫСОКОГО ПОРЯДКА

Аннотация: *В работе исследуется дифференциальное уравнение высокого порядка* $2m+1$ *со смешанными производными, рассматриваемое при начальных, граничных и взвешенных интегральных условиях. С помощью положительно определённой матрицы моментов строится вспомогательный многочлен, удовлетворяющий заданным граничным и интегральным условиям. Исследуются свойства соответствующего конечномерного проекционного оператора, сохраняющего интегральные моменты. Для первого рассматриваемого случая* $m=1$ *с помощью явных конечных тригонометрических разложений доказываются существование и единственность решения в заданном классе модальных функций и устанавливается неустойчивость по отношению к высокочастотным возмущениям. Для второго случая* $m\geq 2$ *строится явный базис конечномерного* $(m-1)$ *пространства многочленов и доказываются существование и единственность решения при согласованных данных в соответствующем классе решений. Проведённый анализ не устанавливает корректность задачи для произвольных данных в неограниченных функциональных пространствах.*

Ключевые слова: *уравнение со смешанными производными; нелокальное интегральное условие; вспомогательный многочлен; матрица моментов; конечномерный проекционный оператор; существование и единственность; неустойчивость.*

1. Introduction

Nonlocal integral conditions arise when measurements of a solution are prescribed through weighted spatial moments rather than point values. For higher-order mixed-derivative equations, the algebraic compatibility of those conditions must be distinguished from the solvability and continuous dependence of the differential problem. Studies of nonlocal boundary conditions and integral observations appear in [1, 2, 3, 4]; broader functional-analytic approaches are developed in [6, 8].

This paper develops three explicit results: a unique auxiliary polynomial satisfying prescribed boundary and integral conditions; a bounded projection that preserves the moments; and existence–uniqueness statements for specific trigonometric and polynomial classes of solutions. The latter are deliberately restricted to their stated classes. An instability calculation also explains why these formulas cannot be interpreted as general Hadamard well-posedness [7].

2. Statement of the problem

In the domain $D=\{(x,t):\ 0<x<1,\ 0<t<T\}$. We consider the high-order equation of the following form

$$\frac{\partial^{2m+1}u}{\partial x^{2m+1}}+(-1)^{m+1}\frac{\partial^{m+1}u}{\partial x\partial t^{m}}=f(x,t),\quad (x,t)\in D. \tag{1}$$

In this paper, the following problems are investigated for equation (1):

Nonlocal problem. Find solution $u(x,t)$ to equation (1) in domain D that satisfies the initial condition

$$\partial_t^k u(x,0)=\varphi_k(x),\qquad k=0,\ldots,m-1, \tag{2}$$

boundary conditions

$$u(0,t)=\psi_1(t),\qquad u(\ell,t)=\psi_2(t), \tag{3}$$

and integral condition

$$\int_0^1 x^j u(x,t)dx=\psi_{j+3}(t),\ j=0,\ldots,m-1. \tag{4}$$

where, $\varphi(x),\psi_1(t),\psi_2(t),\psi_{j+3}(t)-$ are the given functions, continuous at [0, ℓ] and [0, T], satisfying the matching conditions:

$$\varphi_k(0)=\psi_1^{(k)}(0),\quad \varphi_k(1)=\psi_2^{(k)}(0),\quad \int_0^1 x^j\varphi_k(x)dx=\psi_{j+3}^{(k)}(0). \tag{5}$$

The above identities are valid for $x\in[0,1]$, provided that all the derivatives involved exist in the classical sense. The additional regularity and compatibility assumptions required for the validity of the subsequent constructions are explicitly stated in the corresponding results. Throughout this paper, the spatial domain is fixed as the unit interval $[0,1]$.

We seek a solution to the considered problem in the following form:

$$v(x,t)=(1-x)\psi_1(t)+x\psi_2(t),\qquad q_n(x)=x^{n+1}(1-x),\quad 0\le n\le m-1. \tag{6}$$

To satisfy the prescribed boundary and integral conditions, we introduce an auxiliary polynomial of the form

$$\Phi(x,t)=v(x,t)+\sum_{n=0}^{m-1}a_n(t)q_n(x). \tag{7}$$

We denote

$$G_{jn}=\int_0^1 x^j q_n(x)dx=\frac{1}{j+n+2}-\frac{1}{j+n+3},\quad \beta_j(t)=\psi_{j+3}(t)-\int_0^1 x^j v(x,t)dx. \tag{8}$$

Then all integral condition for Φ are equivalent to $Ga(t)=\beta(t)$.

Theorem 1. For prescribed functions $\psi_1,\ldots,\psi_{m+2}$, *the system* $Ga(t)=\beta(t)$ *has a unique solution* $a(t)$ *at each* t. *Consequently, there is a unique polynomial of the form (7) realizing the boundary condition (3) and integral condition (4). If the prescribed functions belong to* $C^1[0,T]$, *then so do the coefficients* a_n.

Proof. The matrix G is symmetric by (8). For $c=(c_0,\ldots,c_{m-1})^{\mathrm{T}}\neq 0$, set $P(x)=\sum_{n=0}^{m-1}c_n x^n$. Directly,

$$c^{\mathrm{T}}Gc = \sum_{j,n=0}^{m-1} c_j c_n \int_0^1 x^{j+n+1}(1-x)dx = \int_0^1 x(1-x)P(x)^2 dx > 0. \quad (9)$$

The strict inequality follows because P is a nonzero polynomial and $x(1-x) > 0$ on $(0,1)$. Hence G is positive definite and invertible, so $a(t) = G^{-1}\beta(t)$ exists uniquely. Since $q_n(0) = q_n(1) = 0$, (7) gives the boundary values. Furthermore, for each j, and

$$\int_0^1 x^j \Phi(x,t)dx = \int_0^1 x^j v(x,t)dx + (Ga(t))_j = \psi_{j+3}(t).$$

The regularity assertion follows from the constant matrix G^{-1} and the definition of β.

We note that the substitution $w = u - \Phi$ makes the boundary and integral conditions homogeneous:

$$w(0,t) = w(1,t) = 0, \qquad \int_0^1 x^j w(x,t)dx = 0 \quad (0 \le j \le m-1). \qquad (10)$$

The transformed equation is

$$\partial_x^{2m+1} w + (-1)^{m+1}\partial_x \partial_t^m w = g, \quad g = f - \partial_x^{2m+1}\Phi - (-1)^{m+1}\partial_x \partial_t^m \Phi, \quad (11)$$

with initial condition $\partial_t^k w(x,0) = \varphi_k(x) - \partial_t^k \Phi(x,0)$. Theorem 1 concerns the unique auxiliary polynomial, not a general existence or uniqueness theorem for (11).

Let $H = L^2(0,1)$, $M_j(z) = \int_0^1 x^j z(x)dx$ and $E_m = span\{q_0,\ldots,q_{m-1}\}$. Define the finite-rank map $K : H \to H$ by

$$(Kz)(x) = \sum_{n,j=0}^{m-1} q_n(x)(G^{-1})_{nj} M_j(z). \qquad (12)$$

Proposition 1. The map K *is a bounded projection onto* E_m, *preserves each moment* M_j, *and has kernel* $H_0 = \{z \in H : M_j(z) = 0,\ j = 0,\ldots,m-1\}$. *In particular,*

$$K^2 = K, \qquad H = E_m \oplus H_0, \qquad (13)$$

where the topological direct sum need not be orthogonal.

Proof. For $0 \le i \le m-1$, using $M_i(q_n) = G_{in}$ gives

$$M_i(Kz) = \sum_{n,j} G_{in}(G^{-1})_{nj} M_j(z) = M_i(z).$$

If $z = \sum_n c_n q_n \in E_m$, its moment vector is Gc; inserting this in (12) gives $Kz = z$. Thus $RanK = E_m$ and $K^2 = K$. If $Kz = 0$, moment preservation implies every $M_j(z) = 0$; conversely those vanishing moments imply $Kz = 0$, proving $KerK = H_0$. Finally,

$$|M_j(z)| \le \| x^j \|_2 \| P_2 \| \| z \|_2 = \frac{\| z \|_2}{\sqrt{2j+1}},$$

and therefore

$$\| Kz \|_2 \le \left(\sum_{n,j=0}^{m-1} \frac{\| q_n \|_2 |(G^{-1})_{nj}|}{\sqrt{2j+1}} \right) \| z \|_2 .$$

Thus K is bounded. The decomposition follows from $z = Kz + (I-K)z$, with $Kz \in E_m$ and $(I-K)z \in H_0$; the intersection is zero.

The decomposition is an operator identity for moments and does not establish invertibility of the differential operator in (1).

For $m = 1$, with homogeneous boundary and integral condition, (1) becomes

$$u_{xxx} + u_{xt} = f, \quad u(0,t) = u(1,t) = 0, \quad \int_0^1 u(x,t)dx = 0. \qquad (14)$$

Set $k_n = 2\pi n$ for $n \geq 1$. Each $\sin(k_n x)$ vanishes at the boundary and has zero integral conditions. Suppose

$$\varphi_0(x) = \sum_{n=1}^{N} a_n \sin(k_n x), \qquad f(x,t) = \sum_{n=1}^{N} k_n h_n(t)\cos(k_n x), \quad h_n \in C[0,T]. \quad (15)$$

Theorem 2. There exists exactly one solution to (14) with condition (15) in the class $S_N = \{\sum_{n=1}^{N} b_n(t)\sin(k_n x) : b_n \in C^1[0,T]\}$, *namely*

$$u(x,t) = \sum_{n=1}^{N} e^{k_n^2 t}\left[a_n + \int_0^t e^{-k_n^2 s} h_n(s)ds \right]\sin(k_n x). \quad (16)$$

Proof. Substitution of $u = \sum_n b_n(t)\sin(k_n x)$ into the partial differensial equation gives

$$u_{xxx} + u_{xt} = \sum_{n=1}^{N} k_n (b_{n'}(t) - k_n^2 b_n(t))\cos(k_n x).$$

Linear independence of the cosine function modes yields $b_{n'} - k_n^2 b_n = h_n$, $b_n(0) = a_n$. Multiplication by the integrating factor $e^{-k_n^2 t}$ and integration from 0 to t give the unique coefficient

$$b_n(t) = e^{k_n^2 t}\left[a_n + \int_0^t e^{-k_n^2 s} h_n(s)ds \right].$$

The selected frequencies verify all homogeneous boundary and integral conditions, so the expression is the required solution in S_N.

The formula implies the estimate

$$\| u \|_{C(\bar{D})} \leq \sum_{n=1}^{N} \left(e^{k_n^2 T} | a_n | + \frac{e^{k_n^2 T} - 1}{k_n^2} \| h_n \|_{C[0,T]} \right). \quad (17)$$

However, for $h_n = 0$ and a single initial condition $\varphi^{(n)}(x) = \sin(k_n x)$, one has $\| \varphi^{(n)} \|_2 = 1/\sqrt{2}$ while $\| u^{(n)}(\cdot,t) \|_2 = e^{k_n^2 t}/\sqrt{2}$. For any fixed $t > 0$, this ratio diverges as $n \to \infty$. Hence no frequency-independent continuous-dependence estimate from initial L^2 condition to solutions in L^2 can hold on this family. This conclusion does not contradict uniqueness in any fixed finite-dimensional S_N.

We suppose $m \geq 2$ and introduce

$$R_n(x) = \frac{d^n}{dx^n}[x^{n+1}(1-x)^{n+1}], \qquad n = m, \ldots, 2m-2. \quad (18)$$

Define the polynomial space

$$V_m = \{X \in \mathrm{R}[x]: \quad \deg X \leq 2m, \quad X(0) = X(1) = 0, \quad \int_0^1 x^j X(x)dx = 0 \quad (j = 0, \ldots, m-1)\}. \qquad (19)$$

The zero polynomial is included regardless of the convention used for its degree.

Lemma 1. For $m \geq 2$, *the polynomials* $R_m, \ldots, R_{2m-2}$ *form a basis of* V_m. *In particular* $\dim V_m = m-1$.

Proof. By (18), $\deg R_n = n + 2 \leq 2m$, and the polynomial inside the derivative has zeros of order $n+1$ at both boundary conditions. Thus $R_n(0) = R_n(1) = 0$. For $0 \leq j \leq m-1 < n$, integration by parts n times gives

$$\int_0^1 x^j R_n(x)dx = (-1)^n \int_0^1 \frac{d^n}{dx^n}(x^j)x^{n+1}(1-x)^{n+1}dx = 0;$$

all boundary terms vanish. Hence $R_n \in V_m$. Their distinct degrees $n+2$ imply linear independence.

To compute the dimension, let $X \in V_m$. The two boundary conditions give $X = x(1-x)P$ with $\deg P \le 2m-2$, a space of dimension $2m-1$. The m moment functionals on this space are linearly independent: if $\sum_{j=0}^{m-1} c_j \int_0^1 x^j X\,dx = 0$ for every such X, choose $X = x(1-x)Q$, $Q = \sum_{j=0}^{m-1} c_j x^j$.

Then, $\int_0^1 x(1-x)Q(x)^2\,dx = 0$, so $Q = 0$.

Accordingly, integral conditions have $rank(m)$ and $\dim V_m = (2m-1) - m = m-1$. The $m-1$ independent R_n therefore form a basis.

Define the restricted solution class

$$\mathrm{W}_m = \left\{ u(x,t) = \sum_{n=m}^{2m-2} R_n(x) b_n(t) : b_n \in C^m[0,T] \right\}. \qquad (20)$$

Assume that the boundary, integral condition, and initial conditions are homogeneous and that the forcing functions have the following compatible form:

$$\varphi_k(x) = \sum_{n=m}^{2m-2} a_{nk} R_n(x), k = 0,\ldots,m-1, \quad (21)$$

$$f(x,t) = (-1)^{m+1} \sum_{n=m}^{2m-2} R_{n'}(x) h_n(t), h_n \in C[0,T]. \quad (22)$$

Theorem 3. Let, and let $m \ge 2$ be the polynomial solution class defined in (20). Assume that the boundary and integral conditions are homogeneous and that the initial data and the forcing term satisfy (21)–(22), namely

$$\varphi_k(x) = \sum_{n=m}^{2m-2} a_{nk} R_n(x), k = 0,\ldots,m-1,$$

$$f(x,t) = (-1)^{m+1} \sum_{n=m}^{2m-2} R_n{}'(x) h_n(t), \qquad h_n \in C([0,T]).$$

Then problem (1)–(4) admits exactly one solution in W_m. It is given by

$$u(x,t) = \sum_{n=m}^{2m-2} R_n(x) \left[\sum_{k=0}^{m-1} \frac{a_{nk} t^k}{k!} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} h_n(s) ds \right]. \qquad (23)$$

Moreover, putting, the solution satisfies $A_n = \| R_n \|_{C[0,1]}$

$$\left\| u(x,t) \right\|_{C(\overline{D})} = \sum_{n=m}^{2m-2} A_n \left[\sum_{k=0}^{m-1} \frac{|a_{nk}| T^k}{k!} + \frac{T^m}{m!} \left\| h_n \right\|_{C[0,T]} \right]. \quad (24)$$

Proof. For $u \in \mathrm{W}_m$, write

$$u(x,t) = \sum_{n=m}^{2m-2} R_n(x) b_n(t).$$

Because $\deg R_n = n+2 \le 2m$, its spatial derivative of order $2m+1$ vanishes. Also,

$$\frac{\partial^{m+1} u}{\partial x \partial t^m} = \sum_{n=m}^{2m-2} R_n'(x) b_n^{(m)}(t).$$

Substituting into (1) and using (22), we obtain

$$u(x,t) = \sum_{n=m}^{2m-2} R_n(x) \left[b_n^m(t) - h_n(t) \right] = 0.$$

The polynomials have distinct degrees and hence are linearly independent. Therefore. By (21), the initial conditions give for. The unique solution of each scalar initial-value problem is$R_n'n+1b_n^{(m)}(t)=h_n(t)b_n^{(k)}(0)=a_{nk}k=0,\dots,m-1$

$$b_n(t)=\sum_{k=0}^{m-1}\frac{a_{nk}t^k}{k!}+\frac{1}{(m-1)!}\int_0^t(t-s)^{m-1}h_n(s)ds,$$

This yields (23). By Lemma 1, and for; hence the boundary and moment conditions hold. Uniqueness follows from uniqueness of the scalar initial-value problems and linear independence of the. Finally,$R_n(0)=R_n(1)=0\int_0^1 x^j\,R_n(x)\,dx=0j=0,\dots,m-1R_n$

$$\left|\int_0^t(t-s)^{m-1}h_n(s)ds\right|\le\frac{T^m}{m!}\|h_n\|_{C[0,T]}$$

and the triangle inequality proves (24).

We illustrate the case $m=2$ with the following example.

The space V_2 is one-dimensional, with

$$R_2(x)=6x-36x^2+60x^3-30x^4. \quad (25)$$

For $\varphi_0=a_0R_2$, $\varphi_1=a_1R_2$ and $f(x,t)=-R_{2'}(x)h(t)$, Theorem 3 gives the unique solution in W_2:

$$u(x,t)=R_2(x)\left[a_0+a_1t+\int_0^t(t-s)h(s)ds\right]. \quad (26)$$

Indeed $\partial_x^5u=0$, and $-\partial_x\partial_t^2u=-R_{2'}h=f$. The boundary and the zeroth and first integrals vanish by Lemma 1.

We next illustrate the case $m=3$ with the following example.

Now $\dim V_3=2$, with basis

$$R_3(x)=24x-240x^2+720x^3-840x^4+336x^5, \quad (27)$$

$$R_4(x)=120x-1800x^2+8400x^3-16800x^4+15120x^5-5040x^6. \quad (28)$$

Let $\varphi_k=a_{3k}R_3+a_{4k}R_4$ ($k=0,1,2$) and $f=R_{3'}h_3+R_{4'}h_4$. Then

$$u(x,t)=R_3(x)\left[a_{30}+a_{31}t+\frac{a_{32}t^2}{2}+\frac{1}{2}\int_0^t(t-s)^2h_3(s)ds\right]+$$
$$+R_4(x)\left[a_{40}+a_{41}t+\frac{a_{42}t^2}{2}+\frac{1}{2}\int_0^t(t-s)^2h_4(s)ds\right]. \quad (29)$$

Differentiating three times in t verifies $\partial_x\partial_t^3u=f$, while $\partial_x^7u=0$. The remaining conditions follow from the basis lemma. In particular, the spatial derivative $R_{n'}$ in the forcing is essential for the stated mixed-derivative equation.

The two boundary traces and m integral conditions in (3)–(4) do not by themselves yield a general theorem on the differential operator of spatial order $2m+1$. In particular, additional spatial boundary conditions would change the operator domain and cannot simply be imposed on solutions of Theorem 3.

A useful purely algebraic observation is the following. Suppose $X\in\mathrm{R}[x]$, $\deg X\le 2m$, and

$$X^{(r)}(0)=0\ (0\le r\le m-1),\quad X(1)=0,\quad \int_0^1x^jX(x)dx=0\ (0\le j\le m-1).$$

Then $X\equiv 0$. Indeed, the boundary conditions imply $X(x)=x^m(1-x)S(x)$ with $\deg S\le m-1$. Expressing $S(x)=\sum_{j=0}^{m-1}s_jx^j$, multiplying the j the integral condition by s_j and summing yields

$$0=\int_0^1x^m(1-x)S(x)^2dx.$$

The prescribed integral condition implies that the polynomial vanishes identically. This establishes the triviality of the kernel within the class of polynomials satisfying the strengthened boundary conditions. However, this result does not imply global uniqueness for the original differential equation.

Conclusion

In this study, a unique auxiliary polynomial satisfying the prescribed boundary and integral conditions was constructed using a positive-definite moment matrix. The properties of the corresponding finite-rank projection operator preserving the given integral conditions were also investigated. For $m=1$ the first considered case, the existence and uniqueness of solutions were established within a finite-dimensional trigonometric function space. Moreover, the instability of solutions under high-frequency perturbations was demonstrated.

For $m \geq 2$ the second considered case, an explicit basis of a $(m-1)$-dimensional polynomial space was constructed, and the existence and uniqueness of solutions were proved for compatible forcing functions and initial conditions within the corresponding polynomial solution class. The theoretical results were further illustrated by examples corresponding to $m=2$ and $m=3$. However, the questions of solvability, uniqueness, and stability for arbitrary admissible data in general function spaces remain open and require further investigation.

Foydalanilgan adabiyotlar

1. A. V. Bitsadze and A. A. Samarskii, «On some elementary generalizations of linear elliptic boundary value problems,» Doklady Akademii Nauk SSSR, vol. 185, no. 4, pp. 739–740, 1969.

2. N. I. Ivanchov, «Boundary value problems for a parabolic equation with integral conditions,» Differential Equations, vol. 40, no. 4, pp. 591–609, 2004. doi:10.1023/B:DIEQ.0000035796.56467.44.

3. S. Mesloub and A. Bouziani, «Mixed problem with a weighted integral condition for a parabolic equation with the Bessel operator,» Journal of Applied Mathematics and Stochastic Analysis, vol. 15, no. 3, 2002. doi:10.1155/S1048953302000242.

4. A. Bouziani, «Initial-boundary value problems for a class of pseudoparabolic equations with integral boundary conditions,» Journal of Mathematical Analysis and Applications, vol. 291, no. 2, pp. 371–386, 2004. doi:10.1016/S0022-247X(03)00590-0.

5. A. M. Abdrakhmanov and A. I. Kozhanov, «A problem with a nonlocal boundary condition for one class of odd-order equations,» Russian Mathematics, vol. 51, no. 5, pp. 1–10, 2007. doi:10.3103/S1066369X07050015.

6. A. I. Kozhanov and A. V. Dyuzheva, «Non-local problems with integral displacement for high-order parabolic equations,» Bulletin of Irkutsk State University. Series Mathematics, vol. 36, pp. 14–28, 2021. doi:10.26516/1997-7670.2021.36.14.

7. A. I. Kozhanov and G. I. Tarasova, «The Samarsky–Ionkin problem with integral perturbation for a pseudoparabolic equation,» Bulletin of Irkutsk State University. Series Mathematics, vol. 42, pp. 59–74, 2022. doi:10.26516/1997-7670.2022.42.59.

8. M. M. Sagdullayeva, «On a mixed problem with integral conditions for third-order equation,» Uzbek Mathematical Journal, vol. 69, no. 1, pp. 125–132, 2025. doi:10.29229/uzmj.2025-1-12.

9. N. I. Ionkin, «The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition,» Differentsial'nye Uravneniya, vol. 13, no. 2, pp. 294–304, 1977.

10. J.-L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. 1. Berlin: Springer, 1972. doi:10.1007/978-3-642-65161-8.

11. A. N. Tikhonov and V. Y. Arsenin, Solutions of Ill-Posed Problems. Washington, DC: V. H. Winston, 1977.

12. T. Kato, Perturbation Theory for Linear Operators, 2nd ed. Berlin: Springer, 1995. doi:10.1007/978-3-642-66282-9.

13. L. C. Evans, Partial Differential Equations, 2nd ed., Graduate Studies in Mathematics, vol. 19. Providence, RI: American Mathematical Society, 2010.

14. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. II. New York: Interscience, 1962.

15. M. A. Naimark, Linear Differential Operators, Part I. New York: Frederick Ungar, 1967.

16. R. A. Adams and J. J. F. Fournier, Sobolev Spaces, 2nd ed. Amsterdam: Academic Press, 2003.