NAMANGAN DAVLAT UNIVERSITETI
ILMIY AXBOROTNOMASI, [2026-10]
ISSN:2181-1458
ISSN:2181-0427

01.00.00-FIZIKA-MATEMATIKA
journal.namdu.uz
PHYSICS AND MATHEMATICS

# P-ADIC QUASI GIBBS MEASURES FOR THE ISING-POTTS MODEL ON A CAYLEY TREE

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***Abstract:*** *In this paper, the p-adic Ising-Potts model is studied on a regular tree. Furthermore, a functional equation is derived and analyzed for translation-invariant measures on a binary tree. Moreover, the phase transition condition is established.*

***Keywords:*** *p-adic numbers, Ising-Potts model, translation-invariant, a phase transition.*

## KELI DARAXTIDA IZING-POTTS MODELI UCHUN P-ADIK KVAZI GIBBS O'LCHOVLARI

***Annotatsiya:*** *Ushbu maqolada regulyar daraxtda p-adik Ising-Potts modeli o'rganilgan. Bundan tashqari, funksional tenglama keltirib chiqarilgan va 2-tartibli daraxtda translatsion-invariant o'lchovlar tahlil qilingan. Qolaversa, faza almashish sharti aniqlangan.*

***Kalit so'zlar:*** *p-adik sonlar, Izing-Potts model, translatsion-invariant, faza almashish.*

## Р-АДИЧЕСКИЕ КВАЗИ-ГИББСОВСКИЕ МЕРЫ ДЛЯ МОДЕЛИ ИЗИНГА-ПОТТСА НА ДЕРЕВЕ КЕЛИ

***Аннотация:*** *В данной работе на регулярном дереве изучена $p$-адическая модель Изинга-Поттса. Кроме того, выведено функциональное уравнение и проанализированы трансляционно-инвариантные меры на дереве 2-го порядка. Более того, определено условие фазового перехода.*

***Ключевые слова:*** *р-адические числа, модель Изинга-Поттса, трансляционно-инвариантный, фазовый переход.*

## Introduction

At present, considerable focus is directed toward investigating mixed-type models. Findings derived from these models can be utilized either independently for separate systems or combined to describe various component states within a binary mixture (such as solid or liquid phases). By tuning model parameters—including molecular interaction energies—scientists are able to model mixture phase behavior and forecast critical points like melting and eutectic transitions.

In [1], [2], [3] the Ising-Potts model were studied over the real field. For a deeper understanding of the physical motivation behind the Ising-Potts model, refer to [5].

Present paper is devoted to study p-adic Ising-Potts model on the Cayley tree.

## I. Preliminaries

***p*-adic numbers and *p*-adic measure.** Let $p$ be any prime number. For any nonzero integer $a$, let $\mathrm{ord}_p a$ be the highest power of $p$ which divides $a$, i.e., the greatest $m$ such that

$$a \equiv 0 \pmod{p^m}.$$

If $a = 0$, we agree to write $\mathrm{ord}_p a = \infty$.

Now, for any rational number $x = \dfrac{a}{b}$, define $\mathrm{ord}_p x = \mathrm{ord}_p a - \mathrm{ord}_p b$. It is easy to check that $\mathrm{ord}_p x$ is well-defined.

Further define a map $|\cdot|_p$ on $\mathbb{Q}$ as follows:

$$|x|_p=\begin{cases} p^{-\mathrm{ord}_p x}, & \text{if } x\neq 0, \\ 0, & \text{if } x=0. \end{cases} \quad (1)$$

The map defined by (1) is a norm. Furthermore, this norm is a non-Archimedean norm, i.e.

$$|x+y|_p \le \max\{|x|_p, |y|_p\}$$

for all $x, y \in \mathbb{Q}$.

The norm defined by (1) is called $p$ *-adic norm*.

The strong triangle inequality property follows that

$$1)\ \text{if } |x|_p \neq |y|_p \ \text{ then } \ |x \pm y|_p = \max\{|x|_p, |y|_p\};$$

$$2)\ \text{if } |x|_p = |y|_p \ \text{ then } \ |x \pm y|_p \le \max\{|x|_p, |y|_p\}.$$

The completion of $\mathbb{Q}$ with respect to the $p$ -adic norm defines the $p$ -adic field $\mathbb{Q}_p$ (see [4]).

Ostrowski proved that every non-trivial norm on $\mathbb{Q}$ is equivalent to either $p$ -adic norm for some $p$, or absolute value. It means that the completion of the field of rational numbers $\mathbb{Q}$ with non-trivial norm is either the field of real numbers $\mathbb{R}$ or one of the fields of $p$ -adic numbers $\mathbb{Q}_p$.

Any $p$ -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}\left(x_0 + x_1 p + x_2 p^2 + \ldots\right), \quad (2)$$

where $x_j, \gamma(x) \in \mathbb{Z}$, $x_0 \neq 0$, $x_j \in \{0,1,\ldots,p-1\}$, $j = 1,2,\ldots$.

In this case $|x|_p = p^{-\gamma(x)}$ (see [4]).

**Theorem 1.** [5] The equation

$$x^2 = a,\ a \neq 0,\ a = p^{\gamma(a)}(a_0 + a_1 p + a_2 p^2 + \ldots),\ a_0 \neq 0,\ a_j \in \{0,1,\ldots,p-1\},\ j \in \mathbb{N},$$

has a solution in $x \in \mathbb{Q}_p$ iff the following statements hold:

i) $\gamma(a)$ is even;

ii) $a_0$ is a quadratic residue of $p$, if $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

Let $(X, B)$ be a measurable space, where $B$ is an algebra of subsets $X$. A function $\mu : B \to \mathbb{Q}_p$ is said to be a $p$ -adic measure if for any $A_1, A_2, \ldots, A_n \in B$ such that $A_i \cap A_j = \varnothing$, $i \neq j$, the following holds:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

A $p$ -adic measure $\mu$ is called bounded if $\sup\{|\mu(A)|_p : A \in B\} < \infty$. It is said that $p$ -adic measure is *probabilistic* if $\mu(X) = 1$.

**Cayley tree.** Let $\Gamma^k = (V, L)$, be a infinite Cayley tree of order, $k \ge 1$ i.e., whose each vertex has exactly $k+1$ edges) (see [6]). Here $V$ is the set of vertices and $L$ is the set of edges. The vertices $x$ and $y$ are called nearest neighbors and they are denoted by $l = \langle x, y \rangle$ if there exists an edge $l$ connecting them. A collection of the pairs $\langle x, x_1 \rangle$, $\langle x_1, x_2 \rangle$ ,..., $\langle x_{d-1}, y \rangle$ is called a path from the point $x$ to the point $y$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from $x$ to $y$. We set

$$W_n = \{x \in V \mid d(x, x^0) = n\},\ V_n = \{x \in V \mid d(x, x^0) \le n\},\ L_n = \{l = < x, y > \in L \mid x, y \in V_n\}.$$

The set of direct successors of $x$ is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1,\ x \in W_n\}.$$

**p-adic Ising-Potts model.** Let $\mathbb{Q}_p$ be the field of $p$-adic numbers [4], $\Gamma^k = (V, L)$ is a Cayley tree of order $k$, $\Phi = \{-1,1\}$ and $\Psi = \{1,2,...,q\}$ $(q \geq 3)$. A configuration $(\sigma, s)$ on $A \subset V$ is then defined as a function $x \in A \mapsto (\sigma(x), s(x)) \in \Phi \times \Psi$. The set of all configurations on $A$ is denoted by $\Omega_A = (\Phi \times \Psi)^A$.

The Hamiltonian of the Ising-Potts model is given by

$$H_n(\sigma_n, s_n) = J_I \sum_{\langle x,y\rangle \in L_n} \sigma(x)\sigma(y) + J_P \sum_{\langle x,y\rangle \in L_n} \delta_{s(x)s(y)}, \quad (3)$$

where $\sigma \in \Phi$, $s \in \Psi$, $J_I, J_P \in B(0,1)$, $\delta_{ij}$ is Kronecker's symbol.

Let us consider $p$-adic probability measure $\mu_n$ on $\Omega_{V_n}$ defined by

$$\mu_n(\sigma_n, s_n) = Z_n^{-1} \exp\{H_n(\sigma_n, s_n)\} \prod_{x \in W_n} h_{\sigma(x),s(x),x}, \quad (4)$$

where $\mathbf{h}_x = \{\left(h_{1,1,x},...,h_{1,q,x},h_{-1,1,x},...,h_{-1,q,x}\right) \in \mathbb{Q}_p^{2q}, x \in V\}$ is a collection of vectors, $Z_n^{-1}$ is the normalizing factor.

The sequence of $p$-adic probability measures $\mu_n$ is called compatibly if for all $n \geq 1$ and

$$(\sigma_{n-1}, s_{n-1}) \in \Omega_{V_{n-1}}, \text{ we have}$$

$$\sum_{(\sigma_{[n]}, s_{[n]}) \in \Omega_{W_n}} \mu_n((\sigma_{n-1}, s_{n-1}) \vee (\sigma_{[n]}, s_{[n]})) = \mu_{n-1}(\sigma_{n-1}, s_{n-1}). \quad (5)$$

If $p$-adic probability measures (4), corresponding to the Hamiltonian (3) are compatibly then a non-Archimedean analogue of the Kolmogorov's extension theorem [7], there exists a unique $p$-adic splitting *quasi Gibbs measure*.

**Theorem 2.** The sequence of $p$-adic probability measures $\mu_n(\sigma_n, s_n)$ satisfy the compatibility condition (5) if and only if for any $x \in V$, $\{x_0\}$ the following equation holds:

If $(i, j) \neq (-1, q)$ then

$$z_{i,j,x} = \prod_{y \in S(x)} \frac{\theta_I^{1-i}\left(\sum_{v=1}^{q}\left(\theta_I^{2i} z_{1,v,y} + z_{-1,v,y}\right) + \left(\theta_P - 1\right)\left(\theta_I^{2i} z_{1,j,y} + z_{-1,j,y}\right)\right)}{\sum_{v=1}^{q-1}\left(z_{1,v,y} + \theta_I^2 z_{-1,v,y}\right) + \theta_P\left(z_{1,q,y} + \theta_I^2\right)}, \quad (6)$$

If $(i, j) = (-1, q)$ then $z_{-1,q,x} = 1$.

Here $\theta_I = \exp_p\{J_I\}$, $\theta_P = \exp_p\{J_P\}$, $i \in \Phi, j \in \Psi$.

**Proof. The proof proceeds as in [8].**

Note that, in [8], these problems were studied over the field of real numbers.

Assume that $\mathbf{h}_x = \mathbf{h}$ for all $x \in V$. If we consider the right-hand side of equation (6) as an operator mapping $\mathbb{Q}_p^{2q-1}$ into itself, then it is easy to see that the following set is invariant under the operator:

$$I = \{\mathbf{z} = (\underbrace{z, z, ..., z, 1}_{q}, z, z, ..., z) \in \mathbb{Q}_p^{2q-1}\}.$$

On the invariant set $I$, (6) has the following form

$$z = \left(\frac{z(\theta_P + q - 2) + 1}{z(q-1) + \theta_P}\right)^k. \quad (7)$$

**Theorem 3.** Let $p > 3$, $|q-1|_p = 1$. Then on a Cayley tree of order two, for the values of $z$ in the invariant set $I$, the following assertions hold for the corresponding $p$-adic quasi Gibbs measures of the Ising-Potts model:

i. If $1-q$ is quadratic residue modulo $p$ then there exist 3 translation-invariant $p$-adic quasi Gibbs measures. Moreover, there exists a phase transition;

ii. If $1-q$ is not quadratic residue modulo $p$ then there exists a unique translation-invariant $p$-adic quasi Gibbs measure.

**Proof. Substitute** $k=2$ into the original equation (7):

$$z=\left(\frac{z(\theta_P+q-2)+1}{z(q-1)+\theta_P}\right)^2 \quad (8)$$

Let us check if $z=1$ is a solution of (8):

$$\text{LHS}=1$$

$$\text{RHS}=\left(\frac{1\cdot(\theta_P+q-2)+1}{1\cdot(q-1)+\theta_P}\right)^2=\left(\frac{\theta_P+q-1}{q-1+\theta_P}\right)^2=1^2=1.$$

Multiply both sides by $(z(q-1)+\theta_P)^2$ (assuming $z(q-1)+\theta_P\neq 0$) of (8):

$$z\left(z(q-1)+\theta_P\right)^2=\left(z(\theta_P+q-2)+1\right)^2$$

or

$$z^3(q-1)^2+z^2\left[2\theta_P(q-1)-(\theta_P+q-2)^2\right]+z\left[\theta_P^2-2(\theta_P+q-2)\right]-1=0. \quad (9)$$

Since $z=1$ is a root, divide the cubic polynomial by $(z-1)$ using polynomial division or factor grouping.

The equation factorizes as:

$$(z-1)\left[(q-1)^2z^2+Bz+1\right]=0 \quad (10)$$

where

$$B=(q-1)^2+2\theta_P(q-1)-(\theta_P+q-2)^2.$$

Simplifying $B$:

$$B=2(q-1)-(\theta_P-1)^2.$$

Thus, the reduced quadratic equation for $z\neq 1$ is:

$$(q-1)^2z^2+\left[2(q-1)-(\theta_P-1)^2\right]z+1=0.$$

Using the quadratic formula $Az^2+Bz+C=0$ with $A=(q-1)^2$, $B=2(q-1)-(\theta_P-1)^2$, and $C=1$:

$$z_{2,3}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

Calculate the discriminant $\Delta=B^2-4AC$:

$$\Delta=\left(2(q-1)-(\theta_P-1)^2\right)^2-4(q-1)^2\cdot 1$$

or

$$\Delta = (\theta_P - 1)^2 \left( (\theta_P - 1)^2 - 4(q-1) \right).$$

Taking the square root of the discriminant:

$$\sqrt{\Delta} = (\theta_P - 1)\sqrt{(\theta_P - 1)^2 - 4(q-1)}\,.$$

According to Theorem 1, $z_{2,3} \in Q_p \Leftrightarrow \sqrt{\Delta} \in Q_p \Leftrightarrow -4(q-1)$ is quadratic residue of modulo $p$ $\Leftrightarrow 1-q$ is quadratic residue of modulo $p$.

Hence, if $1-q$ is quadratic residue modulo $p$ then there exist 3 translation-invariant $p$-adic quasi Gibbs measures $\mu_{z_1}$, $\mu_{z_2}$, $\mu_{z_3}$, otherwise a unique $p$-adic quasi Gibbs measure $\mu_{z_1}$.

It can be shown similarly to the method in [5] that only $\mu_{z_1}$ is bounded and the remaining two $\mu_{z_2}$, $\mu_{z_3}$ are not bounded.

The theorem is proved.

**Corollary 1.** Let $p > 3$, and $q = 3$. If $p \equiv 1,3 \pmod 8$ then for the $(2,3)$-Ising-Potts model on the binary tree there exists a phase transition.

**Proof. Thanks to Theorem 3, in this case if -2 is quadratic residue of modulo $p$**, then a phase transition. It means that $p \equiv 1,3 \pmod 8$ (see [10]).

**Conclusion**

We established the conditions for the existence of translation-invariant $p$-adic quasi Gibbs measures and phase transitions for the Ising-Potts model on a binary tree. Specifically, a phase transition occurs when $1-q$ is a quadratic residue modulo $p$. In particular, for $q = 3$, the occurrence of a phase transition in the $(2,3)$-Ising-Potts model is explicitly characterized by the congruence condition $p \equiv 1,3 \pmod 8$ for $p > 3$.

**Foydalanilgan adabiyotlar**

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