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PROPORTIONAL POPULATION REMOVAL AND EXTINCTION ESTIMATES IN A COMPETITION MODEL WITH A COMMON FREE BOUNDARY

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Abstract: *This study examines a Stefan-type model describing the interaction of two competing populations that share a moving boundary. The model incorporates nonlinear convection and population removal proportional to species density. For classical solutions, the positivity and uniform boundedness of both population densities are proved, together with an upper estimate for the speed of the free boundary. An exact balance relation is derived, linking the size of the habitat to the weighted total masses of the populations. It is shown that when the removal rates of both species exceed their intrinsic growth rates, the populations become extinct exponentially fast, the habitat remains bounded, and the free boundary converges to a finite limiting position. An explicit estimate for the convergence error is also obtained. Finally, a dimensionless example demonstrates how the extinction condition can be applied and provides computable estimates for the limiting habitat size and the time needed for the population densities to fall below a prescribed threshold.*

Keywords: *Stefan problem; common free boundary; Lotka–Volterra competition; nonlinear convection; proportional removal; mass balance; exponential extinction.*

UMUMIY ERKIN CHEGARALI RAQOBAT MODELIDA POPULYATSIYALARNING PROPORSIONAL KAMAYISHI VA YO‘QOLIB KETISH BAHOLARI

Annotatsiya: *Ushbu ishda umumiy harakatlanuvchi chegaraga ega hududda yashovchi ikki raqobatlashuvchi populyatsiyaning o‘zaro ta’sirini ifodalovchi Stefan tipidagi masala o‘rganilgan. Modelda chiziqsiz konveksiya hamda har bir populyatsiya zichligiga mutanosib ravishda individlarning kamaytirilishi hisobga olingan. Klassik yechimlar uchun populyatsiyalar zichligining musbatligi va bir xil chegaralanganligi isbotlangan, shuningdek, erkin chegara harakat tezligining yuqori bahosi olingan. Yashash hududi uzunligini populyatsiyalarning vaznli umumiy massalari bilan bog‘lovchi aniq muvozanat tengligi keltirib chiqarilgan. Agar har ikki tur uchun kamaytirish darajasi ularning tabiiy o‘sish tezligidan yuqori bo‘lsa, populyatsiyalar zichligi eksponensial tezlikda nolga intilishi, yashash hududi cheklangan holda qolishi va erkin chegaraning chekli yakuniy holatga yaqinlashishi isbotlangan. Ushbu yaqinlashish xatoligi uchun ham aniq baho berilgan. Yakunda o‘lchovsiz model misolida yo‘qolib ketish mezonining amaliy qo‘llanilishi ko‘rsatilgan hamda yashash hududining yakuniy uzunligi va populyatsiyalar zichligi belgilangan chegaraviy qiymatgacha kamayishi uchun zarur bo‘lgan vaqtni hisoblash imkonini beruvchi baholar olingan.*

Kalit so‘zlar: *Stefan masalasi; umumiy erkin chegara; Lotka–Volterra raqobati; nochiziqli konveksiya; proporsional kamaytirish; massa balansi; eksponensial yo‘qolib ketish.*

ПРОПОРЦИОНАЛЬНОЕ СОКРАЩЕНИЕ ПОПУЛЯЦИЙ И ОЦЕНКИ ИХ ВЫМИРАНИЯ В КОНКУРЕНТНОЙ МОДЕЛИ С ОБЩЕЙ СВОБОДНОЙ ГРАНИЦЕЙ

Аннотация: *В работе исследуется задача типа Стефана, описывающая взаимодействие двух конкурирующих популяций, обитающих в области с общей подвижной границей. Модель учитывает нелинейную конвекцию и пропорциональное изъятие особей из каждой популяции. Для классических решений доказаны положительность и равномерная ограниченность плотностей обеих популяций, а также получена верхняя оценка скорости движения свободной границы. Выведено точное балансовое соотношение, связывающее размер области обитания со взвешенными суммарными массами популяций. Установлено, что если интенсивность изъятия у обоих видов превышает их естественную скорость роста, то плотности популяций экспоненциально стремятся к нулю, область обитания остаётся ограниченной, а свободная граница приближается к конечному предельному положению. Кроме того, получена явная оценка погрешности этого приближения. На примере безразмерной модели показано практическое применение критерия вымирания и*

вычислены оценки конечного размера области обитания и времени, необходимого для снижения плотностей популяций до заданного уровня.

Ключевые слова: *задача Стефана; общая свободная граница; конкуренция Лотки — Вольтерры; нелинейная конвекция; пропорциональное изъятие; баланс массы; экспоненциальное вымирание.*

Introduction

In population models with a free boundary, the occupied habitat is determined together with the population density. Du and Lin [1] studied this framework for a diffusive logistic equation. Wang and Zhao [2] investigated free boundary problems for a Lotka–Volterra competition system. Nonlinear convection in a scalar free boundary problem was considered by Wang, Wang and Wang [3]. These studies motivate the combination of density dependent transport, interspecific competition and moving habitat boundaries.

The supplied source manuscript considers two competing species with separate moving fronts and nonlinear convection. The present formulation retains the competition and convection mechanisms, introduces proportional removal, and replaces the separate fronts by a common boundary driven by both density gradients. This boundary describes a shared habitat whose expansion depends on the contributions of both populations. The modification changes both the reaction terms and the geometric coupling.

The objective is to derive explicit a priori estimates and a sufficient extinction condition for this problem. The argument combines scalar maximum principles with an exact weighted mass balance. The maximum principle and upper solution framework is standard in nonlinear parabolic theory [4]. The results below are proved directly for the stated model; no claim of priority over the full literature is made. A complete existence and uniqueness theory is outside the scope of this article, and all infinite time conclusions are conditional on the existence of a global classical solution.

Formulation of the problem

Let t denote time and x the spatial coordinate. The population densities are $u(t,x)$ and $v(t,x)$, while $h(t)$ denotes the right endpoint of their common habitat. The equations are considered in the moving domain

$$Q_T = \{(t,x) : 0 < t \le T,\ 0 < x < h(t)\}$$

The competition system with nonlinear convection and proportional removal is

$$u_t = d_1 u_{xx} + m_1 u u_x + u(r_1 - q_1 - b_1 u - c_1 v) \tag{1}$$

$$v_t = d_2 v_{xx} + m_2 v v_x + v(r_2 - q_2 - b_2 v - c_2 u) \tag{2}$$

At the fixed left endpoint we impose homogeneous Neumann conditions. At the moving right endpoint both densities vanish:

$$u_x(t,0) = v_x(t,0) = 0 \tag{3}$$

$$u(t,h(t)) = v(t,h(t)) = 0 \tag{4}$$

The common boundary is driven by the gradients of both populations through the Stefan condition

$$h'(t) = -\mu_1 u_x(t,h(t)) - \mu_2 v_x(t,h(t)) \tag{5}$$

$$h(0) = h_0, \quad u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \tag{6}$$

The diffusion, convection, intrinsic growth, intraspecific competition and boundary mobility coefficients are positive. The removal and interspecific competition coefficients are nonnegative:

$$d_i, m_i, r_i, b_i, \mu_i > 0, \quad q_i, c_i \ge 0, \quad i \in \{1,2\}$$

The initial profiles satisfy the following regularity, positivity and boundary conditions:

$$h_0 > 0, \quad 0 < \alpha < 1, \quad u_0, v_0 \in C^{2+\alpha}([0,h_0])$$

$$u_0(x), v_0(x) > 0 \quad (0 \le x < h_0)$$

$$u_0'(0) = v_0'(0) = 0, \quad u_0(h_0) = v_0(h_0) = 0$$

$$u_0'(h_0) < 0, \quad v_0'(h_0) < 0$$

A classical solution is understood to have a continuous initial trace and to satisfy the differential equations and boundary conditions classically for positive time. First spatial derivatives at the boundary are assumed to exist. Second order compatibility at the initial corners is not imposed, so full parabolic regularity at those corners is not asserted. All boundary derivative arguments below are made at positive times.

The terms $q_1 u$ and $q_2 v$ represent removal proportional to population density. The Neumann condition eliminates the diffusive flux at the left endpoint. In the presence of convection it does not impose zero total flux. The resulting convective contribution is therefore retained explicitly in the mass balance.

Transformation and a priori estimates

To straighten the moving boundary, introduce

$$y = \frac{x}{h(t)}, \quad U(t,y) = u(t,h(t)y), \quad V(t,y) = v(t,h(t)y) \tag{7}$$

The chain rule transforms the first equation into

$$U_t = \frac{d_1}{h^2} U_{yy} + \frac{yh' + m_1 U}{h} U_y + U(r_1 - q_1 - b_1 U - c_1 V) \tag{8}$$

The second equation takes the corresponding form

$$V_t = \frac{d_2}{h^2} V_{yy} + \frac{yh' + m_2 V}{h} V_y + V(r_2 - q_2 - b_2 V - c_2 U) \tag{9}$$

$$U_y(t,0) = V_y(t,0) = 0, \quad U(t,1) = V(t,1) = 0 \tag{10}$$

$$h' = -\frac{\mu_1 U_y(t,1) + \mu_2 V_y(t,1)}{h} \tag{11}$$

The transformed equations hold for $0 < y < 1$, and boundary derivatives in the Stefan condition are evaluated at $y = 1$. On every finite time interval on which $h(t)$ is positive and bounded, the transformed diffusion coefficients remain bounded away from zero.

Lemma 1. For an existing classical solution, both densities are positive in the interior, the common boundary is strictly increasing, and the following bounds hold:

$$0 < u \le K_1, \quad 0 < v \le K_2, \quad h'(t) > 0$$

$$K_1 = \max\{\| u_0 \|_\infty, r_1 / b_1\}$$

$$K_2 = \max\{\| v_0 \|_\infty, r_2 / b_2\}$$

Proof. Each reaction term vanishes when its own density is zero. The maximum principle gives nonnegativity, and the strong maximum principle gives interior positivity. Constant upper solutions yield the density bounds: their spatial derivatives vanish, while removal and competition make the reaction terms no larger than the corresponding logistic reactions. The Hopf boundary lemma gives negative outward spatial derivatives at the right endpoint. The Stefan condition then implies a positive boundary speed.

To bound this speed from above, choose positive constants satisfying

$$N_1 \ge \max\{\| u_0' \|_\infty, r_1 / m_1\}$$

$$N_2 \ge \max\{\| v_0' \|_\infty, r_2 / m_2\}$$

For the first species, set $W = u - N_1(h - x)$. The equation and nonnegativity of the densities give

$$W_t - d_1 W_{xx} - m_1 u W_x \le u(r_1 - m_1 N_1) - N_1 h' \le 0$$

Initially and at the right endpoint, $W \le 0$. A positive maximum cannot occur at the left endpoint because $W_x = N_1 > 0$. Hence the maximum principle yields $u \le N_1(h - x)$. Applying the same argument to the second species and taking the one-sided derivatives at the right endpoint gives

$$0 < h'(t) \le \mu_1 N_1 + \mu_2 N_2$$

An exact balance for mass and habitat length

Define the total masses and their weights by

$$M_1(t) = \int_0^{h(t)} u(t,x)dx, \quad M_2(t) = \int_0^{h(t)} v(t,x)dx$$

$$a_1 = \frac{\mu_1}{d_1}, \quad a_2 = \frac{\mu_2}{d_2}$$

Because the densities vanish at the moving boundary, the endpoint term in the Leibniz differentiation formula is zero. The convection term contributes

$$\int_0^{h(t)} m_1 u u_x \, dx = -\frac{m_1}{2} u(t,0)^2$$

Introduce the integrated reaction terms

$$R_1(t) = \int_0^{h(t)} u(r_1 - q_1 - b_1 u - c_1 v) dx$$

$$R_2(t) = \int_0^{h(t)} v(r_2 - q_2 - b_2 v - c_2 u) dx$$

Direct integration of the two equations gives the exact identities

$$M_1^{'} = d_1 u_x(t,h(t)) - \frac{m_1}{2} u(t,0)^2 + R_1(t)$$

$$M_2^{'} = d_2 v_x(t,h(t)) - \frac{m_2}{2} v(t,0)^2 + R_2(t)$$

Lemma 2. The following weighted functional cancels the diffusive boundary derivatives:

$$\mathrm{H}(t) = h(t) + a_1 M_1(t) + a_2 M_2(t)$$

Indeed, adding the weighted mass identities to the Stefan condition yields

$$\mathrm{H}^{'} = -\frac{a_1 m_1}{2} u(t,0)^2 - \frac{a_2 m_2}{2} v(t,0)^2 + a_1 R_1 + a_2 R_2$$

When removal is at least as large as intrinsic growth, write

$$\delta_i = q_i - r_i \geq 0, \quad i \in \{1,2\}$$

The rate of decrease can then be expressed through nonnegative terms:

$$D_0(t) = \frac{a_1 m_1}{2} u(t,0)^2 + \frac{a_2 m_2}{2} v(t,0)^2$$

$$D_1(t) = a_1 \delta_1 M_1 + a_2 \delta_2 M_2$$

$$D_2(t) = \int_0^{h(t)} \left[a_1 b_1 u^2 + a_2 b_2 v^2 + (a_1 c_1 + a_2 c_2) uv \right] dx .$$

$$\mathrm{H}(t) + \int_0^t \left[D_0(s) + D_1(s) + D_2(s) \right] ds = \mathrm{H}(0)$$

This is an exact integral identity for the model. Omitting the convective loss at the left endpoint would invalidate it. Since all three dissipation terms are nonnegative, the weighted functional is nonincreasing and bounds the habitat length from above. The identity also records how initial weighted mass is divided between boundary advance and cumulative losses.

Exponential extinction and bounded expansion

Theorem. Assume that the problem in Section 2 admits a global classical solution and that removal is strictly stronger than intrinsic growth for both populations:

$$\delta_1 = q_1 - r_1 > 0, \quad \delta_2 = q_2 - r_2 > 0$$

Let $A_1 = \| u_0 \|_\infty$ and $A_2 = \| v_0 \|_\infty$. Then

$$\| u(t,\cdot) \|_\infty \leq A_1 e^{-\delta_1 t}$$

$$\| v(t,\cdot) \|_\infty \leq A_2 e^{-\delta_2 t}$$

Moreover, the boundary converges to a finite limit and satisfies

$$h_0 < h_\infty \leq H_0, \quad H_0 = h_0 + a_1 M_1(0) + a_2 M_2(0)$$

$$0 \leq h_\infty - h(t) \leq H_0 (a_1 A_1 e^{-\delta_1 t} + a_2 A_2 e^{-\delta_2 t})$$

Proof. Consider the spatially constant upper solution

$$z_1(t) = A_1 e^{-\delta_1 t}$$

Substitution into the first differential operator, with all terms moved to the left, produces the nonnegative residual

$$z_1^{'} + z_1(\delta_1 + b_1 z_1 + c_1 v) = z_1(b_1 z_1 + c_1 v) \geq 0$$

The initial and boundary inequalities are satisfied. The scalar comparison principle, with v regarded as the given nonnegative component of the solution, gives $u \le z_1$. This step does not require a cooperative comparison principle for the competitive system. Repeating the argument for the second equation proves its exponential bound.

Lemma 2 gives $h(t) \le \mathrm{H}(t) \le H_0$. By Lemma 1 the boundary is strictly increasing, so a finite limit exists. The masses satisfy

$$M_1(t) \le H_0 A_1 e^{-\delta_1 t}, \quad M_2(t) \le H_0 A_2 e^{-\delta_2 t}$$

Consequently, both masses tend to zero and $\mathrm{H}(t)$ converges to h_∞. Since the functional is nonincreasing,

$$h_\infty \le \mathrm{H}(t) = h(t) + a_1 M_1(t) + a_2 M_2(t)$$

Combining the last two inequalities proves the claimed estimate for convergence of the boundary. This completes the proof.

Corollary. For a prescribed tolerance $\varepsilon > 0$, a sufficient time after which both densities remain below that tolerance is

$$T_\varepsilon = \max\{0, \delta_1^{-1} \ln(A_1/\varepsilon), \delta_2^{-1} \ln(A_2/\varepsilon)\}$$

This is a sufficient time bound rather than an exact first crossing time. Competition, density limitation and boundary losses can make the actual densities fall below the tolerance earlier.

A constructed problem and its analytical solution

Problem. Consider the dimensionless coefficients and initial profiles below. Establish extinction, bound the final habitat length, and find a sufficient time after which both densities do not exceed 0.01 throughout the habitat. As in the theorem, assume a global classical solution exists.

$$h_0 = 2, \quad d_1 = 1, \quad d_2 = 0.5, \quad \mu_1 = 0.2, \quad \mu_2 = 0.1$$
$$m_1 = 0.3, \quad m_2 = 0.2, \quad b_1 = b_2 = 1, \quad c_1 = 0.4, \quad c_2 = 0.6$$
$$r_1 = 1, \quad q_1 = 1.4, \quad r_2 = 0.8, \quad q_2 = 1.1$$
$$u_0(x) = 0.6\cos\left(\frac{\pi x}{4}\right), \quad 0 \le x \le 2$$
$$v_0(x) = 0.4\cos\left(\frac{\pi x}{4}\right), \quad 0 \le x \le 2$$

Solution. The cosine profiles are positive before the right endpoint, have zero derivatives at the left endpoint and vanish at the right endpoint. Their right endpoint derivatives are negative:

$$u_0'(2) = -0.15\pi, \quad v_0'(2) = -0.10\pi$$

The removal margins and mass weights are

$$\delta_1 = 0.4, \quad \delta_2 = 0.3, \quad a_1 = a_2 = 0.2$$

Direct integration gives the initial masses and the habitat bound:

$$M_1(0) = \frac{2.4}{\pi}, \quad M_2(0) = \frac{1.6}{\pi}$$
$$H_0 = 2 + \frac{0.8}{\pi} \approx 2.254648$$

Thus, the limiting habitat length is greater than 2 and no greater than approximately 2.254648. Table 1 lists the calculated density upper bounds from the theorem. These are analytical envelopes evaluated at selected times, not a numerical solution of the partial differential equations.

Time	Upper bound for first density	Upper bound for second density
0	0.600000	0.400000
2	0.269597	0.219525
5	0.081201	0.089252
10	0.010989	0.019915
13	0.003310	0.008097

The sufficient time for the prescribed tolerance is

$$T_{0.01} = \max\{\ln(60)/0.4, \ln(40)/0.3\} \approx 12.296265$$

Therefore, from time 12.30 onward, both densities are at most 0.01 everywhere in the habitat. The full spatial solution has not been computed; the requested extinction and habitat estimates follow analytically from the balance identity and comparison argument.

Discussion

The common Stefan condition couples the species through boundary motion as well as through competition. Weighting each population mass by its mobility coefficient divided by its diffusion coefficient cancels the boundary derivatives. This cancellation allows the maximal habitat expansion to be bounded without solving explicitly for the boundary gradients.

The extinction condition is sufficient, not necessary. Extinction may also occur under weaker removal because of habitat size, diffusion losses or competition, but those regimes are not characterized by the estimates proved here. Nor do the present results establish an ordering between the boundaries of two different solutions when a removal coefficient is changed.

If removal equals intrinsic growth for both species, all dissipation terms in the balance identity remain nonnegative. Boundedness of the habitat therefore persists for a global solution. The exponential estimates based on strictly positive removal margins cannot, however, be extended to this limiting case by direct substitution.

In the constructed example, H_0 depends on the initial masses, diffusion coefficients and Stefan coefficients. Removal coefficients do not appear explicitly in this coarse geometric upper bound, although they determine the exponential decay rates. The upper bound need not coincide with the actual limiting habitat length.

Conclusion

A competition model with proportional removal, nonlinear convection and a common moving boundary has been formulated. Density and boundary speed estimates were established for classical solutions. An exact weighted mass identity provides an upper bound for habitat expansion. Conditional on global classical solvability, removal exceeding intrinsic growth for both species implies exponential extinction and convergence of the boundary to a finite limit. A constructed problem gives explicit bounds for the limiting habitat length and the time needed to reach a specified density tolerance.

Foydalanilgan adabiyotlar

1. Du, Y., and Lin, Z. Spreading-Vanishing Dichotomy in the Diffusive Logistic Model with a Free Boundary. SIAM Journal on Mathematical Analysis, 42(1), 377–405, 2010. https://doi.org/10.1137/090771089

2. Wang, M., and Zhao, J. Free Boundary Problems for a Lotka–Volterra Competition System. Journal of Dynamics and Differential Equations, 26, 655–672, 2014. https://doi.org/10.1007/s10884-014-9363-4

3. Wang, R.-H., Wang, L., and Wang, Z.-C. Free boundary problem of a reaction–diffusion equation with nonlinear convection term. Journal of Mathematical Analysis and Applications, 467(2), 1233–1257, 2018. https://www.sciencedirect.com/science/article/pii/S0022247X18306553

4. Pao, C. V. Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York, 1992. https://doi.org/10.1007/978-1-4615-3034-3

5. Du Y., Lin Z.G. The diffusive competition model with a free boundary: invasion of a superior or inferior competitor // Discrete Contin. Dyn. Syst. Ser. B.— 2014.— 19.— C. 3105–3132.

6. Gu H., Lin Z. G and Lou B. D. Long time behavior for solutions of Fisher-KPP equation with advection and free boundaries // J. Funct. Anal.— 2015.— 269.— C. 1714–1768.

7. Guo J.S., Wu C.H. On a free boundary problem for a two-species weak competition system // J Dyn Diff Equat.— 2012.— 24.— C. 873–895.

8. Guo J.S., Wu C.H. Dynamics for a two-species competition-diffusion model with two free boundaries // Nonlinearity.— 2015.— 28.— C. 1–27.

9. Cantrell R.S., Cosner C. Spatial ecology via reaction-diffusion equations.— England: Wiley, 2003.— P. 428.

10. Ciliberto C. Formule di maggiorazione e teoremi di esistenza per le soluzioni delle equazioni paraboliche in due variabili // Ricerche di Matem.— 1954.— 82.— C. 40–75.

11. Lockwood M.F., Hoopes M. F., Marchetti M. P. Invasion Ecology.— Oxford: Blackwell Publishing, 2013.— P. 444.

12. Takhirov J.O., Rasulov M.S. Problem with free boundary for systems of equations of reaction-diffusion type // Ukr. Math. J.— 2018.— 69.— C. 1968–1980.