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VERIFICATION OF THE GREEN'S FUNCTION REPRESENTATION OF SOLUTION TO A BOUNDARY VALUE PROBLEM WITH THE PRABHAKAR FRACTIONAL DERIVATIVE

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Abstract: *In this paper, we verify the Green's function representation of the solution to the first boundary value problem for a sub-diffusion equation involving the Prabhakar fractional derivative. The solution representation obtained in our previous work [3] is analyzed in detail, and it is proved that the solution satisfies the required regularity conditions and all the assumptions of the main theorem. In the verification process, the properties of the Prabhakar fractional derivative, Prabhakar fractional integral, generalized Mittag-Leffler function, Wright function, and Green's function are extensively utilized.*

Keywords: *Prabhakar fractional derivative, Prabhakar fractional integral, generalized Mittag-Leffler function, Wright function, Green's function.*

PRABHAKAR KASR TARTIBLI HOSILALI CHEGARAVIY MASALA YECHIMINING GRIN FUNKSIYASI ORQALI IFODASINI TEKSHIRISH

Annotatsiya: *Ushbu ishda Prabhakar kasr tartibli hosilasi qatnashgan sub-diffuziya tenglamasi uchun birinchi chegaraviy masala yechimining Grin funksiyasi orqali ifodasi tekshiriladi. [3] ishimizda olingan yechim ifodasi batafsil tahlil qilinib, uning zaruriy regulyarlik shartlari hamda asosiy teorema shartlarini qanoatlantirishi isbotlanadi. Tekshirish jarayonida Prabhakar kasr tartibli hosilasi, Prabhakar kasr tartibli integrali, umumlashgan Mittag-Leffler funksiyasi, Rayt funksiyasi va Grin funksiyasining xossalaridan keng foydalaniladi.*

Kalit so'zlar: *Prabhakar kasr tartibli hosilasi, Prabhakar kasr tartibli integrali, umumlashgan Mittag-Leffler funksiyasi, Rayt funksiyasi, Grin funksiyasi.*

ПРОВЕРКА ПРЕДСТАВЛЕНИЯ РЕШЕНИЯ ЧЕРЕЗ ФУНКЦИЮ ГРИНА ДЛЯ КРАЕВОЙ ЗАДАЧИ С ДРОБНОЙ ПРОИЗВОДНОЙ ПРАБХАКАРА

Аннотация: *В данной работе проводится проверка представления решения через функцию Грина первой краевой задачи для уравнения суб-диффузии, содержащего дробную производную Прабхакара. Представление решения, полученное в нашей предыдущей работе [3], подробно исследуется, и доказывается, что оно удовлетворяет необходимым условиям регулярности и всем условиям основной теоремы. В процессе проверки широко используются свойства дробной производной Прабхакара, дробного интеграла Прабхакара, обобщённой функции Миттага-Леффлера, функции Райта и функции Грина.*

Ключевые слова: *дробная производная Прабхакара, дробный интеграл Прабхакара, обобщённая функция Миттага-Леффлера, функция Райта, функция Грина.*

1. Introduction.

Fractional calculus provides effective mathematical tools for describing processes with memory and nonlocal effects. In particular, fractional diffusion and sub-diffusion equations have been widely studied in connection with anomalous transport and other processes involving memory effects [4, 7]. Among the generalized fractional operators, the Prabhakar fractional derivative, based on the generalized Mittag-Leffler function, provides a flexible framework for modeling such phenomena [1, 2, 8].

Boundary value problems involving Prabhakar fractional operators have recently received increasing attention. In [6], a boundary value problem for a first-order differential equation involving the Prabhakar operator was studied, and an explicit solution representation was obtained. In [12], a Cauchy problem for a sub-diffusion equation with the Prabhakar fractional derivative was considered. In our previous work [3], a first boundary value problem for a sub-diffusion equation was studied, and an explicit Green's function and a corresponding integral representation of the solution were obtained.

The aim of the present paper is to rigorously verify the solution representation obtained in [3]. In particular, we prove that the represented function satisfies the required regularity conditions, the fractional differential equation, the boundary conditions, and all assumptions of the main theorem. The verification relies on the properties of the Prabhakar fractional derivative and integral, the generalized Mittag-Leffler function, the Wright function and the Green's function.

Thus, this work completes the analytical justification of the Green's function representation established in [3].

1. Problem formulation and main result.

Let Ω be a rectangular domain: $\Omega=\{(t,x):0<t<T,0<x<a\},\ 0<a,T<\infty.$ In [3], we consider the first initial-boundary value problem for the following sub-diffusion equation:

$$ {}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x)-u_{xx}(t,x)=f(t,x). \tag{1}$$

Here $f(t,x)$ is a given function, $\alpha,\beta,\gamma,\delta$ are given real numbers such that $\alpha>0,\ 0<\beta\le 1$,

$$ {}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}y(t)=\frac{d}{dt}\,{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}y(t),\quad t>0$$

is the Prabhakar fractional derivative of Riemann-Liouville type,

$$ {}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta}y(t)=\int_0^t (t-s)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta(t-s)^{\alpha}\right]y(s)ds$$

is the Prabhakar fractional integral operator, whereas

$$ E_{\alpha,\beta}^{\gamma}[z]=\sum_{k=0}^{+\infty}\frac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)k!}$$

represents a generalized Mittag-Leffler (Prabhakar) function [8].

Definition. A regular solution of the equation (1) in the domain Ω is called a function $u(t,x)$ with the regularity ${}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u(t,x)\in C(\overline{\Omega})\ \left[t^{1-\beta}u(t,x)\in C(\overline{\Omega})\right],\ u_{xx}(t,x),\ {}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x)\in C(\Omega)$ that satisfies the equation (1) at all points $(t,x)\in\Omega.$

Problem. Find a regular solution $u(t,x)$ of the equation (1) in the domain Ω, satisfying the following boundary and initial conditions:

$$ u(t,0)=\varphi_0(t),\ \ u(t,a)=\varphi_1(t),\ \ 0<t<T, \tag{2}$$

$$ \lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u(t,x)=\tau(x),\quad 0\le x\le a, \tag{3}$$

where $\varphi_0(t),\ \varphi_1(t),\ \tau(x)$ are assumed to possess sufficient smoothness so that the conditions of the following theorem are satisfied [3]:

Theorem. Let $t^{1-\beta}\varphi_0(t),\ t^{1-\beta}\varphi_1(t)\in C[0;T],\ \tau(x)\in C[0;a],\ t^{1-\beta}f(t,x)\in C(\overline{\Omega}),\ f(t,x)$ satisfies the Hölder condition with respect to x and conditions

$$ \lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\varphi_0(t)=\tau(0),\qquad \lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\varphi_1(t)=\tau(a).$$

Then there exists a unique regular solution of the equation (1) in the domain $\Omega,$ satisfying the conditions (2) and (3), represented as

$$ u(t,x)=\int_0^t\varphi_0(\eta)G_{\xi}(t,x,\eta,0)d\eta-\int_0^t\varphi_1(\eta)G_{\xi}(t,x,\eta,a)d\eta+$$

$$ +\int_0^a\tau(\xi)G(t,x,0,\xi)d\xi+\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta. \tag{4}$$

Here

$$G(t,x,\eta,\xi)=\frac{(t-\eta)^{\beta_1-1}}{2}\sum_{n=-\infty}^{\infty}\left[E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi+2an|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)-\right.$$

$$\left.-E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x+\xi+2an|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)\right], \quad (5)$$

$E_{12}(x,y)$ is the bivariate Mittag-Leffler type function [11]:

$$E_{12}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\\ \alpha_2,\beta_2,\delta_2;\alpha_3,\delta_3;\alpha_4,\delta_4;\beta_3,\delta_5\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=$$

$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma(\alpha_1 n+\beta_1 m+\delta_1)x^n y^m}{\Gamma(\alpha_2 n+\beta_2 m+\delta_2)\Gamma(\alpha_3 n+\delta_3)\Gamma(\alpha_4 n+\delta_4)\Gamma(\beta_3 m+\delta_5)},$$

$\left(x,y,\alpha_l,\beta_i,\delta_j\in\mathbb{C};\min\{\alpha_l,\beta_i\}>0;\left(l=\{1,...,4\},i=\{1,2,3\},j=\{1,...,5\}\right)\right)$, in which the double series converges for $x,y\in\mathbb{C}$, if $\Delta_1>0$, and $\Delta_2>0$. Here $\Delta_1=\alpha_2+\alpha_3+\alpha_4-\alpha_1$, $\Delta_2=\beta_2+\beta_3-\beta_1$.

3. Verification of the solution class.

Now, we check that the solution (4) satisfies the conditions of Definition. First, we verify that ${}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u(t,x)\in C(\overline{\Omega})$, that is, we check that $t^{1-\beta}u(t,x)\in C(\overline{\Omega})$. Therefore, we multiply both sides of the solution by t:

$$t^{1-\beta}u(t,x)=t^{1-\beta}\left[\int_0^t\varphi_0(\eta)G_{\xi}(t,x,\eta,0)d\eta-\int_0^t\varphi_1(\eta)G_{\xi}(t,x,\eta,a)d\eta+\right.$$

$$\left.+\int_0^a\tau(s)G(t,x,0,\xi)d\xi+\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta\right]. \quad (6)$$

Now, we consider the situation as $t\to 0$. Let us consider the first integral:

$$\lim_{t\to 0}\left[t^{1-\beta}\int_0^t\eta^{1-\beta}\varphi_0(\eta)\eta^{\beta-1}G_{\xi}(t,x,\eta,0)d\eta\right]=$$

$$=\lim_{t\to 0}\left[t^{1-\beta}\int_0^t\eta^{1-\beta}\varphi_0(\eta)\eta^{\beta-1}\sum_{n=-\infty}^{+\infty}sign(x+2na)\omega(t-\eta,|x+2na|)d\eta\right]=$$

$$=\lim_{t\to 0}\left[t^{1-\beta}\int_0^t\eta^{1-\beta}\varphi_0(\eta)\eta^{\beta-1}\sum_{n=-\infty}^{+\infty}sign(x+2na)(t-\eta)^{-1}\times\right.$$

$$\left.\times E_{12}\left(\begin{matrix}-\gamma_1,1,0;\\ -\beta_1,\alpha,0;-\gamma_1,0;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x+2na|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)d\eta\right]=$$

$$=\lim_{t\to 0}\left[t^{1-\beta}\int_0^t\eta^{1-\beta}\varphi_0(\eta)\eta^{\beta-1}\sum_{n=-\infty}^{+\infty}sign(x+2na)\times\right.$$

$$\left.\times\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k}{k!}(t-\eta)^{-\beta_1 k-1}E_{\alpha,-\beta_1 k}^{-\gamma_1 k}\left[\delta(t-\eta)^{\alpha}\right]d\eta\right].$$

Let us make the substitution $\eta = tz$. Then, by evaluating the limit of $E_{\alpha,-\beta_1 k}^{-\gamma_1 k}\left[\delta t^{\alpha}(1-z)^{\alpha}\right]$ as $t \to 0$, the remaining expression can be represented in terms of the Wright function $e_{\alpha,\beta}^{\mu,\delta}(z)$ [10]:

$$\lim_{t\to 0}\left|\int_0^1 (tz)^{1-\beta}\varphi_0(tz)t^{1-\beta}(tz)^{\beta-1}\times\right.$$

$$\left.\times\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k}{k!}t^{-\beta_1 k}(1-z)^{-\beta_1 k-1}E_{\alpha,-\beta_1 k}^{-\gamma_1 k}\left[\delta t^{\alpha}(1-z)^{\alpha}\right]dz\right| \le$$

$$\le \left\|(tz)^{1-\beta}\varphi_0(tz)\right\|\lim_{t\to 0}\left|\int_0^1 \frac{z^{\beta-1}}{1-z}\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{1}{\Gamma(k+1)\Gamma(-\beta_1 k)}\left(-\frac{|x+2na|}{\left[t(1-z)\right]^{\beta_1}}\right)^k dz\right| =$$

$$= \left\|(tz)^{1-\beta}\varphi_0(tz)\right\|\lim_{t\to 0}\left|\int_0^1 \frac{z^{\beta-1}}{1-z}\sum_{n=-\infty}^{+\infty} sign(x+2na)e_{1,\beta_1}^{1,0}\left(-\frac{|x+2na|}{\left[t(1-z)\right]^{\beta_1}}\right)dz\right|.$$

$$\lim_{t\to 0} e_{1,\beta_1}^{1,0}\left(-\frac{|x+2na|}{\left[t(1-z)\right]^{\beta_1}}\right)=0,$$

Considering that [10] we arrive at the following result:

$$\lim_{t\to 0}\left[t^{1-\beta}\int_0^t \eta^{1-\beta}\varphi_0(\eta)\eta^{\beta-1}G_{\xi}(t,x,\eta,0)\,d\eta\right]=0. \tag{7}$$

By the same method, if we also analyze the integral involving $\varphi_1(\eta)$ at $t=0$, we obtain the following:

$$\lim_{t\to 0}\left[t^{1-\beta}\int_0^t \eta^{1-\beta}\varphi_1(\eta)\eta^{\beta-1}G_{\xi}(t,x,\eta,a)\,d\eta\right]=0. \tag{8}$$

Now we analyze the third integral of (6). Based on the structure of the Green's function, we can write the Green's function as follows

$$G(t,x,\eta,\xi)=$$

$$=\frac{(t-\eta)^{\beta_1-1}}{2}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)+K(t,x,\eta,\xi), \tag{9}$$

where

$$K(t,x,\eta,s)=\frac{(t-\eta)^{\beta_1-1}}{2}\sum_{n=-\infty}^{\infty}{}' E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi+2an|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)-$$

$$-\frac{(t-\eta)^{\beta_1-1}}{2}\sum_{n=-\infty}^{\infty} E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x+\xi+2an|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right).$$

The symbol $\sum_{n=-\infty}^{\infty}{}'$ indicates that the summation is taken over all nonzero integer values of n.

Hence, the third integral of (6) can be expressed as

$$I(t,x)=t^{1-\beta}\int_0^a \tau(\xi)G(t,x,0,\xi)\,d\xi=$$

$$=\frac{1}{2}\int_0^a \tau(\xi)t^{-\beta_1}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi|t^{-\beta_1}\\ \delta t^{\alpha}\end{matrix}\right)d\xi+\int_0^a \tau(\xi)t^{1-\beta}K(t,x,0,\xi)d\xi.$$

Now, we examine the first integral of $I(t,x)$ as $t\to 0.$ For this purpose, we first remove the modulus:

$$\frac{1}{2}\lim_{t\to 0}\int_0^a \tau(\xi)t^{-\beta_1}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi|t^{-\beta_1}\\ \delta t^{\alpha}\end{matrix}\right)d\xi=$$

$$=\frac{1}{2}\lim_{t\to 0}\int_0^x \tau(\xi)t^{-\beta_1}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-(x-\xi)t^{-\beta_1}\\ \delta t^{\alpha}\end{matrix}\right)d\xi+$$

$$+\frac{1}{2}\lim_{t\to 0}\int_x^a \tau(\xi)t^{-\beta_1}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-(\xi-x)t^{-\beta_1}\\ \delta t^{\alpha}\end{matrix}\right)d\xi=$$

$$=\frac{1}{2}\lim_{t\to 0}\left[t^{-\beta_1}\sum_{k=0}^{+\infty}\frac{1}{k!}E^{-\gamma_1k+\gamma_1}_{\alpha,-\beta_1k+\beta_1}[\delta t^{\alpha}]\int_0^x \tau(\xi)\left(-\frac{x-\xi}{t^{\beta_1}}\right)^k d\xi\right]+$$

$$+\frac{1}{2}\lim_{t\to 0}\left[t^{-\beta_1}\sum_{k=0}^{+\infty}\frac{1}{k!}E^{-\gamma_1k+\gamma_1}_{\alpha,-\beta_1k+\beta_1}[\delta t^{\alpha}]\int_x^a \tau(\xi)\left(-\frac{\xi-x}{t^{\beta_1}}\right)^k d\xi\right].$$

By substituting $z_1=-\frac{x-\xi}{t^{\beta_1}}$ in the first integral and $z_2=-\frac{\xi-x}{t^{\beta_1}}$ in the second integral of the last equality, then we calculate the limit of both integrals as t tends to 0:

$$\frac{1}{2}\lim_{t\to 0}\left[\sum_{k=0}^{+\infty}\frac{1}{k!}E^{-\gamma_1k+\gamma_1}_{\alpha,-\beta_1k+\beta_1}[\delta t^{\alpha}]\int_{-\frac{x}{t^{\beta_1}}}^0 \tau\left(x+t^{\beta_1}z_1\right)z_1{}^k dz_1\right]+$$

$$+\frac{1}{2}\lim_{t\to 0}\left[\sum_{k=0}^{+\infty}\frac{1}{k!}E^{-\gamma_1k+\gamma_1}_{\alpha,-\beta_1k+\beta_1}[\delta t^{\alpha}]\int_{\frac{x-a}{t^{\beta_1}}}^0 \tau\left(x-t^{\beta_1}z_2\right)z_2^k dz_2\right]=$$

$$=\frac{1}{2}\lim_{t\to 0}\left[\sum_{k=0}^{+\infty}\frac{\tau(x)}{k!}E^{-\gamma_1k+\gamma_1}_{\alpha,-\beta_1k+\beta_1}[\delta t^{\alpha}]\int_{-\infty}^0 z_1{}^k dz_1\right]+$$

$$+\frac{1}{2}\lim_{t\to 0}\left[\sum_{k=0}^{+\infty}\frac{\tau(x)}{k!}E^{-\gamma_1k+\gamma_1}_{\alpha,-\beta_1k+\beta_1}[\delta t^{\alpha}]\int_{-\infty}^0 z_2^k dz_2\right].$$

Now we use the Wright function and its property

$$\lim_{|z|\to\infty} ze^{\mu,\delta}_{\alpha,\beta}(z)=-\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)} \tag{10}$$

as

$$\frac{\tau(x)}{2}\left[\sum_{k=0}^{+\infty}\frac{z_1{}^{k+1}}{(k+1)!\Gamma(-\beta_1k+\beta_1)}\Bigg|_{-\infty}^0+\sum_{k=0}^{+\infty}\frac{z_2{}^{k+1}}{(k+1)!\Gamma(-\beta_1k+\beta_1)}\Bigg|_{-\infty}^0\right]=$$

$$=\frac{\tau(x)}{2}\left[z_1e^{2,\beta_1}_{1,\beta_1}(z_1)\Big|_{-\infty}^0+z_2e^{2,\beta_1}_{1,\beta_1}(z_2)\Big|_{-\infty}^0\right]=\frac{\tau(x)}{\Gamma(\beta)}.$$

Next, we consider the following limit:

$$\lim_{t\to 0}\left[t^{1-\beta}K(t,x,0,\xi)\right]=$$

$$= \lim_{t\to 0}\left[\frac{t^{-\beta_1}}{2} {\sum_{n=-\infty}^{\infty}}' E_{12}\left(\begin{matrix} -\gamma_1,1,\gamma_1; \\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1 \end{matrix} \middle| \begin{matrix} -\left|x-\xi+2an\right|t^{-\beta_1} \\ \delta t^{\alpha} \end{matrix} \right) - \right.$$

$$\left. - \frac{t^{-\beta_1}}{2} \sum_{n=-\infty}^{\infty} E_{12}\left(\begin{matrix} -\gamma_1,1,\gamma_1; \\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1 \end{matrix} \middle| \begin{matrix} -\left|x+\xi+2an\right|t^{-\beta_1} \\ \delta t^{\alpha} \end{matrix} \right) \right] =$$

$$= \lim_{t\to 0}\left[\frac{t^{-\beta_1}}{2} {\sum_{n=-\infty}^{+\infty}}' {\sum_{k=0}^{+\infty}}' \frac{(-1)^k \left|x-\xi+2na\right|^k}{k!} t^{-\beta_1 k} E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta t^{\alpha}\right] - \right.$$

$$\left. - \frac{t^{-\beta_1}}{2} \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k \left|x+\xi+2na\right|^k}{k!} t^{-\beta_1 k} E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta t^{\alpha}\right] \right] =$$

$$= \lim_{t\to 0}\left[\frac{t^{-\beta_1}}{2} {\sum_{n=-\infty}^{+\infty}}' {\sum_{k=0}^{+\infty}}' \frac{(-1)^k \left|x-\xi+2na\right|^k t^{-\beta_1 k}}{k!} - \frac{t^{-\beta_1}}{2} \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k \left|x+\xi+2na\right|^k t^{-\beta_1 k}}{k!} \right] \times$$

$$\times \lim_{t\to 0}\left[\frac{1}{\Gamma(\beta_1-\beta_1 k)} + \frac{(\gamma_1-\gamma_1 k)_1}{\Gamma(\alpha+\beta_1-\beta_1 k)} \delta t^{\alpha} + \frac{(\gamma_1-\gamma_1 k)_2}{2\Gamma(2\alpha+\beta_1-\beta_1 k)} \delta^2 t^{2\alpha} + \ldots \right].$$

By evaluating the last limit in the final equality, the remaining expression can be represented in terms of the Wright function:

$$\lim_{t\to 0}\left[t^{1-\beta} K(t,x,0,\xi) \right] = \lim_{t\to 0}\left[\frac{t^{-\beta_1}}{2} {\sum_{n=-\infty}^{+\infty}}' {\sum_{k=0}^{+\infty}}' \frac{1}{\Gamma(k+1)\Gamma(\beta_1-\beta_1 k)} \left(-\frac{\left|x-\xi+2na\right|}{t^{\beta_1}} \right)^k - \right.$$

$$\left. - \frac{t^{-\beta_1}}{2} \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{\Gamma(k+1)\Gamma(\beta_1-\beta_1 k)} \left(-\frac{\left|x+\xi+2na\right|}{t^{\beta_1}} \right)^k \right] =$$

$$= \frac{1}{2} \lim_{t\to 0}\left[{\sum_{n=-\infty}^{+\infty}}' \left(t^{-\beta_1} e_{1,\beta_1}^{1,\beta_1}\left(-\frac{\left|x-\xi+2na\right|}{t^{\beta_1}} \right) \right) - \sum_{n=-\infty}^{+\infty} \left(t^{-\beta_1} e_{1,\beta_1}^{1,\beta_1}\left(-\frac{\left|x+\xi+2na\right|}{t^{\beta_1}} \right) \right) \right].$$

We express the last equality in the following form and apply (10):

$$\frac{1}{2} \lim_{t\to 0}\left[{\sum_{n=-\infty}^{+\infty}}' \left(-\frac{1}{\left|x-\xi+2na\right|} \right) \left(-\frac{\left|x-\xi+2na\right|}{t^{\beta_1}} \right) e_{1,\beta_1}^{1,\beta_1}\left(-\frac{\left|x-\xi+2na\right|}{t^{\beta_1}} \right) - \right.$$

$$\left. - \sum_{n=-\infty}^{+\infty} \left(-\frac{1}{\left|x+\xi+2na\right|} \right) \left(-\frac{\left|x+\xi+2na\right|}{t^{\beta_1}} \right) e_{1,\beta_1}^{1,\beta_1}\left(-\frac{\left|x+\xi+2na\right|}{t^{\beta_1}} \right) \right] = 0.$$

Thus, $\lim_{t\to 0}\left[t^{1-\beta} K(t,x,0,\xi) \right] = 0.$

We conclude that

$$\lim_{t\to 0} I(t,x) = \frac{\tau(x)}{\Gamma(\beta)}. \tag{11}$$

Next, we analyze the final integral of (6) as $t \to 0$:

$$\lim_{t\to 0}\left[t^{1-\beta} \int_0^t \int_0^a \eta^{1-\beta} f(\eta,\xi) \eta^{\beta-1} G(t,x,\eta,\xi) d\xi d\eta \right] =$$

$$= \frac{1}{2}\lim_{t\to 0}\Bigg[t^{1-\beta}\int_0^t\int_0^a \eta^{1-\beta} f(\eta,\xi)\eta^{\beta-1}(t-\eta)^{\beta_1-1}\times$$

$$\times\sum_{n=-\infty}^{\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[|x-\xi+2na|^k-|x+\xi+2na|^k\right]}{k!}(t-\eta)^{-\beta_1 k}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]d\xi d\eta\Bigg].$$

We perform the substitution $\eta = tz.$ Following the same procedure as above, we find the limit of the Prabhakar function and write the remaining expression through the Wright function:

$$\frac{1}{2}\lim_{t\to 0}\Bigg| t^{\beta_1}\int_0^1\int_0^a (tz)^{1-\beta} f(tz,\xi)z^{\beta-1}(1-z)^{\beta_1-1}\times$$

$$\times\sum_{n=-\infty}^{\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[|x-\xi+2na|^k-|x+\xi+2na|^k\right]}{k!}t^{-\beta_1 k}(1-z)^{-\beta_1 k}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta t^\alpha(1-z)^\alpha\right]d\xi dz\Bigg| \le$$

$$\le \frac{1}{2}\left\|(tz)^{1-\beta} f(tz,\xi)\right\|\lim_{t\to 0}\Bigg| t^{\beta_1}\int_0^1\int_0^a z^{\beta-1}(1-z)^{\beta_1-1}\times$$

$$\times\sum_{n=-\infty}^{\infty}\left(e^{1,\beta_1}_{1,\beta_1}\left(-\frac{|x-\xi+2na|}{[t(1-z)]^{\beta_1}}\right)-e^{1,\beta_1}_{1,\beta_1}\left(-\frac{|x+\xi+2na|}{[t(1-z)]^{\beta_1}}\right)\right)d\xi dz\Bigg|.$$

Since $\lim_{t\to 0} e^{1,\beta_1}_{1,\beta_1}\left(-\frac{|x\pm\xi+2na|}{[t(1-z)]^{\beta_1}}\right)=0$ and the exponent of t is positive, it follows that this limit equals zero:

$$\lim_{t\to 0}\left[t^{1-\beta}\int_0^t\int_0^a \eta^{1-\beta} f(\eta,s)\eta^{\beta-1}G(t,x,\eta,s)dsd\eta\right]=0. \tag{12}$$

According to $t^{1-\beta}\varphi_0(t),\ t^{1-\beta}\varphi_1(t)\in C[0;T],\ \tau(x)\in C[0;a],\ t^{1-\beta} f(t,x)\in C(\overline{\Omega})$ and the equalities (7), (8), (11), (12), the expression (6) is continuous at the point $t=0.$

Now, we examine the case at $\eta\to t$. Let us begin by examining the term in (6) that involves the function φ_0:

$$\lim_{\eta\to t}\left[\varphi_0(\eta)G_\xi(t,x,\eta,0)\right]=$$

$$=\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k}{k!}(t-\eta)^{-\beta_1 k-1}E^{-\gamma_1 k}_{\alpha,-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]\right]$$

$$=\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k}{k!}(t-\eta)^{-\beta_1 k-1}\right]\times$$

$$\times\lim_{\eta\to t}\left[\frac{1}{\Gamma(-\beta_1 k)}+\frac{(-\gamma_1 k)_1}{\Gamma(\alpha-\beta_1 k)}\delta(t-\eta)^\alpha+\frac{(-\gamma_1 k)_2}{2\Gamma(2\alpha-\beta_1 k)}\delta^2(t-\eta)^{2\alpha}+\ldots\right]=$$

$$=\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k}{k!\Gamma(-\beta_1 k)}(t-\eta)^{-\beta_1 k-1}\right].$$

We rewrite the final expression using the Wright function and turn to the following lemma [10]:

Lemma. If $\delta_1 < 1$ and for arbitrary positive x and y, $x_0 \in (0;x)$, $\beta_0 \in [\beta;1]$, $\theta \in \left(\frac{1}{2}, \min\left\{1, \frac{1}{2\beta}\right\}\right)$, the following inequality holds:

$$\left| y^{\delta_1 - 1} e^{1,\delta_1}_{1,\beta}\left(-\frac{x}{y^\beta}\right)\right| \le \frac{1}{\beta\pi}\left(\frac{\cos\beta\theta\pi}{\beta\pi}\right)^{\frac{\delta_1-1}{\beta}} \Gamma\left(\frac{1-\delta_1}{\beta}\right)(x-x_0)^{\frac{\delta_1-1}{\beta}} e^{1,1}_{1,\beta}\left(-\frac{x_0}{y^\beta}\right).$$

According to Lemma, we find that its value is equal to zero:

$$\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)(t-\eta)^{-1} e^{1,0}_{1,\beta_1}\left(-\frac{|x+2na|}{(t-\eta)^{\beta_1}}\right)\right] = 0.$$

Hence,

$$\lim_{\eta\to t}\left[\varphi_0(\eta) G_\xi(t,x,\eta,0)\right] = 0. \tag{13}$$

By the same method,

$$\lim_{\eta\to t}\left[\varphi_1(\eta) G_\xi(t,x,\eta,a)\right] = 0. \tag{14}$$

Next, we examine the final integrand of (6) the case $\eta \to t$:

$$\lim_{\eta\to t}\left[f(\eta,\xi) G(t,x,\eta,\xi)\right] = \lim_{\eta\to t}\Bigg[f(\eta,\xi)(t-\eta)^{\beta_1-1} \times$$

$$\times \sum_{n=-\infty}^{\infty}\sum_{k=0}^{+\infty} \frac{(-1)^k\left[|x-\xi+2na|^k - |x+\xi+2na|^k\right]}{k!}(t-\eta)^{-\beta_1 k} E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]\Bigg].$$

Using the equality $\lim_{\eta\to t} E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right] = \frac{1}{\Gamma(\beta_1-\beta_1 k)}$, the sum over k can be represented again in terms of the Wright function. By Lemma, it is not difficult to verify that this limit is equal to zero:

$$\frac{1}{2}\lim_{\eta\to t}\left[f(\eta,\xi)(t-\eta)^{\beta_1-1}\sum_{n=-\infty}^{\infty}\left[e^{1,\beta_1}_{1,\beta_1}\left(-\frac{|x-\xi+2na|}{(t-\eta)^{\beta_1}}\right) - e^{1,\beta_1}_{1,\beta_1}\left(-\frac{|x+\xi+2na|}{(t-\eta)^{\beta_1}}\right)\right]\right] = 0.$$

Thus,

$$\lim_{\eta\to t}\left[f(\eta,\xi) G(t,x,\eta,\xi)\right] = 0. \tag{15}$$

From the equalities (13), (14), and (15), it follows that expression (6) is also continuous at the point $\eta = t$.

Now we verify that $u_{xx}(t,x) \in C(\Omega)$. To do this, we first compute the second derivative of the solution with respect to x:

$$u_{xx}(t,x) = \int_0^t \varphi_0(\eta)\frac{\partial^2}{\partial x^2} G_\xi(t,x,\eta,0)\,d\eta - \int_0^t \varphi_1(\eta)\frac{\partial^2}{\partial x^2} G_\xi(t,x,\eta,a)\,d\eta +$$

$$+\int_0^a \tau(\xi)\frac{\partial^2}{\partial x^2} G(t,x,0,\xi)\,d\xi + \int_0^t\int_0^a f(\eta,\xi)\frac{\partial^2}{\partial x^2} G(t,x,\eta,\xi)\,d\xi d\eta. \tag{16}$$

We first compute $\frac{\partial^2}{\partial x^2} G_\xi(t,x,\eta,0)$ and $\frac{\partial^2}{\partial x^2} G(t,x,\eta,\xi)$:

$$1)\ \frac{\partial^2}{\partial x^2} G_\xi(t,x,\eta,0) = \frac{\partial^2}{\partial x^2}\sum_{n=-\infty}^{+\infty} sign(x+2na)\omega(t-\eta,|x+2na|) =$$

$$= \frac{\partial^2}{\partial x^2}\left[-\sum_{n=-\infty}^{-1} \omega(t-\eta,-x-2na) + \sum_{n=0}^{+\infty} \omega(t-\eta,x+2na) \right] =$$

$$= \frac{\partial^2}{\partial x^2} \sum_{n=-\infty}^{-1} \left(-(t-\eta)^{-1} E_{\alpha,0}^{0}\left[\delta(t-\eta)^{\alpha} \right] - \sum_{k=1}^{+\infty} \frac{(x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-1} E_{\alpha,-\beta_1 k}^{-\gamma_1 k}\left[\delta(t-\eta)^{\alpha} \right] \right) +$$

$$+ \frac{\partial^2}{\partial x^2} \sum_{n=0}^{+\infty} \left((t-\eta)^{-1} E_{\alpha,0}^{0}\left[\delta(t-\eta)^{\alpha} \right] + \sum_{k=1}^{+\infty} \frac{(-1)^k (x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-1} E_{\alpha,-\beta_1 k}^{-\gamma_1 k}\left[\delta(t-\eta)^{\alpha} \right] \right) =$$

$$= \frac{\partial}{\partial x} \sum_{n=-\infty}^{-1} \left(-\sum_{k=0}^{+\infty} \frac{(x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-\beta_1-1} E_{\alpha,-\beta_1 k-\beta_1}^{-\gamma_1 k-\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] \right) +$$

$$+ \frac{\partial}{\partial x} \sum_{n=0}^{+\infty} \left(-\sum_{k=0}^{+\infty} \frac{(-1)^k (x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-\beta_1-1} E_{\alpha,-\beta_1 k-\beta_1}^{-\gamma_1 k-\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] \right) =$$

$$= \frac{\partial}{\partial x} \sum_{n=-\infty}^{-1} \left(-(t-\eta)^{-\beta_1-1} E_{\alpha,-\beta_1}^{-\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] - \right.$$

$$\left. -\sum_{k=1}^{+\infty} \frac{(x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-\beta_1-1} E_{\alpha,-\beta_1 k-\beta_1}^{-\gamma_1 k-\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] \right) +$$

$$+ \frac{\partial}{\partial x} \sum_{n=0}^{+\infty} \left(-(t-\eta)^{-\beta_1-1} E_{\alpha,-\beta_1}^{-\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] - \right.$$

$$\left. -\sum_{k=1}^{+\infty} \frac{(-1)^k (x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-\beta_1-1} E_{\alpha,-\beta_1 k-\beta_1}^{-\gamma_1 k-\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] \right) =$$

$$= -\sum_{n=-\infty}^{-1} \sum_{k=0}^{+\infty} \frac{(x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-2\beta_1-1} E_{\alpha,-\beta_1 k-2\beta_1}^{-\gamma_1 k-2\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] +$$

$$+ \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k (x+2na)^k}{k!} (t-\eta)^{-\beta_1 k-2\beta_1-1} E_{\alpha,-\beta_1 k-2\beta_1}^{-\gamma_1 k-2\gamma_1}\left[\delta(t-\eta)^{\alpha} \right] =$$

$$= \sum_{n=-\infty}^{+\infty} sign(x+2na) \sum_{k=0}^{+\infty} \frac{(-1)^k \left| x+2na \right|^k (t-\eta)^{-\beta_1 k-\beta-1}}{k!} E_{\alpha,-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^{\alpha} \right];$$

$$2) \frac{\partial^2}{\partial x^2} G(t,x,\eta,\xi) =$$

$$= \frac{\partial^2}{\partial x^2} \left(\frac{(t-\eta)^{\beta_1-1}}{2} \sum_{n=-\infty}^{\infty} \left[E_{12}\left(\begin{matrix} -\gamma_1,1,\gamma_1; \\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1 \end{matrix} \middle| \begin{matrix} -\left| x-\xi+2an \right|(t-\eta)^{-\beta_1} \\ \delta(t-\eta)^{\alpha} \end{matrix} \right) - \right. \right.$$

$$\left. \left. -E_{12}\left(\begin{matrix} -\gamma_1,1,\gamma_1; \\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1 \end{matrix} \middle| \begin{matrix} -\left| x+\xi+2an \right|(t-\eta)^{-\beta_1} \\ \delta(t-\eta)^{\alpha} \end{matrix} \right) \right] \right) =$$

$$= \frac{\partial^2}{\partial x^2} \left(\sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k \left[\left| x-\xi+2na \right|^k - \left| x+\xi+2na \right|^k \right]}{2\Gamma(k+1)} (t-\eta)^{-\beta_1 k+\beta_1-1} E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta(t-\eta)^{\alpha} \right] \right) =$$

$$= \frac{\partial^2}{\partial x^2} \left(\sum_{n=-\infty}^{-1} \sum_{k=0}^{+\infty} \frac{\left[(x-\xi+2na)^k - (x+\xi+2na)^k \right]}{2\Gamma(k+1)} (t-\eta)^{-\beta_1 k+\beta_1-1} E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta(t-\eta)^{\alpha} \right] \right) +$$

$$+\frac{\partial^2}{\partial x^2}\left(\sum_{k=0}^{+\infty}\frac{(-1)^k(x-\xi)^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^{\alpha}\right]+\right.$$

$$+\sum_{k=0}^{+\infty}\frac{(-1)^k(\xi-x)^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^{\alpha}\right]-$$

$$\left.-\sum_{k=0}^{+\infty}\frac{(-1)^k(x+\xi)^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^{\alpha}\right]\right)+$$

$$+\frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[(x-\xi+2na)^k-(x+\xi+2na)^k\right]}{2\Gamma(k+1)}(t-\eta)^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(t-\eta)^{\alpha}\right]\right)=$$

$$=\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[|x-\xi+2na|^k-|x+\xi+2na|^k\right]}{2\Gamma(k+1)}(t-\eta)^{-\beta_1 k-\beta_1-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta(t-\eta)^{\alpha}\right].$$

Next, we evaluate the limits of $\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,0)$, $\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,a)$ and $\frac{\partial^2}{\partial x^2}G(t,x,\eta,\xi)$ as $\eta\to t$:

$$\lim_{\eta\to t}\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,0)=$$

$$=\lim_{\eta\to t}\sum_{n=-\infty}^{+\infty}sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k(t-\eta)^{-\beta_1 k-\beta-1}}{k!}E^{-\gamma-\gamma_1 k}_{\alpha,-\beta_1 k-\beta}\left[\delta(t-\eta)^{\alpha}\right].$$

Considering that $\lim_{\eta\to t}E^{-\gamma-\gamma_1 k}_{\alpha,-\beta_1 k-\beta}\left[\delta(t-\eta)^{\alpha}\right]=\frac{1}{\Gamma(-\beta_1 k-\beta)}$, the remaining expression can be written in terms of the Wright function:

$$\lim_{\eta\to t}\sum_{n=-\infty}^{+\infty}sign(x+2na)t^{-\beta-1}e^{1,-\beta}_{1,\beta_1}\left(-\frac{|x+2na|}{(t-\eta)^{\beta_1}}\right).$$

It is easy to see that this also equals zero according to the Lemma:

$$\lim_{\eta\to t}\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,0)=0.$$

As above, it follows that $\lim_{\eta\to t}\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,a)=0$ and $\lim_{\eta\to t}\frac{\partial^2}{\partial x^2}G(t,x,\eta,\xi)=0.$ Hence, $\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,0)$, $\frac{\partial^2}{\partial x^2}G_{\xi}(t,x,\eta,a)$ and $\frac{\partial^2}{\partial x^2}G(t,x,\eta,\xi)$ are continuous $\eta\to t.$ There are no singularities at the remaining points. Therefore, $u_{xx}(t,x)\in C(\Omega).$

Let us confirm that ${}^{PRL}D^{\alpha,\beta,\gamma,\delta}_{0t}u(t,x)\in C(\Omega).$ For this, we apply the Prabhakar fractional derivative to both sides of the solution $u(t,x)$ by t:

$${}^{PRL}D^{\alpha,\beta,\gamma,\delta}_{0t}u(t,x)={}^{PRL}D^{\alpha,\beta,\gamma,\delta}_{0t}\left[\int_0^t\varphi_0(\eta)G_{\xi}(t,x,\eta,0)d\eta-\int_0^t\varphi_1(\eta)G_{\xi}(t,x,\eta,a)d\eta+\right.$$

$$\left.+\int_0^a\tau(\xi)G(t,x,0,\xi)d\xi+\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta\right]. \tag{17}$$

Let us begin by calculating the term involving the function $\varphi_0(\eta)$:

$$ {}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}\left(\int_0^t \varphi_0(\eta)G_\xi(t,x,\eta,0)d\eta\right)=\frac{\partial}{\partial t}\,{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\left(\int_0^t \varphi_0(\eta)G_\xi(t,x,\eta,0)d\eta\right)= $$

$$ =\frac{\partial}{\partial t}\int_0^t (t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]dy\int_0^y \varphi_0(\eta)G_\xi(y,x,\eta,0)d\eta= $$

$$ =\frac{\partial}{\partial t}\int_0^t \varphi_0(\eta)d\eta\int_\eta^t (t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]G_\xi(y,x,\eta,0)dy. $$

Now we compute the integral with respect to y separately:

$$ \int_\eta^t (t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]G_\xi(y,x,\eta,0)dy=\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k}{k!}\times $$

$$ \times\int_\eta^t (t-y)^{-\beta}(y-\eta)^{-\beta_1 k-1}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]E_{\alpha,-\beta_1 k}^{-\gamma_1 k}\left[\delta(y-\eta)^{\alpha}\right]dy. $$

Next, we use [5]

$$ \sum_{k=0}^{+\infty}a_k\sum_{m=0}^{+\infty}b_m=\sum_{k=0}^{+\infty}\sum_{m=0}^{k}a_m b_{k-m} \tag{18} $$

for the generalized Mittag-Leffler functions and perform the substitution $y=(t-\eta)z+\eta$:

$$ \sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k}{k!}\sum_{i=0}^{+\infty}\sum_{j=0}^{i}\frac{\delta^i(-\gamma)_j(-\gamma_1 k)_{i-j}}{j!(i-j)!\Gamma(\alpha j+1-\beta)\Gamma(\alpha i-\alpha j-\beta_1 k)}\times $$

$$ \times\int_\eta^t (t-y)^{\alpha j-\beta}(y-\eta)^{\alpha i-\alpha j-\beta_1 k-1}dy= $$

$$ =\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k}{k!}\sum_{i=0}^{+\infty}\frac{\delta^i(t-\eta)^{\alpha i-\beta_1 k-\beta}}{\Gamma(\alpha i-\beta_1 k-\beta+1)}\sum_{j=0}^{i}\frac{(-\gamma)_j(-\gamma_1 k)_{i-j}}{j!(i-j)!}. $$

In accordance with [9]

$$ \sum_{m=0}^{k}\frac{(\delta)_m(\gamma)_{k-m}}{m!(k-m)!}=\frac{(\delta+\gamma)_k}{k!} \tag{19} $$

we obtain

$$ \sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k(t-\eta)^{-\beta_1 k-\beta}}{k!}\sum_{i=0}^{+\infty}\frac{(-\gamma-\gamma_1 k)_i\delta^i(t-\eta)^{\alpha i}}{i!\Gamma(\alpha i-\beta_1 k-\beta+1)}= $$

$$ =\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k(t-\eta)^{-\beta_1 k-\beta}}{k!}E_{\alpha,1-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^{\alpha}\right]. $$

Next, we insert the derived result into the preceding equality and proceed with the computation:

$$ \frac{\partial}{\partial t}\int_0^t \varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k(t-\eta)^{-\beta_1 k-\beta}}{k!}E_{\alpha,1-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^{\alpha}\right]d\eta= $$

$$ =\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k(t-\eta)^{-\beta_1 k-\beta}}{k!}E_{\alpha,1-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^{\alpha}\right]\right]+ $$

$$ +\int_0^t \varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+2na\right|^k(t-\eta)^{-\beta_1 k-\beta-1}}{k!}E_{\alpha,-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^{\alpha}\right]d\eta. $$

Now let us carefully evaluate the limit that arises from taking the derivative:

$$\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k(t-\eta)^{-\beta_1 k-\beta}}{k!}\right]\lim_{\eta\to t}E_{\alpha,1-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^\alpha\right]=$$

$$=\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)(t-\eta)^{-\beta}\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k(t-\eta)^{-\beta_1 k}}{k!\Gamma(1-\beta_1 k-\beta)}\right].$$

We rewrite the final expression using the Wright-type function and make use of its property [10] $\lim_{|z|\to\infty} z^2 e_{\alpha,\beta}^{\alpha-k,\delta}(z)=\frac{1}{\Gamma(-k-\alpha)\Gamma(\delta+2\beta)}$ as

$$\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)(t-\eta)^{-\beta}e_{1,\beta_1}^{1,1-\beta}\left(-\frac{|x+2na|}{(t-\eta)^{\beta_1}}\right)\right]=$$

$$=\lim_{\eta\to t}\left[\varphi_0(\eta)\sum_{n=-\infty}^{+\infty}\frac{sign(x+2na)}{(x+2na)^2}\left(-\frac{|x+2na|}{(t-\eta)^{\beta_1}}\right)^2 e_{1,\beta_1}^{1,1-\beta}\left(-\frac{|x+2na|}{(t-\eta)^{\beta_1}}\right)\right]=0.$$

Therefore,

$${}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}\left(\int_0^t\varphi_0(\eta)G_\xi(t,x,\eta,0)d\eta\right)=\int_0^t\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+2na)\times$$

$$\times\sum_{k=0}^{+\infty}\frac{(-1)^k|x+2na|^k(t-\eta)^{-\beta_1 k-\beta-1}}{k!}E_{\alpha,-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^\alpha\right]d\eta. \tag{20}$$

Similarly, for the term involving $\varphi_1(\eta)$ in (17), if we repeat the same calculations as above, we obtain the following:

$${}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}\left(\int_0^t\varphi_1(\eta)G_\xi(t,x,\eta,a)d\eta\right)=\int_0^t\varphi_0(\eta)\sum_{n=-\infty}^{+\infty} sign(x+(2n+1)a)\times$$

$$\times\sum_{k=0}^{+\infty}\frac{(-1)^k|x+(2n+1)a|^k(t-\eta)^{-\beta_1 k-\beta-1}}{k!}E_{\alpha,-\beta_1 k-\beta}^{-\gamma-\gamma_1 k}\left[\delta(t-\eta)^\alpha\right]d\eta. \tag{21}$$

Next, we proceed to the term involving $\tau(\xi)$:

$${}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}\left(\int_0^a\tau(\xi)G(t,x,0,\xi)d\xi\right)=\frac{\partial}{\partial t}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\left(\int_0^a\tau(\xi)G(t,x,0,\xi)d\xi\right)=$$

$$=\frac{\partial}{\partial t}\int_0^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]dy\int_0^a\tau(\xi)G(y,x,0,\xi)d\xi=$$

$$=\frac{\partial}{\partial t}\int_0^a\tau(\xi)d\xi\int_0^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]G(y,x,0,\xi)dy. \tag{22}$$

Now let us evaluate the following integral:

$$\int_0^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]G(y,x,0,\xi)dy=\int_0^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]\times$$

$$\times\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[|x-\xi+2na|^k-|x+\xi+2na|^k\right]}{2\Gamma(k+1)}y^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta y^\alpha\right]dy=$$

$$=\int_0^t(t-y)^{-\beta}\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[|x-\xi+2na|^k-|x+\xi+2na|^k\right]}{2\Gamma(k+1)}y^{-\beta_1 k+\beta_1-1}\times$$

$$\times\sum_{j=0}^{+\infty}\frac{(-\gamma)_j\delta^j(t-y)^{\alpha j}}{j!\Gamma(\alpha j+1-\beta)}\sum_{i=0}^{+\infty}\frac{(\gamma_1-\gamma_1 k)_i\delta^i y^{\alpha i}}{i!\Gamma(\alpha i+\beta_1-\beta_1 k)}dy.$$

We apply (18) to the sums with respect to i and j and perform the substitution $y=tz$:

$$\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}\times$$

$$\times\sum_{j=0}^{+\infty}\sum_{i=0}^{j}\frac{(-\gamma)_i(\gamma_1-\gamma_1 k)_{j-i}\delta^j}{i!(j-i)!\Gamma(\alpha i+1-\beta)\Gamma(\alpha j-\alpha i+\beta_1-\beta_1 k)}\int_0^t(t-y)^{\alpha i-\beta}y^{\alpha j-\alpha i-\beta_1 k+\beta_1-1}dy=$$

$$=\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\sum_{i=0}^{j}\frac{(-\gamma)_i(\gamma_1-\gamma_1 k)_{j-i}\delta^j t^{\alpha j-\beta_1-\beta_1 k}}{i!(j-i)!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}.$$

Using (19), we obtain

$$\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j t^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}.$$

According to (21), we take the derivative of the last equality with respect to t and express the obtained result in terms of the generalized Mittag-Leffler function:

$$\frac{\partial}{\partial t}\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j t^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}=$$

$$=\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j t^{\alpha j-\beta_1-\beta_1 k-1}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k)}=$$

$$=\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta t^\alpha\right].$$

Thus, we determine that

$${}^{PRL}D^{\alpha,\beta,\gamma,\delta}_{0t}\left(\int_0^a\tau(\xi)G(t,x,0,\xi)d\xi\right)=$$

$$=\int_0^a\tau(\xi)\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta t^\alpha\right]d\xi. \quad (23)$$

Finally, we compute the last term of equation (17):

$${}^{PRL}D^{\alpha,\beta,\gamma,\delta}_{0t}\left(\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta\right)=$$

$$=\frac{\partial}{\partial t}{}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\left(\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta\right)=$$

$$=\frac{\partial}{\partial t}\int_0^t(t-y)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]dy\int_0^a d\xi\int_0^y f(\eta,\xi)G(y,x,\eta,\xi)d\eta=$$

$$=\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)d\eta\int_\eta^t(t-y)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]G(y,x,\eta,\xi)dy.$$

Here we replace the function $G(y,x,\eta,\xi)$ by its representation given in (9):

$$\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)\Bigg[\int_\eta^t (t-y)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]\times$$

$$\times\frac{(y-\eta)^{\beta_1-1}}{2}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi|(y-\eta)^{-\beta_1}\\ \delta(y-\eta)^\alpha\end{matrix}\right)dy+$$

$$+\int_\eta^t (t-y)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]K(y,x,\eta,\xi)dy\Bigg]d\eta. \tag{24}$$

Now we compute the first part of (24):

$$\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)d\eta\int_\eta^t (t-y)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]\times$$

$$\times\frac{(y-\eta)^{\beta_1-1}}{2}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi|(y-\eta)^{-\beta_1}\\ \delta(y-\eta)^\alpha\end{matrix}\right)dy=$$

$$=\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{2\Gamma(k+1)}d\eta\times$$

$$\times\int_\eta^t (t-y)^{-\beta}(y-\eta)^{-\beta_1 k+\beta_1-1}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(y-\eta)^\alpha\right]dy.$$

Let us evaluate the integral with respect to y. We apply (18) to the Prabhakar functions and perform the substitution $y=(t-\eta)z+\eta.$ Using (19), we obtain

$$\int_\eta^t (t-y)^{-\beta}(y-\eta)^{-\beta_1 k+\beta_1-1}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\left[\delta(y-\eta)^\alpha\right]dy=$$

$$=\sum_{j=0}^{+\infty}\sum_{i=0}^{j}\frac{(-\gamma)_i(\gamma_1-\gamma_1 k)_{j-i}\delta^j}{i!(j-i)!\Gamma(\alpha i+1-\beta)\Gamma(\alpha j-\alpha i+\beta_1-\beta_1 k)}\int_\eta^t (t-y)^{\alpha i-\beta}(y-\eta)^{\alpha j-\alpha i-\beta_1 k+\beta_1-1}dy=$$

$$=\sum_{j=0}^{+\infty}\sum_{i=0}^{j}\frac{(-\gamma)_i(\gamma_1-\gamma_1 k)_{j-i}\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1 k}}{i!(j-i)!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}=\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}.$$

We continue the computation using the obtained result:

$$\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}d\eta=$$

$$=\frac{1}{2}\frac{\partial}{\partial t}\int_0^x d\xi\int_0^t f(\eta,\xi)(t-\eta)^{-\beta_1}\sum_{k=0}^{+\infty}\frac{1}{\Gamma(k+1)}\left(-\frac{x-\xi}{(t-\eta)^{\beta_1}}\right)^k E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]d\eta+$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\int_x^a d\xi\int_0^t f(\eta,\xi)(t-\eta)^{-\beta_1}\sum_{k=0}^{+\infty}\frac{1}{\Gamma(k+1)}\left(-\frac{\xi-x}{(t-\eta)^{\beta_1}}\right)^k E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]d\eta.$$

Next, we differentiate the resulting expression with respect to the parameter t:

$$\frac{1}{2}\lim_{\eta\to t}\int_0^x f(\eta,\xi)(t-\eta)^{-\beta_1}\sum_{k=0}^{+\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x-\xi}{(t-\eta)^{\beta_1}}\right)^k E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]d\xi+$$

$$+\frac{1}{2}\lim_{\eta\to t}\int_x^a f(\eta,\xi)(t-\eta)^{-\beta_1}\sum_{k=0}^{+\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{\xi-x}{(t-\eta)^{\beta_1}}\right)^k E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]d\xi+$$

$$+\frac{1}{2}\int_0^a d\xi\int_0^t f(\eta,\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1k-1}E_{\alpha,-\beta_1-\beta_1k}^{-\gamma_1-\gamma_1k}\left[\delta(t-\eta)^\alpha\right]d\eta.$$

In the first and second integrals of the final expression, we perform the substitutions $z_1=\frac{x-\xi}{(t-\eta)^{\beta_1}}$ and $z_1=\frac{\xi-x}{(t-\eta)^{\beta_1}}$, respectively:

$$\frac{1}{2}\lim_{\eta\to t}\int_0^{\frac{x}{(t-\eta)^{\beta_1}}} f\left(\eta,x-(t-\eta)^{\beta_1}z_1\right)\sum_{k=0}^{+\infty}\frac{(-1)^k z_1^{\,k}}{\Gamma(k+1)}E_{\alpha,1-\beta_1-\beta_1k}^{-\gamma_1-\gamma_1k}\left[\delta(t-\eta)^\alpha\right]dz_1+$$

$$+\frac{1}{2}\lim_{\eta\to t}\int_0^{\frac{a-x}{(t-\eta)^{\beta_1}}} f\left(\eta,x+(t-\eta)^{\beta_1}z_2\right)\sum_{k=0}^{+\infty}\frac{(-1)^k z_2^{\,k}}{\Gamma(k+1)}E_{\alpha,1-\beta_1-\beta_1k}^{-\gamma_1-\gamma_1k}\left[\delta(t-\eta)^\alpha\right]dz_2+$$

$$+\frac{1}{2}\int_0^a d\xi\int_0^t f(\eta,\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1k-1}E_{\alpha,-\beta_1-\beta_1k}^{-\gamma_1-\gamma_1k}\left[\delta(t-\eta)^\alpha\right]d\eta=$$

$$=\frac{f(t,x)}{2}\int_0^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k z_1^{\,k}}{\Gamma(k+1)\Gamma(1-\beta_1-\beta_1k)}dz_1+$$

$$+\frac{f(t,x)}{2}\int_0^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k z_2^{\,k}}{\Gamma(k+1)\Gamma(1-\beta_1-\beta_1k)}dz_2+$$

$$+\frac{1}{2}\int_0^a d\xi\int_0^t f(\eta,\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1k-1}E_{\alpha,-\beta_1-\beta_1k}^{-\gamma_1-\gamma_1k}\left[\delta(t-\eta)^\alpha\right]d\eta.$$

After computing the integrals with respect to z_1 and z_2 and applying (10), we arrive at the following result:

$$f(t,x)+\frac{1}{2}\int_0^a d\xi\int_0^t f(\eta,\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1k-1}E_{\alpha,-\beta_1-\beta_1k}^{-\gamma_1-\gamma_1k}\left[\delta(t-\eta)^\alpha\right]d\eta. \tag{25}$$

Next, we evaluate the second term of (24)

$$\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)\,d\eta\int_\eta^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]K(y,x,\eta,\xi)\,dy \tag{26}$$

using the same approach. Firstly, we calculate the following integral:

$$\int_\eta^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]K(y,x,\eta,\xi)\,dy=\int_\eta^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]\times$$

$$\times\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi+2na|^k}{2\Gamma(k+1)}(y-\eta)^{-\beta_1k+\beta_1-1}E_{\alpha,\beta_1-\beta_1k}^{\gamma_1-\gamma_1k}\left[\delta(y-\eta)^\alpha\right]dy-$$

$$-\int_\eta^t(t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^\alpha\right]\times$$

$$\times\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k|x+\xi+2na|^k}{2\Gamma(k+1)}(y-\eta)^{-\beta_1k+\beta_1-1}E_{\alpha,\beta_1-\beta_1k}^{\gamma_1-\gamma_1k}\left[\delta(y-\eta)^\alpha\right]dy=$$

$$=\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi+2na|^k}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1k)_j\,\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1k}}{j!\Gamma(\alpha j-\beta_1-\beta_1k+1)}-$$

$$-\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}.$$

We substitute the obtained result into (26) and differentiate with respect to the parameter t:

$$\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)\left[\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x-\xi+2na\right|^k}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}-\right.$$

$$\left.-\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}\sum_{j=0}^{+\infty}\frac{(-\gamma_1-\gamma_1 k)_j\delta^j(t-\eta)^{\alpha j-\beta_1-\beta_1 k}}{j!\Gamma(\alpha j-\beta_1-\beta_1 k+1)}\right]d\eta=$$

$$=\int_0^a\lim_{\eta\to t}\left[f(\eta,\xi)\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x-\xi+2na\right|^k}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k}E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]-\right.$$

$$\left.-f(\eta,\xi)\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k}E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]\right]d\xi+$$

$$+\int_0^a d\xi\int_0^t f(\eta,\xi)\left[\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x-\xi+2na\right|^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]-\right.$$

$$\left.-\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]\right]d\eta.$$

At this point, the value of the following limit is 0, according to the Lemma:

$$\lim_{\eta\to t}\left[\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x-\xi+2na\right|^k}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k}E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]-\right.$$

$$\left.-\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k}E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]\right]=0.$$

Hence,

$$\frac{\partial}{\partial t}\int_0^a d\xi\int_0^t f(\eta,\xi)d\eta\int_\eta^t(t-y)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}\left[\delta(t-y)^\alpha\right]K(y,x,\eta,\xi)dy=$$

$$=\int_0^a d\xi\int_0^t f(\eta,\xi)\left[\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x-\xi+2na\right|^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]-\right.$$

$$\left.-\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}(t-\eta)^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]\right]d\eta. \tag{27}$$

Summing expressions (25) and (27), we arrive at the following result:

$${}^{PRL}D^{\alpha,\beta,\gamma,\delta}_{0t}\left(\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta\right)=$$

$$=f(t,x)+\int_0^a\int_0^t f(\eta,\xi)\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\left|x-\xi+2na\right|^k-\left|x+\xi+2na\right|^k\right]}{2\Gamma(k+1)}\times$$

$$\times(t-\eta)^{-\beta_1-\beta_1 k-1}E^{-\gamma_1-\gamma_1 k}_{\alpha,-\beta_1-\beta_1 k}\left[\delta(t-\eta)^\alpha\right]d\eta d\xi. \tag{28}$$

If we examine the limiting behavior of (20), (21) and (28) at the point $\eta = t$, it is not difficult to see that all of them are equal to 0 according to the Lemma. In (23), there is no singular point.

Therefore, for our solution, the condition ${}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x) \in C(\Omega)$ is satisfied. We have thoroughly verified the class to which our solution belongs.

4. Verification of the solution satisfying the equation.

Now we proceed to check whether the obtained solution indeed satisfies the equation (1).

Let us introduce the following notation:

$$u_0(t,x) = u(t,x) - u_1(t,x), \qquad u_1(t,x) = \int_0^t \int_0^a f(\eta,\xi) G(t,x,\eta,\xi) d\xi d\eta.$$

First, we show that $Lu_0 = 0.$ Let us begin by calculating the term involving the function $\varphi_0(\eta).$ Since we have computed the value of $\frac{\partial^2}{\partial x^2} G_\xi(t,x,\eta,0)$ above, we can write the following accordingly:

$$\frac{\partial^2}{\partial x^2} \int_0^t \varphi_0(\eta) G_\xi(t,x,\eta,0) d\eta =$$

$$= \int_0^t \varphi_0(\eta) \sum_{n=-\infty}^{+\infty} sign(x+2na) \sum_{k=0}^{+\infty} \frac{(-1)^k |x+2na|^k (t-\eta)^{-\beta_1 k - \beta - 1}}{k!} E_{\alpha,-\beta_1 k-\beta}^{-\gamma-\gamma_1 k} \left[\delta (t-\eta)^\alpha\right] d\eta.$$

Comparing the last equality with (20), we can see that they are equal. So their difference equals zero. The term containing $\varphi_1(\eta)$ can be demonstrated analogously to the term containing $\varphi_0(\eta)$.

Hence, we proceed to the term involving $\tau(\xi).$ Using the value of $\frac{\partial^2}{\partial x^2} G(t,x,\eta,\xi)$, we obtain the following expression:

$$\frac{\partial^2}{\partial x^2} \int_0^a \tau(\xi) G(t,x,0,\xi) d\xi =$$

$$= \int_0^a \tau(\xi) \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k \left[|x-\xi+2na|^k - |x+\xi+2na|^k \right]}{2\Gamma(k+1)} t^{-\beta_1 k - \beta_1 - 1} E_{\alpha,-\beta_1-\beta_1 k}^{-\gamma_1-\gamma_1 k} \left[\delta t^\alpha\right] d\xi.$$

We also see that the difference between (23) and the last equality is equal to zero. Thus we have proven that $Lu_0 = 0.$

From the value of $\frac{\partial^2}{\partial x^2} G(t,x,\eta,\xi)$, we determine that

$$\frac{\partial^2}{\partial x^2} \int_0^t \int_0^a f(\eta,\xi) G(t,x,\eta,\xi) d\xi d\eta =$$

$$= \int_0^t \int_0^a f(\eta,\xi) \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k \left[|x-\xi+2na|^k - |x+\xi+2na|^k \right]}{2\Gamma(k+1)} \times$$

$$\times (t-\eta)^{-\beta_1 k - \beta_1 - 1} E_{\alpha,-\beta_1-\beta_1 k}^{-\gamma_1-\gamma_1 k} \left[\delta (t-\eta)^\alpha\right] d\xi d\eta. \tag{29}$$

By subtracting (29) from (28), it follows that $Lu_1 = f(t,x).$

5. Verification of the solution satisfying boundary conditions.

We verify that the solution (4) satisfies the boundary conditions (2). For this purpose, we let $x \to 0$ in (4). From (5), it follows that

$$\lim_{x\to 0} G(t,x,\eta,\xi) = 0, \quad \lim_{x\to 0} G_\xi(t,x,\eta,a) = 0, \quad \lim_{x\to 0} G(t,x,0,\xi) = 0.$$

As a result, we get

$$\lim_{x\to 0} u(t,x) = \lim_{x\to 0}\int_0^t \varphi_0(\eta) G_\xi(t,x,\eta,0)\,d\eta.$$

Now, let us evaluate this equality directly:

$$\lim_{x\to 0}\int_0^t \varphi_0(\eta) G_\xi(t,x,\eta,0)\,d\eta = \lim_{x\to 0}\int_0^t \varphi_0(\eta)\sum_{n=-\infty}^{\infty} sign(x+2na)\,\omega\big(t-\eta,|x+2na|\big)d\eta =$$

$$= \lim_{x\to 0}\int_0^t \varphi_0(\eta)\,\omega(t-\eta,x)\,d\eta -$$

$$-\lim_{x\to 0}\int_0^t \varphi_0(\eta)\sum_{n=1}^{\infty}\Big[\omega(t-\eta,2na-x)-\omega(t-\eta,2na+x)\Big]d\eta = = \lim_{x\to 0}\int_0^t \varphi_0(\eta)\,\omega(t-\eta,x)\,d\eta.$$

We replace the function ω with its value and apply the substitution $s = \dfrac{x}{(t-\eta)^{\beta_1}}$:

$$\lim_{x\to 0} u(t,x) = \lim_{x\to 0}\int_0^t \varphi_0(\eta)\,\omega(t-\eta,x)\,d\eta =$$

$$= \lim_{x\to 0}\int_0^t \varphi_0(\eta)\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{n!}(t-\eta)^{-\beta_1 n-1} E^{-\gamma_1 n}_{\alpha,-\beta_1 n}\Big[\delta(t-\eta)^{\alpha}\Big]d\eta =$$

$$= \lim_{x\to 0}\int_0^{+\infty}\varphi_0\left[t-\left(\frac{x}{s}\right)^{\frac{1}{\beta_1}}\right]\sum_{n=0}^{+\infty}\frac{(-1)^n s^{n-1}}{\beta_1 n!}E^{-\gamma_1 n}_{\alpha,-\beta_1 n}\left[\delta\left(\frac{x}{s}\right)^{\frac{\alpha}{\beta_1}}\right]ds =$$

$$= \varphi_0(t)\int_0^{+\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n s^{n-1}}{n!\beta_1\Gamma(-\beta_1 n)}ds = -\varphi_0(t)\sum_{n=0}^{+\infty}\frac{(-1)^n s^n}{\Gamma(n+1)\Gamma(1-\beta_1 n)}\Bigg|_0^{+\infty} =$$

$$= -\varphi_0(t)\lim_{s\to+\infty} e^{1,1}_{1,\beta_1}(-s) + \varphi_0(t) = \varphi_0(t).$$

Similarly, we have

$$\lim_{x\to a} u(t,x) = \varphi_1(t).$$

6. Verification of the solution satisfying initial condition.

We now check that our solution (4) fulfills the initial condition (3):

$$\lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t} u(t,x) = \lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\int_0^a \tau(\xi) G(t,x,0,\xi)\,d\xi =$$

$$= \lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\int_0^a \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi|^k}{2\Gamma(k+1)}t^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\Big[\delta t^{\alpha}\Big]d\xi +$$

$$+\lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\int_0^a \tau(\xi) K(t,x,0,\xi)\,d\xi =$$

$$= \lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\int_0^x \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(x-\xi)^k}{2\Gamma(k+1)}t^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\Big[\delta t^{\alpha}\Big]d\xi +$$

$$+\lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\int_x^a \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(\xi-x)^k}{2\Gamma(k+1)}t^{-\beta_1 k+\beta_1-1}E^{\gamma_1-\gamma_1 k}_{\alpha,\beta_1-\beta_1 k}\Big[\delta t^{\alpha}\Big]d\xi +$$

$$+\lim_{t\to 0} {}^{P}I^{\alpha,1-\beta,-\gamma,\delta}_{0t}\int_0^a \tau(\xi) K(t,x,0,\xi)\,d\xi.$$

From the last expression, we evaluate the term containing the integral with respect to s from 0 to x:

$$\lim_{t\to 0} {}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\int_0^x \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(x-\xi)^k}{2\Gamma(k+1)}t^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta t^{\alpha}\right]d\xi =$$

$$=\lim_{t\to 0}\int_0^t (t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]dy\int_0^x \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(x-\xi)^k}{2\Gamma(k+1)}y^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta y^{\alpha}\right]d\xi =$$

$$=\lim_{t\to 0}\int_0^x \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(x-\xi)^k}{2\Gamma(k+1)}d\xi\int_0^t y^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta y^{\alpha}\right](t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]dy.$$

After computing the integral with respect to y, we obtain

$$\int_0^t y^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta y^{\alpha}\right](t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]dy = t^{-\beta_1-\beta_1 k}E_{\alpha,1-\beta_1-\beta_1 k}^{-\gamma_1-\gamma_1 k}[\delta t^{\alpha}].$$

Next, we proceed to evaluate the integral with respect to ξ. We make the substitution $\frac{x-\xi}{t^{\beta_1}}=z$:

$$\lim_{t\to 0}\int_0^x \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k}{2\Gamma(k+1)}\left(\frac{x-\xi}{t^{\beta_1}}\right)^k t^{-\beta_1}E_{\alpha,1-\beta_1-\beta_1 k}^{-\gamma_1-\gamma_1 k}[\delta t^{\alpha}]d\xi =$$

$$=\frac{1}{2}\lim_{t\to 0}\int_0^{\frac{x}{t^{\beta_1}}}\tau\left(x-t^{\beta_1}z\right)\sum_{k=0}^{+\infty}\frac{(-1)^k z^k}{\Gamma(k+1)}E_{\alpha,1-\beta_1-\beta_1 k}^{-\gamma_1-\gamma_1 k}[\delta t^{\alpha}]dz.$$

Now we compute the limit and use (10):

$$\lim_{t\to 0} {}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\int_0^x \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(x-\xi)^k}{2\Gamma(k+1)}t^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta t^{\alpha}\right]d\xi =$$

$$=\frac{\tau(x)}{2}\int_0^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k z^k}{\Gamma(k+1)\Gamma(1-\beta_1-\beta_1 k)}dz = -\frac{\tau(x)}{2}(-z)e_{1,\beta_1}^{2,1-\beta_1}(-z)\Big|_0^{+\infty}=\frac{\tau(x)}{2}. \tag{30}$$

Similarly,

$$\lim_{t\to 0} {}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\int_x^a \tau(\xi)\sum_{k=0}^{+\infty}\frac{(-1)^k(\xi-x)^k}{2\Gamma(k+1)}t^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta t^{\alpha}\right]d\xi=\frac{\tau(x)}{2}. \tag{31}$$

Now we turn to the following term:

$$\lim_{t\to 0} {}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\int_0^a \tau(\xi)K(t,x,0,\xi)d\xi =$$

$$=\lim_{t\to 0}\int_0^t (t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]dy\times$$

$$\times\int_0^a \tau(\xi)\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi+2na|^k}{2\Gamma(k+1)}y^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta y^{\alpha}\right]d\xi -$$

$$-\lim_{t\to 0}\int_0^t (t-y)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-y)^{\alpha}\right]dy\times$$

$$\times\int_0^a \tau(\xi)\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k|x+\xi+2na|^k}{2\Gamma(k+1)}y^{-\beta_1 k+\beta_1-1}E_{\alpha,\beta_1-\beta_1 k}^{\gamma_1-\gamma_1 k}\left[\delta y^{\alpha}\right]d\xi =$$

$$=\lim_{t\to 0}\int_0^a \tau(\xi)\sum_{n=-\infty}^{+\infty}{}'\sum_{k=0}^{+\infty}\frac{(-1)^k|x-\xi+2na|^k}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k}E_{\alpha,1-\beta_1-\beta_1 k}^{-\gamma_1-\gamma_1 k}[\delta t^{\alpha}]d\xi -$$

$$-\lim_{t\to 0}\int_0^a \tau(\xi)\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\left|x+\xi+2na\right|^k}{2\Gamma(k+1)}t^{-\beta_1-\beta_1 k}E^{-\gamma_1-\gamma_1 k}_{\alpha,1-\beta_1-\beta_1 k}[\delta t^\alpha]d\xi.$$

When evaluating the limits in the last equality, by using the Lemma, we see that their values are equal to 0. Consequently, according to (30) and (31),

$$\lim_{t\to 0}{}^P I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u(t,x)=\tau(x),\quad 0\le x\le a.$$

It is verified in detail that the solution (4) satisfies all the conditions stated in Theorem and Definition.

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