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ISAACS' PROBLEM FOR ACCELERATED PLAYERS UNDER INTEGRAL-GEOMETRIC RESTRICTIONS ON CONTROLS

Annotation: In the given work, Isaacs' problem ("Life-line" game) is addressed for the case when a control of pursuer has integral constraint, and that of evader has geometric constraint, which these players move with accelerations.

Keywords: Differential game, pursuer, evader, stretegy, Life-line.

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BOSHQARUVLARI INTEGRAL-GEOMETRIK CHEGARALANISHGA EGA TEZLANISHLI O'YINCHILAR UCHUN AYZEKS MASALASI

Annotatsiya: Ushbu ishda tezlanish bilan harakat qiluvchi o'yinchilar uchun quvlochining boshqaruvi integral, qochuvchining boshqaruvi geometrik chegaralanishga ega Ayzeks masalasi ("Qutulish chizig'i" o'yini) hal qilingan.

Kalit so'zlar: Differensial o'yin, quvlovchi, qochuvchi, strategiya, Qutilish chizig'i.

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ЗАДАЧА АЙЗЕКСА ДЛЯ ИГРОКОВ С УСКОРЕНИЯМИ, ПРИ НАЛИЧИИ ИНТЕГРАЛЬНО-ГЕОМЕТРИЧЕСКИХ ОГРАНИЧЕНИЙ НА УПРАВЛЕНИЕ

Аннотация: В данной работе рассматривается задача Айзекса (игра с "линия жизни") в случае, когда управление преследователя подчинено интегральному ограничению, а управление убегающего-геометрическому ограничению, при этом оба игрока движутся с ускорениями.

Ключевые слова: Дифференциальные игры, преследователь, убегающий, стратегия, Линия жизни.

Introduction.

In the middle of the twentieth century, the term “Differential games” was first apparent in R. Isaacs’s monograph [4] including exclusive conflict and game problems and also, additional open problems for future studies. In view of of the basic approaches in the theory of differential games constructed by L.S.Pontryagin [10], N.N.Krasovskii [8], B.N.Pshenichnyi [11] and others, a differential game is regarded as an optimal control problem from the standpoint of either a pursuer or an evader, and in consequence, the differential game comes to study a pursuit problem and to an evasion problem, respectively.

The number of the problems in [4] were solved by L.A.Petrosjan, A.A.Azamov and et al. In particular, in [9] L.A.Petrosjan completely solved the “Life-line” problem when players’ controls obeying to geometrical constraints, and to solve that problem, he first established the parallel approach method. Subsequently, the parallel pursuit strategy (the Π-strategy) based on this method was proposed. In the middle of the eighteenth years A.A.Azamov [1] first suggested an analytical solution of the “Life-line” game with multiple pursuers and one evader by using support function of the multi-valued mapping. A.A.Azamov [1] investigated the structure of phase space of differential pursuit-evasion games for the case where an evader has the discriminated information and here, an alternative for differential pursuit-evasion games in an infinite time was established by the transfinite iteration method of Pshenichnyi’s operator. Thereafter, for the cases in which controls of both objects adhering to integral, linear, Gronwall type or mixed constraints, the pursuit-evasion problem and the “Life-line” game have been thoroughly considered in the works of B.T.Samatov and et al [5,6,7,14,15]

We have considered the pursuit-evasion problems in a differential game with one pursuer and one evader that each of them has inertial movement. The control of the pursuer is subject to integral constraint and that of the evader is subject to geometric constraint. In the pursuit problem, we propose the parallel approach strategy, which is known as the _-strategy, for the pursuer, and we define sufficient solvability condition of pursuit. In the evasion problem, a special admissible strategy is implemented for the evader and sufficient solvability condition of evasion is obtained. Besides, the lower boundary of distance between the pursuer and evader is shown.

Statement of problems

Suppose that a player P, namely the Pursuer, with a control u goes after another player E, namely the Evader, with a control v in R^n. Let x represent the position of the Pursuer, and let y represent that of the Evader. Then we are going to discuss a differential game where players' motions take place in accordance with the following second order differential equations:

$$E : \ddot{y} = v, y(0) = y_0, \dot{y}(0) = y_1 \quad (1)$$

$$P : \ddot{x} = u, x(0) = x_0, \dot{x}(0) = x_1 \quad (2)$$

where $x, y, u, v \in R^n, n \geq 2; x_0, y_0$ are players' initial positions, and x_1, y_1 are their initial velocity vectors, respectively. Assume that the considered game is given under the conditions $x_0 \neq y_0$ and $x_1 = y_1$.

The control parameter u is the acceleration vector, and this is considered as a measurable function $u(\cdot) = R_{+¿ \to R^n ¿}$, and on that function, impose the integral constraint (for short, the I-constraint) in the form

$$\int_0^t \square (t-s) \vee u(s) ¿^2 ds \leq p_0, t \geq 0 (3)$$

where ρ_0 is a given positive number describing the amount of Pursuer's energy at time $t=0$. In the sequel, by U_I we infer the class of all control functions $u(\cdot)$ fulfilling the constraint (3).

In the same way, the control parameter v is the acceleration vector, and this is taken ass a measurable function $v(\cdot) : R_{+¿ \to R^n ¿}$, and on that function, impose the geometrical constraint (for short, the G-constraint) of the form

$$|v(t)| \le \beta \text{ almost everywhere } t \ge 0 \qquad (4)$$

where β is a given positive number indicating the maximum value of Evader's acceleration. In the sequel, by V_G we infer the class of all control functions $v(\cdot)$ fulfilling the constraint (4).

Definition 1. For each pair $(\rho_0, u(\cdot)), u(\cdot) \in U_f$, we call the quantity

$$\rho(t)=\rho_0-\int_0^t \square(t-s)|u(s)|^2 ds, \rho(0)=\rho_0 \qquad (5)$$

the residual resource of the Pursuer at the current time $t, t \ge 0$.

Definition 2. It is said that the Pursuer wins by the Π-strategy on a finite time interval $[0, T]$ if, for any $v(\cdot) \in V_G$: a) some instant $t^* \in [0, T]$, from which $z(t^*)=0$ is derived, can be determined; b) $u(v(\cdot)) \in U_l$ is valid on the interval $[0; t^*]$. In this occasion, the number T is called the guaranteed capture time.

Pursuit and Evasion Problems

Let us take the vector $p=(\rho_0, \beta, |z_0|) \in R_+^3$ on the parameters ρ_0, β, σ as a parametric state of the IG-Game. Then we establish the following nonempty connected sets of such states p :

$$P_{IG}=\{p: \rho_0 \ge 4\beta|z_0|, \beta \ge 0, \rho_0>0, |z_0| \ge 0\}$$

$$E_{IG}=\{p: \rho_0<4\beta|z_0|, \beta>0, \rho_0 \ge 0, |z_0|>0\}$$

which $P_{IG} \cup E_{IG}=R_+^3, P_{IG} \cap E_{IG}=\varnothing$ are satisfied.

Note that the geometrical representations of the sets P_{IG} and E_{IG} in R_+^3 are the following: the set of the points $p=(\rho_0, \beta, |z_0|)$ whose coordinates satisfy $\rho_0=4\beta|z_0|$ is a second-order saddle surface centered at the origin, while P_{IG} is located in the inner part and E_{IG} is located in the outer part of that set.

Definition 3. The control function

$$u_{IG}(v)=v-\lambda_{IG}(v)\xi_0 \qquad (6)$$

is said to be the Π_{IG}-strategy of the Pursuer in the IG-Pursuit Game, where

$$\lambda_{IG}(v)=\langle v,\xi_0\rangle+\frac{\eta_0}{2}+\sqrt{\left(\langle v,\xi_0\rangle+\frac{\eta_0}{2}\right)^2-¿v¿^2},\xi_0=\frac{z_0}{|z_0|},\eta_0=\frac{\rho_0}{|z_0|},$$

and $\langle v,\xi_0\rangle$ is the inner product of the vectors v and ξ_0 in R^n. Here $\lambda_{IG}(v)$ is usually called the resolving function.

Thanks to equations (1)-(2), for arbitrary $v(\cdot)\in V_G$ and for the function $u_{IG}(v(\cdot))\in U_I$ it is obtained that from the triad $(x_0,x_1,u_{IG}(v(\cdot)))$ Pursuer's trajectory

$$x(t)=x_0+x_1t+\int_0^t \square(t-s)u_{IG}(v(s))ds \qquad (7)$$

and from the triad $(y_0,y_1,v(t))$ Evader's trajectory

$$y(t)=y_0+y_1t+\int_0^t \square(t-s)v(s)ds \, (8)$$

In this case, the goal of Pursuer P is to capture Evader E, i.e. to achieve $x(t)=y(t)$, and while Evader E strives to avoid-an encounter, i.e. to continue $x(t)\neq y(t)$ for all $t\geq 0$ and if this can not be done, to postpone the encounter time as long as possible.

Theorem 1[12]**.** If $p\in P_{IG}$ in the IG-Pursuit Game, then the Pursuer wins by Π_{IG}-strategy (6) on the time interval $[0,T_{IG}]$, where $T_{IG}=\sqrt{2|z_0|/\theta_1}$ and θ_1 is the same with (7).

In the current section, an admissible control will be offered for the Evader, by which it is proved that $\Pi_{IG^{-¿}¿}$ strategy (6) is an optimal strategy and the T_{IG} is an optimal time of pursuit.

Let in the IG-Evasion Game the Evader employ the following control as the strategy:

$$v_{IG}^{¿}(t)=-\beta\xi_0,\xi_0=\frac{z_0}{|z_0|} \qquad (9)$$

Theorem 2 [12]**.** a) Let $p\in P_{IG}$. Then the Evader wins by the strategy (9) in the interval $[0,T_{IG})$, where T_{IG} is the guaranteed capture time stated in Theorem 1.

b) Let $p \in E_{IG}$. Then the Evader wins by the strategy (9) in the interval ¿ and the distance between the Pursuer and the Evader satisfies the inequality

$$¿ z(t) \vee \geq |z_0| - \frac{\rho_0}{4\beta}$$

An attainability domain of the Evader

If $p \in P_{IG}$ in the IG-Game, then Theorem 1 asserts that by virtue of Π_{IG}-strategy (6) the Pursuer is able to capture the Evader at some point in R^n. Ergo, the players P and E will meet at various points according to the choice of the control $v(\cdot) \in V_G$.

As in the work [2,3], let $M_{IG}(x, y, \rho)$ be the set consisting of all points μ where the Pursuer moving from the position x and consuming the resource ρ should encounter the Evader moving from the position y, i.e.

$$M_{IG}(x, y, \rho) = ¿$$

When $\rho \neq \beta$, the set $M_{IG}(x, y, \rho)$ is bounded by

$$\partial M_{IG}(x, y, \rho) = ¿$$

Remark 1. In the works [2], it was shown that the geometrical representation of the set $M_{IG}(x, y, \rho)$ in the plane is Descartes' Oval or Pascal's Snail.

And now, let the Pursuer hold Π_{IG}-strategy (6) while the Evader employs arbitrary control $v(\cdot) \in V_G$. Then the triad $(x_0, x_1, u_{IG}(v(\cdot)))$ gives the trajectory (7) and the triad ($y_0, y_1, v(\cdot)$) yields the trajectory (8), where $t \in [0, t^¿], 0 < t^¿ \leq T_{IG}$, and $t^¿$ is players' meeting time, i.e. $x(t^¿) = y(t^¿)$. Then for each triad $(x(t), y(t), \rho(t))$, $t \in [0, t^¿]$, we build the following sets:

$$M_{IG}(x(t), y(t), \rho(t)) = ¿$$

$$M_{IG}(x_0, y_0, \rho_0) = \left\{ \mu : |\mu - x_0|^2 \geq \frac{\rho_0}{\beta} |\mu - y_0| \right\} (10)$$

Lemma 1[12]**.** The multi-valued mapping $M_{IG}(x(t), y(t), \rho(t)) - t x_1$ is monotonically decreasing in respect to the inclusion when $t \in [0,$ t*], i.e. if $t_1, t_2 \in [0, t^¿]$ and $t_1 < t_2$, then $M_{IG}(x(t_2), y(t_2), \rho(t_2))\ t_2 x_1 \subset M_{IG}(x(t_1), y(t_1), \rho(t_1)) - t_1 x_1$.

Corollary 1. It can be inferred from directly Lemma 1 that [12]:

a) $M_{IG}(x(t), y(t), \rho(t)) \subset M_{IG}(x_0, y_0, \rho_0) + t x_1$ at each $t \in [0, t^{¿}]$;

b) $y(t) \in M_{IG}(x_0, y_0, \rho_0) + t x_1$ for all $t \in [0, t^{¿}]$.

The IG-Game with a "Life-line"

The current section investigates the Isaacs problem playing a special role in the theory of differential games. Let a non-empty and closed subset L, which is named the "Life-line", in space R^n be given. In this occasion, Pursuer P intends to intercept Evader E, viz. to accomplish $x(t_{¿}) = y(t_{¿})$ at a finite time $t_{¿} > 0$ when Evader E doesn't leave the area $R^n \setminus L$. As for Evader E aims to get to the area L without an encounter with Pursuer P or to continue the condition $x(t) \neq y(t)$ for all $t \geq 0$, and if there is no chance of doing this, then Evader E strives to maximize the moment of the encounter with Pursuer P. We have to mention that the area L doesn't limit the motion of Pursuer P. Further, it is required that the conditions $x_0 \neq y_0$ and $y_0 \notin L$ are satisfied for the initial positions x_0 and y_0 at the beginning of the IG-Game with the "Life-line" L.

Definition 4. Π_{IG}-strategy (6) is said to be winning on the time interval $[0, T_{IG}]$ in the IG-Game with the "Life-line" L, if for Evader's arbitrary control $v(\cdot) \in V_G$ there exists an instant $t_{¿} \in [0, T_{IG}]$ such that:

1. $x(t_{¿}) = y(t_{¿})$;
2. $y(t) \notin L$ at each $t \in [0, t_{¿}]$.

Theorem 3. Suppose $p \in P_{IG}$ and $M_{IG}^{¿}(x_0, y_0, \rho_0, T_{IG}) \cap L = \varnothing$. Then Π_{IG}-strategy (6) will be winning on the time interval $[0, T_{IG}]$ in the IG-Game with the "Life-line" L, where T_{IG} is the guaranteed capture time shown in Theorem1 [13].

Proof. The proof immediately comes from Theorem 1, Lemma 1 and corollary 1.

We now turn to solve the IG-Game with the "Life-line" L to the benefit of Evader E

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Definition 5. A control $v_E(\cdot) \in V_G$ is said to be winning in the IG-Game with the "Life-line" L if for . Pursuer's arbitrary control $u(\cdot) \in U_I$ at least one of the given conditions is satisfied:

1) a finite instant ϑ, when $y(\vartheta) \in L$ and $x(t) \neq y(t)$ for every $t \in ¿$, exists;
2) $x(t) \neq y(t)$ for every $t \geq 0$.

Let's define the set

$$M_E(\mu, y_0) = \left\{ \mu^{¿} : \mu^{¿} = \sqrt{\frac{2|\mu - y_0|}{\beta}} x_1 + \mu, \mu \in M(x_0, y_0, \rho_0) \right\} (11)$$

Definition 6. The set $M_E(\mu, y_0)$ in (11) is said to be the reachable set of the Evader in the IG-Game with the "Life-line" L [13].

Theorem 4. Suppose $p \in P_{IG}$ and $M_E(\mu, y_0) \cap L \neq \varnothing$. Then Evader E has a control $v_E(\cdot) \in V_G$ which is winning in the IG-Game with the "Life-line" L.

Proof. The second condition of Theorem 4 asserts that a point $\mu^{¿} \in M_E(\mu, y_0) \cap L$, which satisfies $\mu^{¿} = \sqrt{2|\mu - y_0|/\beta}\, x_1 + \mu, \mu \in M(x_0, y_0, \rho_0)$, exists. Then for Evader E, offer the constant control

$$v_E = \frac{\mu - y_0}{|\mu - y_0|} \beta (12)$$

It is firstly needed to substantiate that Evader E, with the help of the control (12), is able to reach the aimed point $\mu^{¿}$ at the moment $\vartheta = \sqrt{2|\mu - y_0|/\beta}$. To this end, replacing (8) into (10) and employing $x_1 = y_1$. yields

$$y(\vartheta) = y_0 + y_1 \vartheta + \int_0^{\vartheta} \square (\vartheta - s) v_E ds = y_0 + y_1 \vartheta + \frac{\vartheta^2}{2} v_E = ¿$$

$$¿\, y_0 + x_1 \sqrt{2|\mu - y_0|/\beta} + \mu - y_0 = \mu^{¿}$$

We are now in a position to show the validity of the condition a) in Definition 5, viz, Evader E will not encounter with Pursuer P until the time $\mu^{¿}$. Let be opposite, i.e. for Pursuer P, a control $u_P(\cdot) \in U_I$ realizing the equality $x(\delta) = y(\delta)$ at a time $\delta \in (0, \vartheta)$ exists. Then, since $x_1 = y_1$ and $z(t) = x(t) - y(t)$, it proceeds from equations (1)-(2) that

$$z(\delta)=z_0+\int_0^{\delta} \square(\delta-s)\left(u_P(s)-v_E\right)ds=0 \quad (13)$$

By virtue of the Cauchy-Bunyakovskii integral inequality, the right-hand side of $z(\delta)$ can be absolutely evaluated as

$$\left|z_0-\int_0^{\delta} \square(\delta-s)v_E\,ds\right|\leq\int_0^{\delta} \square(\delta-s)\left|u_P(s)\right|ds\leq\sqrt{\frac{\rho_0}{2}}\,\delta$$

or

$$\left|z_0-\frac{\delta^2}{2}v_E\right|\leq\sqrt{\frac{\rho_0}{2}}\,\delta \qquad (14)$$

Taking square both sides of (14) gives the biquadratic inequality of δ

$$\frac{\beta^2}{4}\delta^4-\left(\langle z_0,v_E\rangle+\frac{\rho_0}{2}\right)\delta^2+\left|z_0\right|^2\leq 0 \qquad (15)$$

and its solution will be

$$\delta\geq\frac{1}{\beta}\sqrt{2\langle z_0,v_E\rangle+\rho_0-\sqrt{\left(2\langle z_0,v_E\rangle+\rho_0\right)^2-4\beta^2\left|z_0\right|^2}} \qquad (16)$$

As $\delta<\vartheta$ and $\vartheta=\sqrt{2\left|\mu-y_0\right|/\beta}$, it is derived from (16) that

$$2\beta\left|\mu-y_0\right|>2\langle z_0,v_E\rangle+\rho_0-\sqrt{\left(2\langle z_0,v_E\rangle+\rho_0\right)^2-4\beta^2\left|z_0\right|^2} \qquad (17)$$

On account of $z_0=x_0-y_0$, the left-hand side of (17) is evaluated as

$$2\beta\left|\mu-y_0\right|=2\beta\left|\mu-x_0+x_0-y_0\right|\geq 2\beta\left(\left|\mu-x_0\right|-\left|z_0\right|\right). \qquad (18)$$

It is not hard to make sure that the right-hand side of (17) is well-defined and non-negative when $p\in P_{IG}$. Hence introducing $g(\zeta)=\zeta-\sqrt{\zeta^2-\gamma^2},\zeta=2\langle z_0,v_E\rangle+\rho_0,\gamma=2\beta\left|z_0\right|$ we determine $\frac{dg(\zeta)}{d\zeta}<0$, and in consequence, obtain that

$$g_{max}(\zeta)=g(\zeta_{min})=\rho_0-2\beta\left|z_0\right|-\sqrt{\rho_0^2-4\beta\left|z_0\right|\rho_0} \qquad (19)$$

Thus, we can say that the inequality (17) is always true if the relation, which is generated by combining (18) with (19),

$$2\beta\left(\left|\mu-x_0\right|-\left|z_0\right|\right)>\rho_0-2\beta\left|z_0\right|-\sqrt{\rho_0^2-4\beta\left|z_0\right|\rho_0}$$

is satisfied. From the latter we get

$$\rho_0 - 2\beta|\mu - x_0| < \sqrt{\rho_0^2 - 4\beta|z_0|\rho_0} \tag{20}$$

Squaring both sides of (20) produces

$$4\beta^2|\mu - x_0|^2 - 4\beta\rho_0|\mu - x_0| < 4\beta|z_0|\rho_0 \tag{21}$$

As $z_0 = x_0 - y_0$, we have

$$|z_0| = |x_0 - y_0| = |x_0 - \mu + \mu - y_0| \geq |\mu - x_0| - |\mu - y_0| \tag{22}$$

Applying (22) to the right-hand side of (21) we derive

$$|\mu - x_0|^2 < \frac{\rho_0}{\beta}|\mu - y_0|$$

and this is contrary to our assumption owing to (10). The proof is complete.

Remark 2. Relying on the definitions of $M_{IG}^{¿}(x_0, y_0, \rho_0, T_{IG})$ and $M_E(\mu, y_0)$ we can confirm the validity of the relation $M_E(\mu, y_0) \subset M_{IG}^{¿}(x_0, y_0, \rho_0, T_{IG})$ in the IG-Game with the "Life-line" L.

Conclusion

In the present paper, we have discussed the pursuit–evasion games for the inertial motion dynamics of the Pursuer and the Evader with integral and geometrical constraints on the controls. In the IG-Pursuit Game, we have defined the parallel convergence strategy, i.e. the Π-strategy for the Pursuer, and we have found the sufficient solvability condition of pursuit. In the IG-Evasion Game, we have given the constant control to the Evader as the strategy, and we have obtained the sufficient solvability condition of escape. In addition, we have demonstrated that the Π_{IG}-strategy is optimal and T_{IG} is the optimal capture time. We have built the attainability domains of the players and defined sufficient conditions solving the IG-Game with a “Life-line”.

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