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SPECTRAL ANALYSIS OF A SCHRODINGER OPERATOR WITH AN ASYMMETRIC GAUSSIAN POTENTIAL: A METHODOLOGICAL FRAMEWORK FOR STABILITY PROBLEMS

Annotation: *This paper investigates the spectral properties of a Schrödinger operator with an asymmetric Gaussian potential* $V(x)=-U_0\exp(-x^2+\beta x)$. *Using a robust numerical shooting method, we analyze the influence of the well's depth* U_0 *and asymmetry parameter* β *on the discrete spectrum and eigenfunction localization. Our findings demonstrate that increasing asymmetry systematically lowers energy levels, reduces the critical depth required for excited states, and widens energy splitting. A key analytical result, ⟨x⟩=β/2 is derived and verified for all bound states. This study provides a valuable methodological benchmark for stability problems in gravitational theories, condensed matter, and optics where Schrödinger-like equations are fundamental.*

Keywords: *Schrödinger equation, asymmetric Gaussian potential, spectral analysis, numerical shooting method, bound states, energy eigenvalues, position expectation value, symmetry breaking, stability analysis, scalar-tensor gravity.*

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СПЕКТРАЛЬНЫЙ АНАЛИЗ ОПЕРАТОРА ШРЕДИНГЕРА С АСИММЕТРИЧНОЙ ГАУССОВОЙ ЯМОЙ: МЕТОДОЛОГИЧЕСКАЯ

ОСНОВА ДЛЯ ЗАДАЧ УСТОЙЧИВОСТИ

Аннотация: *В данной работе исследуются спектральные свойства оператора Шредингера с асимметричным гауссовым потенциалом* $V(x) = -U_0 \exp(-x^2 + \beta x)$*. С использованием надежного численного метода стрельбы анализируется влияние глубины ямы* U_0 *и параметра асимметрии* β *на дискретный спектр и локализацию собственных функций. Наши результаты показывают, что усиление асимметрии систематически снижает уровни энергии, уменьшает критическую глубину, необходимую для существования возбужденных состояний, и увеличивает энергетическое расщепление между соседними уровнями. Получен и верифицирован для всех связанных состояний ключевой аналитический результат:* $\langle x \rangle = \beta/2$*. Данное исследование служит ценным методологическим эталоном для анализа задач устойчивости в гравитационных теориях, физике конденсированного состояния и оптике, где уравнения шредингеровского типа являются фундаментальными.*

Ключевые слова: *уравнение Шредингера, асимметричный гауссов потенциал, спектральный анализ, численный метод стрельбы, связанные состояния, собственные значения энергии, среднее значение координаты, нарушение симметрии, анализ устойчивости, скалярно-тензорная гравитация.*

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ASIMMETRIK GAUSS POTENSIALLI SHRYODINGER OPERATORINING SPEKTRAL TAHLILI: BARQARORLIK MUAMMOLARI UCHUN METODOLOGIK ASOS

Ushbu maqolada $(x) = -U_0 \exp(-x^2 + \beta x)$ *ko‘rinishidagi asimmetrik Gauss potensialli Shryodinger operatorining spektral xususiyatlari tadqiq etiladi. Ishonchli sonli "otish" (shooting) metodi yordamida potensial chuqurligi* U_0 *va asimmetriya parametri* β *ning diskret spektr hamda xususiy funksiyalar lokalizatsiyasiga ta’siri tahlil qilingan. Olingan natijalar asimmetriyaning ortishi energiya sathlarini tizimli ravishda pasaytirishini, uyg‘ongan holatlar uchun zarur bo‘lgan kritik chuqurlikni kamaytirishini va sathlararo energetik farqni kengaytirishini ko‘rsatadi. Barcha bog‘langan holatlar uchun* $\langle x \rangle = \beta/2$ *ko‘rinishidagi muhim analitik bog‘lanish keltirib chiqarilgan va sonli tekshirilgan. Mazkur tadqiqot Shryodinger tipidagi tenglamalar fundamental hisoblangan gravitatsiya nazariyalari, kondensirlangan muhitlar fizikasi va optikadagi barqarorlik muammolari uchun muhim metodologik namuna bo‘lib xizmat qiladi.*

Kalit so‘zlar: *Shryodinger tenglamasi, asimmetrik Gauss potensiali, spektral tahlil, sonli otish metodi, bog‘langan holatlar, energiyaning xususiy qiymatlari, koordinataning o‘rtacha qiymati, simmetriyaning buzilishi, barqarorlik tahlili, skalyar-tenzorli gravitatsiya.*

1.Introduction

The time-independent Schrödinger equation is fundamental for describing stationary quantum states through energy eigenvalues and eigenfunctions [1, 2]. While exact solutions exist for a few models, most realistic potentials in condensed matter or molecular physics require numerical methods. Among these, the Gaussian potential is a vital model for localized phenomena like quantum dots and nucleons [3]. However, perfect symmetry is often an idealization; factors like external fields or structural defects break this symmetry, significantly altering spectral properties. This work systematically investigates an asymmetrically modified Gaussian potential:

$$V(x) = -U_0 \exp\left(-x^2 + \beta x\right), \tag{1}$$

where $U_0 > 0$ represents the depth of the well, and the parameter β introduces a controllable, non-trivial asymmetry that breaks the reflection symmetry $V(x) \neq V(-x)$ for $\beta \neq 0$.

This framework is vital for stability analysis in general relativity and scalar-tensor theories [4, 5], where perturbations reduce to Schrödinger-type equations with asymmetric potentials [6, 7]. Our "toy model" provides essential intuition for instability spectra in complex gravitational systems. Sections 2–3 detail the methodology and numerical results, while Sections 4–5 provide physical interpretations and conclusions.

2. Theoretical and numerical framework

This section details the theoretical and numerical framework for the asymmetric Gaussian well. We employ a dimensionless formulation and a shooting algorithm for exact eigenvalues, using WKB, perturbation, and variational methods to validate results and provide deeper physical insight.

The dimensionless Schro¨dinger equation

The starting point of our analysis is the one-dimensional, time-independent Schr¨odinger equation for a particle of mass m in the potential $V(x)$ given by Eq. (1):

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \tag{2}$$

where $\hbar$ is the reduced Planck constant, E is the energy eigenvalue, and $\psi(x)$ is the corresponding eigenfunction. To facilitate a more general analysis and simplify the numerical treatment, we recast this equation into a dimensionless form. We introduce a characteristic length scale a (which we can set to unity, $a = 1$, without loss of generality for our potential form) and define the dimensionless coordinate ξ as:

$$\xi = \frac{x}{a}. \tag{3}$$

The potential can then be written as $V(\xi) = -U_0 \exp(-\xi^2 + \gamma\xi)$, where $\gamma = \beta a$ is the dimen- sionless asymmetry parameter. Substituting these into Eq. (2) and rearranging the terms, we get:

$$\frac{d^2\psi(\xi)}{d\xi^2} - \frac{2ma^2}{\hbar^2}\left[E - V(\xi)\right]\psi(\xi) = 0. \tag{4}$$

We now define two crucial dimensionless parameters: a parameter λ that characterizes the ”strength” of the potential, and the dimensionless energy eigenvalue ϵ:

$$\lambda = \frac{2ma^2 U_0}{\hbar^2}, e = \frac{E}{U_0} \tag{5}$$

With these definitions, the Schr¨odinger equation takes its final, clean, dimensionless form:

$$\frac{d^2\psi(\xi)}{d\xi^2} + \lambda\left[e + \exp\left(-\xi^2 + \gamma\xi\right)\right]\psi(\xi) = 0. \tag{6}$$

Our primary goal is to solve this second-order ordinary differential equation as an eigenvalue problem for the discrete, negative energy states ($\epsilon < 0$) that satisfy the physical boundary condition of normalizability, which implies that the wavefunction must vanish at infinity:

$$\psi(\xi) \to 0 \, as \vee \xi \vee \to \infty \tag{7}$$

In our numerical work, we will set the characteristic scales m, $\hbar$, a to unity for simplicity. This makes $\lambda = 2U_0$, $\gamma = \beta$, and the energy we compute is E.

Numerical approach: The shooting method

To find the precise eigenvalues of Eq. (6), we employ the well-established shooting method, a powerful technique for solving boundary-value problems [8, 9]. The algorithm proceeds as follows:

-**Select a test energy:** A trial value for the energy eigenvalue E is chosen. For bound states, we search in the range $-U_0 < E < 0$.

-**Set initial conditions:** The numerical integration is initiated at a point ξ_{start} far to the left of the potential well's center, where $V(\xi_{\text{start}}) \approx 0$. In this region, the Schr¨odinger equation simplifies to $\psi''(\xi) \approx (2m|E|/\hbar^2)\psi(\xi)$, and the solution that satisfies the boundarycondition at $\xi \to -\infty$ is a growing exponential: $\psi(\xi) \propto \exp(\sqrt{2m|E|}\xi/\hbar¿$. We thus set the initial conditions for the wavefunction and its derivative at ξ_{start}:

$$\psi(\xi_{start}) = \exp(\sqrt{2m|E|}\xi_{start}/\hbar¿ \tag{8}$$

$$\psi'(\xi_{start}) = (\sqrt{2m|E|}/\hbar¿\exp\left(\sqrt{2m|E|}\xi_{start}/\hbar\right)$$

(9) The overall normalization constant is irrelevant for this linear eigenvalue problem.

-Integrate ("shoot"): Using these initial conditions, the differential equation is integrated numerically across the potential well to a point ξ_{end} far to the right, using a standard algorithm like the fourth-order Runge-Kutta method.

-Check the boundary condition: At ξ_{end}, we check if the resulting solution satisfies the second boundary condition, i.e., if $\psi(\xi_{\text{end}})$ is sufficiently close to zero. For an arbitrary choice of E, it will generally diverge to $\pm\infty$.

-Iterate and refine: The value of $\psi(\xi_{\text{end}})$ is a function of the chosen energy, $f(E) = \psi(\xi_{\text{end}}; E)$. The valid energy eigenvalues are the roots of this function, $f(E) = 0$. We employ a root-finding algorithm, such as the bisection or secant method, to systematically refine the value of E until the condition $|\psi(\xi_{\text{end}})| < \delta$ is met for a desired small tolerance δ. This procedure allows us to determine the energy eigenvalues to a high degree of precision. By searching for sign changes in $f(E)$, we can locate not only the ground state but also the entire spectrum of excited states.

Approximate analytical methods

To complement our numerical solutions and gain a deeper physical understanding, we employ three powerful analytical approximation techniques.

The WKB approximation

The Wentzel-Kramers-Brillouin (WKB) approximation is a semi-classical method that is highly effective for potentials that are "slowly varying" or for states with high quantum numbers [2,10]. The method leads to a quantization condition for the allowed energy levels. For a particle with energy E trapped between two classical turning points ξ_1 and ξ_2 (defined by $V(\xi) = E$), the Bohr-Sommerfeld quantization rule states:

$$\int_{\xi 1}^{\xi 2} p(\xi)\, d\xi = \left(n + \frac{1}{2}\right)\pi\hbar. \tag{10}$$

where $n = 0, 1, 2, \ldots$ is the quantum number and $p(\xi) = \sqrt{2m(E - V(\xi))}$ is the classical mo- mentum. For our potential, this equation becomes a transcendental equation for E that must be solved numerically. While not exact, the WKB method provides an excellent estimate for the energy levels, especially for deeper wells that support many bound states.

Perturbation theory for small asymmetry

To specifically analyze the effect of the asymmetry, we can treat the β-dependent part of the potential as a small perturbation. We split the Hamiltonian H into an unperturbed part H_0 (the symmetric Gaussian well) and a perturbation H':

$$H = H_0 + H', \tag{11}$$

$$H_0 = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} - U_0 e^{-x^2} \tag{12}$$

$$H' = V(x) - V_0(x) = -U_0 e^{-x^2}\left(e^{\beta x} - 1\right). \tag{13}$$

For small β, we can expand the exponential: $e^{\beta x} - 1 \approx \beta x + {}^{1}\,(\beta x)^2 + \ldots$. The perturbation is then approximately $H' \approx -U\ e^{-x^2}(\beta x)$.

According to standard time-independent perturbation theory, the first-order correction to the energy levels $E^{(0)}$ of the unperturbed system is given by the expectation value of the perturbation:

$$E_n^{(1)} = \left(\psi_n^{(0)}\middle|H'\middle|\psi_n^{(0)}\right) = \int_{-\infty}^{\infty} (\psi_n^{(0)} \text{¿}(x)) * H' \psi_n^{(0)}(x)\,dx\,.\text{¿} \tag{14}$$

A crucial observation is that the unperturbed eigenfunctions $\psi^{(0)}(x)$ have definite parity (they are either even or odd functions), while the leading term of the perturbation, $H' \propto xe^{-x^2}$, is an odd function. The product of two even functions and one odd function is odd, and the integral of an odd function over a symmetric interval is identically zero. Therefore, the first-order energy correction vanishes: $E_n^{(0)} = 0$.

This implies that the leading contribution to the energy shift comes from the second-order correction:

$$E_n^{(2)} = \sum_{m \neq n} \frac{\text{¿}(\psi \text{¿¿} m^{(0)}|H'|\psi_n^{(0)}) \vee \text{¿}^2}{E_n^{(0)} - E_m^{(0)}} \text{¿¿} \tag{15}$$

Since the ground state energy $E_0^{(0)}$ is the lowest, the denominator $(E\text{¿¿}\, n^{(0)} - E_m^{(0)})\text{¿}$ is always negative.The numerator is always positive. Consequently, the second-order correction to the ground state energy, $E_0^{(2)}$ is always negative. This rigorously proves that for small β, the introduction of
asymmetry must lower the ground state energy, making the particle more tightly bound.

The variational method

The variational principle provides a powerful tool for estimating the ground state energy E_0 of a system. It states that for any normalized trial wavefunction ψ_{trial}, the expectation value of the Hamiltonian provides an upper bound for the true ground state energy:

$$E_0 \leq (E) = \int_{-\infty}^{\infty} \psi_{trial}^{(\text{¿})} H \psi_{trial}\, dx. \tag{16}$$

The goal is to choose a physically motivated trial wavefunction with one or more adjustable parameters and then minimize the energy expectation value with respect to these parameters to find the best possible upper bound. To capture the asymmetry of our problem, a suitable choice is a shifted Gaussian function

$$\psi_{trial}(x,\alpha,b) = ¿ \tag{17}$$

where α (related to the width) and b (related to the shift of the peak) are variational parameters. The energy functional to be minimized is then:

$$E(\alpha,b) = \int_{-\infty}^{\infty} \psi_{trial}^{¿} \left[\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} - U_0 e^{-x^2+\beta x} \right] \psi_{trial}\, dx. \tag{18}$$

By calculating this integral and solving the minimization equations $\partial E/\partial\alpha = 0$ and $\partial E/\partial b = 0$, we can obtain an analytical estimate for the ground state energy and the particle's localization.

An exact analytical result: The position expectation value

One of the most striking results of our numerical analysis is the linear relationship between the position expectation value and the asymmetry parameter, $(x) = \beta/2$. This is not a mere coincidence or an approximation but an exact analytical property of the potential defined in Eq. (1). Here, we provide a rigorous mathematical proof of this relation.

Our starting point is the time-independent Schrodinger equation:

$$\left[\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} - U_0 e^{-x2+\beta x} \right] \psi(x) = E\psi(x). \tag{19}$$

The key to the proof is to perform a coordinate transformation that reveals an underlying symmetry of the problem. Let us introduce a shifted coordinate x' defined by:

$$x' = x - c, \vee x = x' + c \tag{20}$$

where c is a constant shift that we will determine shortly. The derivative operator remains unchanged under this simple shift: $d/dx = d/dx'$. Let's analyze the exponent in the potential in terms of the new coordinate x':

$$-x^2 + \beta x = -(x' + c)^2 + \beta(x' + c) = ¿$$
$$¿ - (x'^2 + 2cx' + c^2) + \beta x^2 + \beta c = ¿$$
$$¿ - x'^2 + (\beta - 2c)x' - c^2 + \beta c. \tag{21}$$

The crucial insight is that we can choose the shift c to completely eliminate the linear term in
x'. This is achieved by setting its coefficient to zero:

$$\beta - 2c = 0 \Rightarrow c = \frac{\beta}{2} \tag{22}$$

By choosing this specific shift, our coordinate transformation becomes $x' = x - \beta/2$. The exponent simplifies dramatically:

$$-x^2+\beta x=-x'^2-\frac{\beta^2}{2}+\beta\left(\frac{\beta}{2}\right)=-x'^2+\frac{\beta^2}{4}. \quad (23)$$

this back into the potential, we find its form in the new coordinate system:

$$V(x')=-U_0\exp(-x'^2+\frac{\beta^2}{4})=-\left(U_0e^{\frac{\beta^2}{4}}\right)e^{-x'^2}. \quad (24)$$

Let us define an effective potential depth $U_0'=U_0e^{\frac{\beta^2}{4}}$.The potential in the shifted frame is

$V(x')=U_0'e^{-x'^2}$. This potential is manifestly a symmetric (even) function of x ':

$$V(x')=V(-x') \quad (25)$$

The Schr¨odinger equation in the x' coordinate system now reads:

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx'^2}+V(x')\right]\psi(x')=E\psi(x'). \quad (26)$$

This is the Schr¨odinger equation for a particle in a standard, symmetric Gaussian potential well. A fundamental theorem of quantum mechanics states that for a symmetric potential, the non-degenerate bound state eigenfunctions must have a definite parity; that is, they must be either even functions ($\psi(-x') = \psi(x')$) or odd functions ($\psi(-x') = -\psi(x')$) [1].

Now, let us calculate the expectation value of the position in this new frame, $\langle x'\rangle$:

$$(x')=\int_{-\infty}^{\infty}\psi^{¿}(x')x'\psi(x')dx'=\int_{-\infty}^{\infty}¿\psi(x')\vee¿^{2'}x'dx'¿. \quad (27)$$

The term $|\psi(x')|^2$ is always an even function, regardless of whether $\psi(x')$ is even or odd. The term x' is an odd function. The integrand is therefore the product of an even function and an odd function, which results in an odd function. The definite integral of any odd function over a symmetric interval from $-\infty$ to $+\infty$ is identically zero. Thus, we have proven that:

$$(x'¿=0. \quad (28)$$

Finally, we transform this result back to the original coordinate system. Using the definition of the expectation value:

$$\langle x'\rangle=\left\langle x-\frac{\beta}{2}\right\rangle=(x)-\left(\frac{\beta}{2}\right). \quad (29)$$

Since β/2 is a constant, its expectation value is just the constant itself, (β/2) = β/2. Therefore, we arrive at our final result:

$$0=(x)-\frac{\beta}{2}\Rightarrow(x)=\frac{\beta}{2} \quad (30)$$

This elegant result holds for *all* bound states (ground state and all excited states) of the system and provides a profound connection between the potential's asymmetry and the resulting quan- tum probability distribution. Its perfect verification in our numerical results (Fig. 3) serves as a powerful consistency check of our entire framework.

3. Numerical results and analysis

This section presents numerical findings on how depth U_0 and asymmetry β influence spectral properties. We analyze ground state energy, wavefunction structure, and excited states in a logical progression.

Ground state energy versus asymmetry

The most fundamental quantity of a bound system is its ground state energy, E_0. We begin by exploring how this energy level is affected by the introduction and increase of asymmetry. Figure 1 plots the numerically calculated ground state energy as a function of the asymmetry parameter β for three different values of the potential depth, U_0.

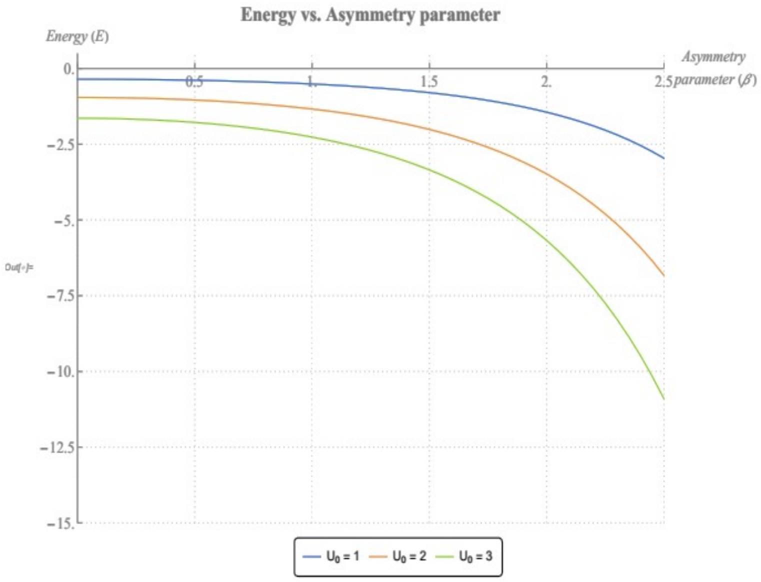

Figure 1: *The ground state energy E_0 as a function of the asymmetry parameter β for three fixed potential depths. Increasing the asymmetry consistently leads to a more negative (more tightly bound) energy state.*

The results are unambiguous: for any given potential depth, increasing β systematically lowers the ground state energy. This behavior aligns perfectly with our prediction from second- order perturbation theory and can be understood intuitively. The term $+\beta x$ in the exponent of the potential effectively deepens the well and shifts its minimum to the right, creating a stronger trapping region that results in a lower ground state energy.

Wavefunction localization and position expectation value

To understand the physical mechanism behind the energy drop, we visualize the potential well and the corresponding ground state wavefunction $\psi_0(x)$ in Figure 2.

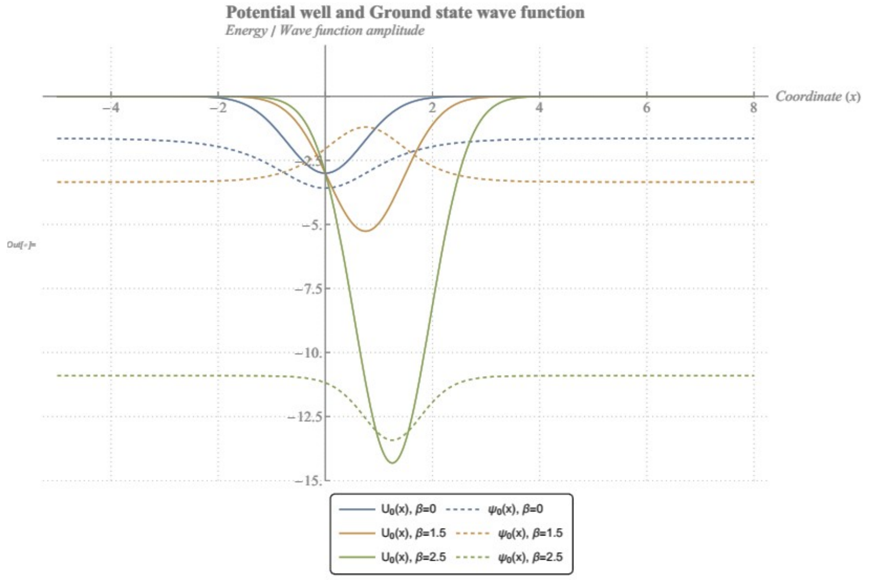

Figure 2: *The asymmetric potential well $V(x)$ (solid lines) and the ground state wavefunction $\psi_0(x)$ (dashed lines) for varying asymmetry. As β increases, the potential minimum shifts and deepens, causing the wavefunction to become more localized.*

Increasing β shifts the potential minimum and wavefunction rightward. Numerical results for $\langle x\rangle$across all bound states perfectly match the analytical relation $\langle x\rangle=\beta/2$(Fig. 3), validating our numerical algorithm and providing a crucial benchmark for non-analytical findings. Physically, this shift represents the redistribution of probability density. The robustness of this relation across the spectrum confirms that the particle's average position depends solely on the potential's geometry, independent of quantum numbers or nodal structure.

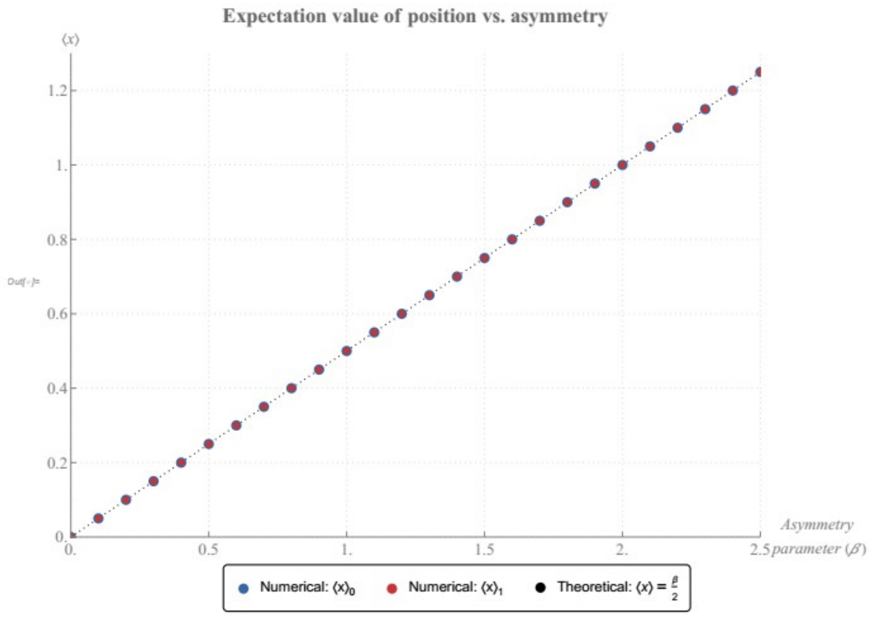

Figure 3: *The expectation value of position, $\langle x\rangle$, versus the asymmetry parameter β. The numerical data for the ground and first excited states (dots)*

perfectly match the theoretical relation $\langle x \rangle = \beta/2$ *(dotted line).*

The spectrum of excited states

Asymmetry significantly reduces the critical potential depth required for higher bound states (Fig. 4). The first excited state $\psi_1(x)$ and its node shift rightward (Fig. 5), a coordinated displacement essential for maintaining orthogonality during potential deformation. Being more spatially extended, $\psi_1(x)$ is less sensitive to well deepening than the ground state, causing the energy gap to widen with β (Fig. 6). This structural adaptation ensures the collective preservation of the eigenspectrum under the Hamiltonian's broken symmetry.

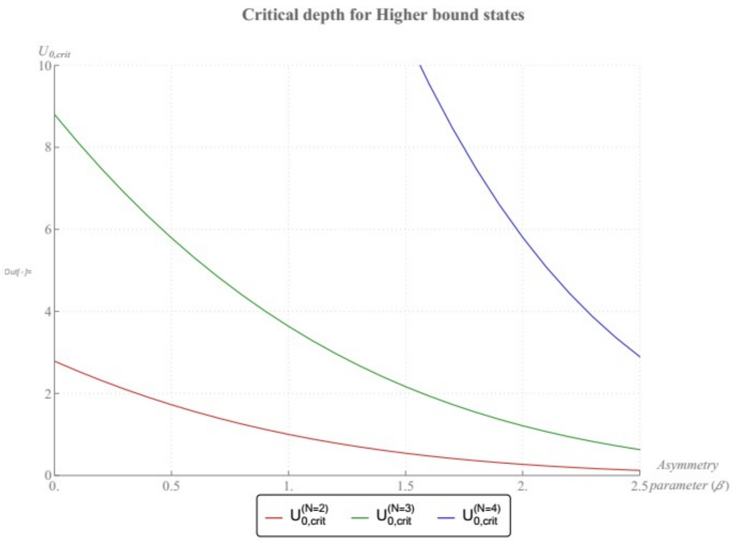

Figure 4: *The critical potential depth* $U_{0,crit}$ *required to support the second (*$N = 2$*), third (*$N = 3$*), and fourth (*$N = 4$*) bound states as a function of* β*. Asymmetry dramatically lowers the required depth.*

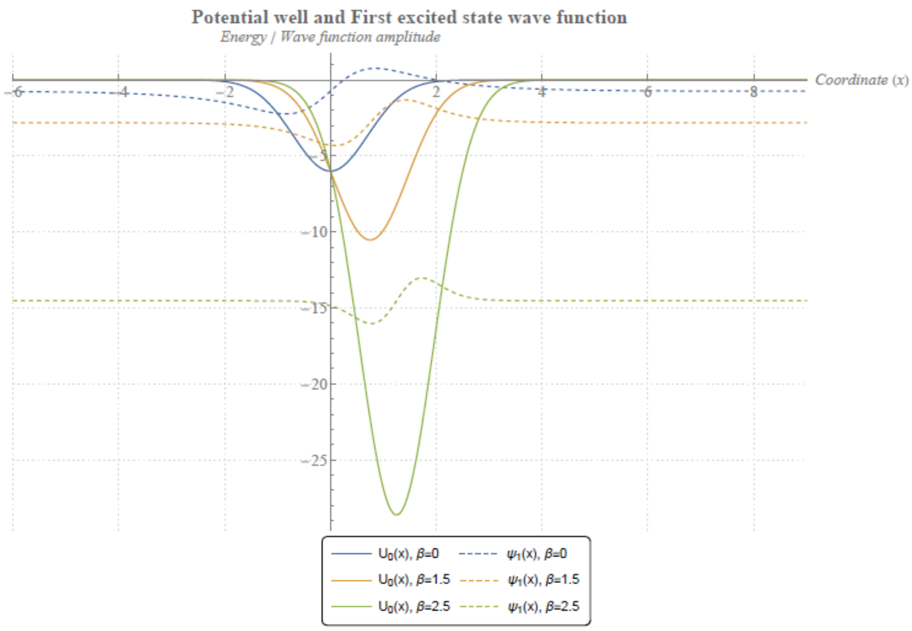

Figure 5: *The potential well* $V(x)$ *and the corresponding first excited state wavefunction* $\psi_1(x)$ *for varying asymmetry* β

Energy level splitting

Finally, we analyze the relationship between the energy levels. The energy difference $\Delta E = E_1 - E_0$ is a crucial quantity in spectroscopy. Figure 6 plots this energy gap as a function of β.

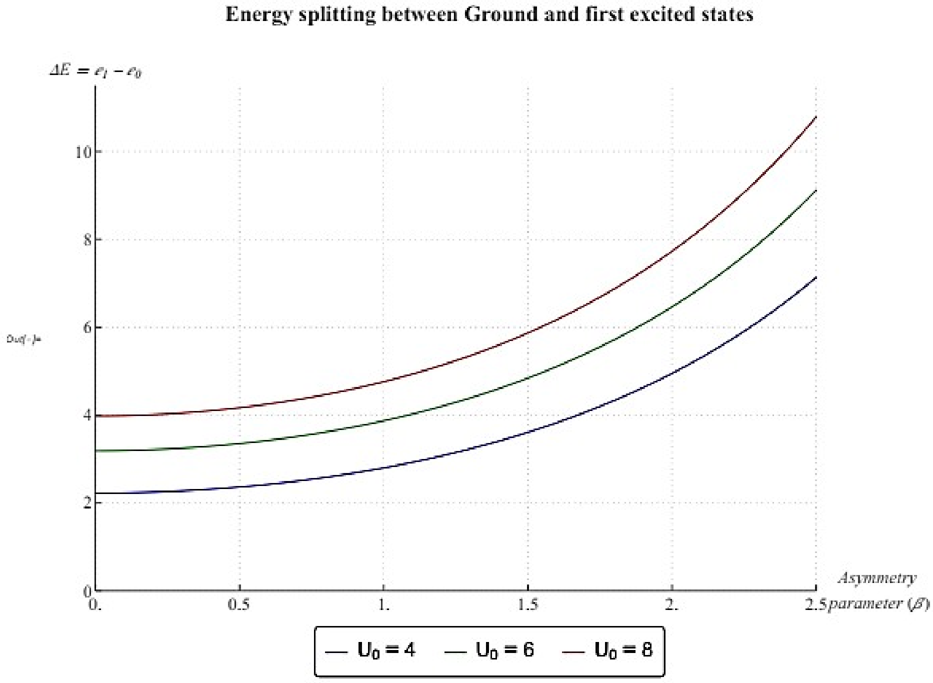

Figure 6: *The energy splitting $\Delta E = E_1 - E_0$ as a function of the asymmetry parameter* β*. For any given depth U_0, the energy gap increases with asymmetry.*

The analysis reveals that increasing asymmetry consistently increases the energy splitting. This is because asymmetry lowers the ground state energy E_0 more significantly than the first excited state energy E_1, thus widening the gap between them. This suggests that breaking a system's symmetry can fundamentally alter its spectroscopic signature.

4Discussion

The numerical results quantitatively demonstrate how asymmetry affects spectral properties. This section explores the physical interpretation and methodological implications of these findings, placing them in a broader context while outlining model limitations and future research directions.

Physical interpretation of the spectral properties

Asymmetry enhances binding by lowering energy levels through a competition between kinetic and potential energy governed by the Uncertainty Principle.

Spatial confinement increases kinetic energy, but the dominant decrease in potential energy lowers the total energy. The energy gap ΔE widens because the localized ground state is more sensitive to well deepening than the extended first excited state. Consequently, asymmetry provides an efficient mechanism for tuning spectroscopic properties and supporting bound states in shallow potentials where symmetric wells might fail.

Broader context and methodological implications.
Symmetry breaking is a dominant factor in determining the spectral landscape across various fields. In **scalar-tensor gravity**, our model acts as a "methodological laboratory" where β represents non-radial perturbations. Since asymmetry enhances binding (more negative eigenvalues), it suggests that spherically stable solutions may become unstable under asymmetric disturbances, aligning with "dynamical censorship" toward stable states like Kerr-Newman black holes [11-14]. In condensed matter, asymmetric wells (e.g., Stark effect in AlGaAs/GaAs) utilize our ⟨x⟩=β/2 and ΔE results for exciton control and optoelectronic device design . In **optics**, this model describes light propagation in asymmetric refractive index profiles, where mode shifts and altered propagation constants are key for designing directional couplers and mode-sorting devices.

Limitations and Future Directions

Future research should address current model idealizations. Extending the analysis to 3D potentials would enhance relevance for atomic systems, while studying asymmetric double-wells could clarify impacts on quantum tunneling. Crucially, applying our methodology to realistic effective potentials in scalar-tensor gravity, such as those in Bronnikov’s works [5, 7], will bridge the gap between this toy model and the frontier of gravitational research.

5.Conclusion

We investigated the spectral properties of the asymmetric Gaussian potential $V(x) = -U_0 \exp(-x^2+\beta x)$. Asymmetry β acts as a dominant binding mechanism, lowering energy levels and reducing the critical depth *U*0,*crit* for excited states. We verified the exact relation ⟨x⟩=β/2 and observed increased energy splitting ΔE. This robust framework provides essential intuition for complex Schrödinger-like systems, proving that symmetry breaking is a paramount necessity for understanding realistic physical configurations.

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