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KAPUTO KASR HOSILASI VA RIMAN–LYUVILL INTEGRALINI O'Z ICHIGA OLUVCHI BUZULADIGAN IKKINCHI TARTIBLI TENGLAMA UCHUN BOSHLANG'ICH-CHEGARAVIY MASALASI

Annotatsiya. Ushbu maqolada to'rtburchak sohada ikkita buzilish chiziqlariga ega bo`lgan va Kaputo kasr hosilasi va Riman–Lyuvill integralini o'z ichiga oluvchi xususiy hosilali differensial tenglamalar uchun boshlang'ich-chegara masala o'rganilgan. Masalaning yechimi yagonaligi energiya integrallari usuli bilan isbotlangan. Masalaning mavjudligi esa, Furye usuli bilan ko`rsatilgan.

Kalit so'zlar: *boshlang'ich-chegaraviy masalasi, Kaputo kasr hosilasi, buziladigan differensial tenglama.*

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INITIAL-BOUNDARY VALUE PROBLEM FOR A DEGENERATE SECOND ORDER EQUATION WITH FRACTIONAL CAPUTO DERIVATIVE AND RIEMANN-LIOUVILLE INTEGRAL

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Abstract. This article studies an initial-boundary value problem for a partial differential equation involving the Caputo fractional derivative and the Riemann–Liouville integral, with two lines of discontinuity in a rectangular domain. The uniqueness of the solution is proved using the method of energy integrals. The existence of the solution is demonstrated by means of the Fourier method.

Key words: initial-boundary problem, Caputo fractional derivative, degenerate differential equation.

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НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ ВЫРОЖДЕННОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ДРОБНОЙ ПРОИЗВОДНОЙ КАПУТО И ИНТЕГРАЛОМ РИМАНА-ЛИУВИЛЛЯ

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Аннотация. В данной статье исследована начально-краевая задача для дифференциального уравнения в частных производных, содержащего дробную производную Капуто и интеграл Римана–Лиувилля, с двумя линиями разрыва в прямоугольной области. Единственность решения задачи доказана с помощью метода энергетических интегралов. Существование решения показано с использованием метода Фурье.

Ключевые слова: начально-краевая задача, дробная производная Капуто, вырожденное дифференциальное уравнение.

Introduction

It is well known that the theory of differential equations has a long and rich history. Until the last quarter of the twentieth century, this theory primarily focused on differential equations of integer order. With the development of fractional (differential and integral) analysis in the late twentieth century, researchers began to study differential equations involving fractional derivatives. At present, numerous scientific articles have been published that address initial, boundary, and spectral problems for differential equations (both ordinary and partial) containing differential operators of fractional order in various forms (see, for example, [1]–[5], and the references therein). The books [6] and [7] have played a significant role in the development of this area.

Recently, there has been growing interest in the study of boundary value problems for second-order mixed-type equations involving fractional-order differential operators (see, for example, [8]–[14]).

In the aforementioned and other related works, only non-degenerate equations have been considered. However, both local and non-local boundary value problems for degenerate partial differential equations involving fractional derivatives of the unknown function remain largely unstudied. The investigation of boundary value problems for such equations is of great importance not only from a theoretical standpoint but also from a practical one, as these equations frequently arise in the mathematical modeling of various problems in gas and hydrodynamics, the theory of small surface bendings, mathematical biology, and other scientific fields.

Problem statement

Consider the following degenerate partial integro-differential equation in the domain $\Omega = \{(t,x): 0 < x < 1, 0 < t < T\}$:

$$ {}^{c}D_{0t}^{\alpha} u(t,x) + a I_{0t}^{\beta} u(t,x) + b u(t,x) = [x^{\alpha_1}(1-x)^{\beta_1} u_x(t,x)], \quad (1)$$

where $T, \alpha, \beta, \alpha_1, \beta_1, a, b$, are given real numbers, $T > 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \alpha_1 < 1$, $0 < \beta_1 < 1$, $u(t,x)$ - is an unknown function,

$$ {}^{c}D_{0t}^{\alpha} u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-z)^{-\alpha} u_z(z,x)\,dz $$

- fractional derivative in the sense of Caputo [15] of the function $u(t,x)$, with respect to the argument t,

$$I_{0t}^{\beta}u(t,x)=\frac{1}{\Gamma(\beta)}\int_0^t (t-z)^{\beta-1}u_z(z,x)dz$$

-fractional integral in the Riemann-Liouville [15] sense of the function $u(t,x)$ with respect to the argument t.

Let us study the following initial-boundary value problem for equation (1).

Problem A**.** Find a function $u(t,x)$ with the following properties:

1) $u(t,x)$, $x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)\in C(\overline{\Omega})$, $\left[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)\right]_x$, ${}^cD_{0t}^{\alpha}u(t,x)\in C(\Omega)$;

2) in the domain Ω satisfies equation (1);

3) The following boundary conditions hold on $\partial\Omega$:

$$u(0,t)=0,\ \left[(1-x)^{\beta_1}u_x(t,x)\right]|_{x=1}=0, t\in[0,T]; \tag{2}$$

$$u(0,x)=\varphi(x), x\in[0,1], \tag{3}$$

where $\varphi(x)$ is a given continuous function.

Uniqueness of the solution problem A

Before proceeding to prove the uniqueness of the solution to the problem A, we present the following auxiliary lemmas.

Lemma 1. *If* $V(0,x)=0$ *and* $\alpha\in(0,1)$*, then the following inequality holds [19]:*

$$I(x)=\int_0^t V(t,x)\,{}^cD_{0t}^{\alpha}V(t,x)dt\geq 0.$$

Lemma 2. *If* $\beta\in(0,1)$*, then the following inequality holds [19]:*

$$I(x)=\int_0^t V(t,x) I_{0t}^{\beta} V(t,x)\,dt \geq 0$$

Theorem 1. *If* $a \geq 0, b \geq 0,$ *then problem* A *cannot have more than one solution.*

Proof. Let us assume that there are two solutions $u_1(t,x)$ and $u_2(t,x)$ of the problem A. Their difference will be denoted by $u(t,x)$. Then the function $u(t,x)$ is a solution of equation (1) under homogeneous boundary conditions.

We multiply equation (1) by the function $u(t,x)$ and integrate over the domain Ω:

$$\int_0^1 dx \int_o^T u(x,t)\, {}^c D_{0t}^{\alpha} u(x,t)\,dt + a\int_0^t dx \int_0^T u(x,t) I_{0t}^{\beta} u(t,x)\,dt + b\int_0^1 dx \int_0^T u^2(t,x)\,dt =$$

$$= \int_0^T dt \int_0^1 u(t,x) \left[x^{\alpha_1} (1-x)^{\beta_1} u_x(t,x) \right]_x dx$$

Applying the rule of integration by parts to the inner integral of the last term and taking into account conditions (2), we have

$$\int_0^T dt \int_0^1 x^{\alpha_1} (1-x)^{\beta_1} u_x^2(t,x)\,dx =$$

$$= \int_0^1 dx \int_{0-}^T u(t,x)\, {}^c D_{0t}^{\alpha} u(t,x)\,dt - a\int_0^1 dx \int_0^T u(t,x) I_{0t}^{\beta} u(t,x)\,dt - b\int_0^1 dx \int_0^T u^2(t,x)\,dt$$
,

from which, by virtue of Lemma 1, 2 and $a \geq 0,\ b \geq 0$ the equality follows:

$$\int_0^T dt \int_0^1 x^{\alpha_1} (1-x)^{\beta_1} u_x^2(t,x)\,dx = 0$$

Therefore, $x^{\alpha_1}(1-x)^{\beta_1}u_x^2(t,x)=0$ i.e. $u_x(t,x)=0$ for $(t,x)\in\Omega$. Then $u(t,x)=\tau(t)$, where $\tau(t)$ - is an arbitrary function. Satisfying this function with the conditions $u(t,0)=0$, and taking into account, we obtain $u(t,x)\equiv 0,\ (t,x)\in\Omega$. Since $u(t,x)\in C(\overline{\Omega})$, then $u(t,x)\equiv 0$, $(t,x)\in\Omega$. Then, u $u_1(t,x)=u_2(t,x)$, $(t,x)\in\overline{\Omega}$

The proof of Theorem 1 is complete.

Study of the spectral problem

With the formal application of the Fourier method to the stated problem Apq, the following spectral problem arises: find those values of the parameter λ for which nontrivial solutions of the equation

$$Mv\equiv-\left[x^{\alpha_1}(1-x)^{\beta_1}v'(x)\right]'=\lambda v(x),\ 0<x<1, \tag{4}$$

exist, satisfying the following conditions:

$$\begin{cases} v(x),\ x^{\alpha_1}(1-x)^{\beta_1}v'(x)\in C[0,1]; \\ v(0)=0,\ \left[(1-x)^{\beta_1}v'(x)\right]|_{x=1}=0 \end{cases}. \tag{5}$$

It is easy to show that the spectral problem {(4),(5)} has a solution $v(x)\not\equiv 0$ only for, $\lambda>0$ and it is equivalent to the following integral equation with symmetric kernel:

$$v(x)=\lambda\int_0^1 G(x,s)v(s)ds \tag{6}$$

where the function $G(x,s)$ is defined by formula

$$G(x,s)=\begin{cases}\int\limits_0^x \frac{dz}{z^{\alpha_1}(1-z)^{\beta_1}},\ x\le s;\\ \int\limits_0^s \frac{dz}{z^{\alpha_1}(1-z)^{\beta_1}},\ x\ge s,\end{cases}$$

where $\sigma_1=\left[\Gamma(1-\alpha_1)\Gamma(1-\beta_1)\right]/\Gamma(2-\alpha_1-\beta_1)$.

Then, according to the theory of integral equations, the integral equation (6) therefore, the problem (4),(5) has a countable number of eigenvalues $0<\lambda_1\le\lambda_2\le\lambda_3\le...\le\lambda_k,...$ condensing at infinity, and the corresponding eigenfunctions $v_1(x),\ v_2(x)v_3(x),...,v_k(x),...$ form orthonormal system in the space $L_2(0,1)$.

The following lemmas can be proven on their own. Therefore, we present them without proof.

Lemma 3. *Let the function* $h(x)$ *satisfy the following conditions:*

$$h(x)\in C[0,1] \tag{7}$$

$$x^{\alpha_1}(1-x)^{\beta_1}h'(x)\in C[0,1], \tag{8}$$

$$h(0)=0, \tag{9}$$

$$\left[(1-x)^{\beta_1}h'(x)\right]|_{x=1}=0, \tag{10}$$

$Mh(x)\in L_2(0,1)$*. Then, on the segment* $[0,1]$ *it can be expanded into an absolutely and uniformly convergent series by the system of eigen functions* $\left\{v_k(x)\right\}_{k=1}^{\infty}$ *of the problem (4), (5):*

$$h(x)=\sum_{k=1}^{+\infty} h_k v_k(x),$$

where $h_k=\int_0^1 h(x)v_k(x)dx$.

Lemma 4. The following series converge uniformly on the segment $[0,1]$:

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[x^{\alpha}(1-x)^{\beta}v'_k(x)\right]^2}{\lambda_k^2}.$$

Lemma 5. *If function* $h(x)$ *the conditions (7), (9),* $x^{\alpha_1/2}(1-x)^{\beta_1/2}\left[Mh(x)\right]' \in L_2(0,1)$ *hold true, then the inequality*

$$\sum_{k=1}^{+\infty}\lambda_k h_k^2 \le \int_0^1 x^{\alpha_1}(1-x)^{\beta_1}\left[h'(x)\right]^2 dx,$$

is valid, where $h_k=\int_0^1 h(x)v_k(x)dx$.

Lemma 6. *If function h(x) the conditions (7)-(10), and* $Mh(x)$ *function conditions (7), (9)* $x^{\alpha_1/2}(1-x)^{\beta_1/2}\left[Mh(x)\right]' \in L_2(0,1)$ *hold true, then the inequality*

$$\sum_{k=1}^{+\infty}\lambda_k^3 h_k^2 \le \int_0^1 x^{\alpha_1}(1-x)^{\beta_1}\left\{\left[Mh(x)\right]'\right\}^2 dx,$$

is valid, where $h_k=\int_0^1 h(x)v_k(x)dx$.

Uniqueness and stability of a solution to problem A

The solution to the problem A is formally sought in the form

$$u(x,t)=\sum_{k=1}^{+\infty}u_k(t)v_k(x), \tag{11}$$

where $v_k(x),\ k\in N$ are the eigen functions of the problem (4), (5) and $u_k(t),\ k\in N$ are the unknown functions to be determined.

Substituting (11) into (1) and (3), with respect to $u_k(t),$ we obtain the following problem:

$${}^{c}D_{0t}^{\alpha}u_k(t)+aI_{0t}^{\beta}u_k(t)+(\lambda_k+b)u_k(t)=0,\ 0<t<T, \tag{12}$$

$$u_k(0)=\varphi_k, \tag{13}$$

where $\varphi_k=\int_0^1\varphi(x)v_k(x)dx,\ k\in N$.

To solve this problem, we apply the operator I α 0t to equation (12) and, using the properties $I_{0t}^{\alpha}\,{}^{c}D_{0t}^{\alpha}u_k(t)=u_k(t)-u_k(0)$ and $I_{0t}^{\alpha}I_{0t}^{\beta}u_k(t)=I_{0t}^{\alpha+\beta}u_k(t)$ and condition (13), we obtain the Volterra integral equation of the second kind:

$$u_k(t)+aI_{0t}^{\alpha+\beta}u_k(t)+(\lambda_k+b)I_{0t}^{\alpha}u_k(t)=\varphi_k, \tag{14}$$

To solve equation (14), we apply the method of successive approximations:

$$u_{k,m}(t)=u_{k,0}(t)-aI_{0t}^{\alpha+\beta}u_{k,m}(t)-(\lambda_k+b)I_{0t}^{\alpha}u_{k,m}(t),\ m=1,2,3,...,\ u_{k,0}(t)=\varphi_k. \tag{15}$$

Using formula $I_{0t}^{\alpha}I_{0t}^{\beta}u_k(t)=I_{0t}^{\alpha+\beta}u_k(t)$, we calculate $u_{k,m}(t)$:

$$u_{k,m}(t)=\sum_{n=0}^{m}\sum_{j=0}^{n}C_n^j(-a)^{n-j}(-\lambda_k-b)^j I_{0t}^{\beta(n-j)+\alpha j}u_{k,0}(t), \tag{16}$$

where $\gamma = \alpha + \beta$, $C_n^j = \dfrac{n!}{j!(n-j)!}$.

According to the theory of integral equations [18], if there exists $\lim\limits_{m \to +\infty} u_{k,m}(t)$ uniformly in t, then its limit function is a solution of the integral equation (14).

Passing to the limit at $m \to +\infty$ in (16) and substituting the expression $u_{k,0}(t)$, we obtain a solution to equation (14) in the form

$$u_k(t) = \varphi_k \sum_{n=0}^{+\infty} \sum_{j=0}^{n} C_n^j (-a)^{n-j} (-\lambda - b)^j I_{0t}^{\gamma(n-j)+\alpha j}(1) \quad , \tag{17}$$

Using the formula $I_{0t}^{\alpha_2}(1) = \dfrac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)}$, we can write (17) in the form

$$u_k(t) = \varphi_k \sum_{n=0}^{+\infty} \sum_{j=0}^{n} C_n^j (-a)^{n-j} (-\lambda_k - b)^j \frac{t^{\gamma(n-j)+\alpha j}}{\Gamma(\gamma(n-j)+\alpha j+1)} \tag{18}$$

We will prove that the series (18) is convergent a little later.

Using the formula [20] $\sum\limits_{k=0}^{+\infty} \sum\limits_{n=0}^{k} B_{k,n} = \sum\limits_{n=0}^{+\infty} \sum\limits_{k=n}^{+\infty} B_{k,n}$, we can write (18) in the form

$$u_k(t) = \varphi_k \sum_{j=0}^{+\infty} \sum_{n=j}^{+\infty} C_n^j (-a)^{n-j} (-\lambda_k - b)^j \frac{t^{\gamma(n-j)+\alpha j}}{\Gamma(\gamma(n-j)+\alpha j+1)}.$$

If we introduce the notation $n = j + m$ here, we can write the last equation in the form

$$u_k(t) = \varphi_k \sum_{j=0}^{+\infty} \sum_{m=0}^{+\infty} C_{m+j}^j (-a)^m (-\lambda_k - b)^j \frac{t^{\gamma m+\alpha j}}{\Gamma(\gamma m+\alpha j+1)}.$$

This equation can be written using the Mittag-Leffer function of two variables as follows:

$$u_k(t)=\varphi_k E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\left|\begin{matrix}-\alpha t^{\gamma}\\ -(\lambda_k+b)t^{\alpha}\end{matrix}\right.\right),\ t\in[0,T], \tag{19}$$

where E_2 -Mittag-Leffer function of two variables [21]

$$E_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right)=$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1+\alpha_3 m+\beta_2 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_4 m)}\frac{y^n}{\Gamma(\delta_3+\beta_3 m)}, \tag{20}$$

$x,y,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3\in R$,

$$(\gamma_1)_{\alpha_1 m+\beta_1 n}=\frac{\Gamma(\gamma_1+\alpha_1 m+\beta_1 n)}{\Gamma(\gamma_1)}.$$

If $\alpha_3+\alpha_4-\alpha_1-\alpha_2>0$ and $\beta_2+\beta_3-\beta_1>0$, then the series (20) is absolutely and uniformly convergent for arbitrary $\forall(x,y)$ [22].

It is not difficult to show that the following formulas are valid for the function E_2:

$$I_{0t}^{\beta}\left(E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\left|\begin{matrix}-\alpha t^{\gamma}\\ -(\lambda_k+b)t^{\alpha}\end{matrix}\right.\right)\right)=t^{\beta}E_2\left(\begin{matrix}1,1,1;1,0;\\ 1+\beta,\gamma,\alpha;1,1;1,1;\end{matrix}\left|\begin{matrix}-\alpha t^{\gamma}\\ -(\lambda_k+b)t^{\alpha}\end{matrix}\right.\right),$$

$${}^{c}D_{0t}^{\alpha}\left(E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\left|\begin{matrix}-\alpha t^{\gamma}\\ -(\lambda_k+b)t^{\alpha}\end{matrix}\right.\right)\right)=-(\lambda_k+b)E_{\alpha,1}\left(-(\lambda_k+b)t^{\alpha}\right)-$$

$$-\alpha t^{\gamma-\alpha}E_2\left(\begin{matrix}2,1,1;1,0;\\ 1+\gamma-\alpha,\gamma,\alpha;2,1;1,1;\end{matrix}\left|\begin{matrix}-\alpha t^{\gamma}\\ -(\lambda_k+b)t^{\alpha}\end{matrix}\right.\right),$$

$$E_2\left(\begin{array}{c|c} 1,1,1;1,0; & -\alpha t^{\gamma} \\ 1,\gamma,\alpha;1,1;1,1; & -(\lambda_k+b)t^{\alpha}\end{array}\right)=$$

$$=E_{\alpha,1}\left(-(\lambda_k+b)t^{\alpha}\right)-\alpha t^{\gamma}E_2\left(\begin{array}{c|c} 2,1,1;1,0; & -\alpha t^{\gamma} \\ \gamma+1,\gamma,\alpha;2,1;1,1; & -(\lambda_k+b)t^{\alpha}\end{array}\right),$$

$$E_2\left(\begin{array}{c|c} 1,1,1;1,0; & -\alpha t^{\gamma} \\ 1+\gamma-\alpha,\gamma,\alpha;2,1;1,1; & -(\lambda_k+b)t^{\alpha}\end{array}\right)-E_2\left(\begin{array}{c|c} 2,1,1;1,0; & -\alpha t^{\gamma} \\ 1+\gamma-\alpha,\gamma,\alpha;2,1;1,1; & -(\lambda_k+b)t^{\alpha}\end{array}\right)=$$

$$=(\lambda_k+b)t^{\alpha}E_2\left(\begin{array}{c|c} 2,1,1;1,0; & -\alpha t^{\gamma} \\ \gamma+1,\gamma,\alpha;2,1;1,1; & -(\lambda_k+b)t^{\alpha}\end{array}\right),$$

where $E_{\alpha_2,\beta_2}(z)$ - Mittag-Leffler function [15]:

$$E_{\alpha_2,\beta_2}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha_2 k+\beta_2)},\ \alpha_2>0.$$

Using these formulas, it can be proven that (19) is a solution to problem (12), (13).

It can be shown directly that (19) can be written in the form

$$u_k(t)=\varphi_k\int_0^1\int_0^{+\infty}\frac{e^{-\eta}}{\sqrt{\xi(1-\xi)}}e_{1,-\alpha}^{1,1/2}\left[-\alpha t^{\gamma}(1-\xi)^{\gamma}\eta\right]$$

$$e_{1,-\gamma}^{1,1/2}\left[-(\lambda_k+b)t^{\alpha}\xi^{\gamma}\eta\right]d\eta d\xi,\ t\in[0,T] \tag{21}$$

where $e_{\alpha_3,\beta_3}^{\mu_3,\delta_3}(z)$ - Wright-type function [5]:

$$e_{\alpha_3,\beta_3}^{\mu_3,\delta_3}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha_3 k+\mu_3)\Gamma(\delta_3-\beta_3 k)},\ \alpha_3>0,\ \alpha_3>\beta_3. \tag{22}$$

If $z<0$, $\mu_3 \geq 0$, $\delta_3 \geq \beta_3$ the following inequality holds for function (22) [see page 87 in [5]:

$$e^{\mu_3,\delta_3}_{\alpha_3,\beta_3}(z) < \frac{1}{\Gamma(\mu_3)\Gamma(\delta_3)} \tag{23}$$

If we say $a \geq 0$, $b \geq 0$ and note that $\lambda_k > 0$, then using inequality (23), we obtain the following inequality for (21):

$$|u_k(t)| \leq |\varphi_k|, \; t \in [0,T]. \tag{24}$$

Substituting (19) into (11), we find a formal solution to the problem A_{pq}:

$$u(t,x) = \sum_{k=1}^{+\infty} \varphi_k E_2 \left(\begin{matrix} 1,1,1;1,0; \\ 1,\gamma,\alpha;1,1;1,1; \end{matrix} \middle| \begin{matrix} -\alpha t^{\gamma} \\ -(\lambda_k + b)t^{\alpha} \end{matrix} \right) v_k(x), \; \gamma = \alpha + \beta \tag{25}$$

Theorem 2. *If* $a>0$, $b>0$, $p \neq q$ *and function* $\varphi(x)$ *satisfy the conditions of Lemma 6, then of the series (25) determines the unique solution of the problem* A_{pq}.

Proof. To prove the theorem, it is sufficient to prove that the series (25) and the series corresponding to the functions $x_1^{\alpha}(1-x)^{\beta_1} u_x(x,t)$ converge uniformly in $\overline{\Omega}$, and the series corresponding to the functions $\left[x^{\alpha_1}(1-x)^{\beta_1} u_x(t,x)\right]_x$ ${}^{c}D^{\alpha}_{0t}u(t,x) \in C(\Omega)$ converges uniformly on any compact $\Omega_1 \subset \Omega$.

Consider the series corresponding to the function $\left[x^{\alpha_1}(1-x)^{\beta_1} u_x(t,x)\right]_x$.

Using inequality (24), then from (25) follows the inequality

$$\left|\left[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)\right]_x\right| \le \sum_{k=1}^{+\infty}|\varphi_k|\left|\left[x^{\alpha_1}(1-x)^{\beta_1}v_k'(x)\right]'\right|.$$

Hence, by virtue of equation (4), on any compact set $\Omega_1 \subset \Omega$, we have

$$\left|\left[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)\right]_x\right| \le \sum_{k=1}^{+\infty}|\lambda_k\varphi_k v_k(x)|. \tag{26}$$

Based on the Cauchy-Schwartz inequality, we have

$$\sum_{k=1}^{+\infty}|\lambda_k\varphi_k v_k(x)| = \sum_{k=1}^{+\infty}\left|\lambda_k^{3/2}\varphi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le \left(\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right)^{1/2}.$$

Here, by Lemmas 6 and 4, the series on the right-hand side converge uniformly with respect to $x\in[0,1]$. Then the series on the left-hand side converges uniformly with respect to $x\in[0,1]$. Therefore, the series (26) converges absolutely and uniformly on the compact set $\Omega_1 \subset \Omega$. The convergence of the remaining series are proved similarly. Theorem 2 has been proved.

Theorem 3. *If the conditions of Theorem 2 are met, then the following estimate is true for the solution of problem* A*:*

$$\|u(x,t)\|_{C(\overline{\Omega})} \le C_1\|\varphi'(x)\|_{L_2,r} \tag{27}$$

where $C_1 = \sup_{[0,1]}\sqrt{G(x,x)}$, $\|f(x)\|_{L_2,r} = \int_0^1 r(x)f^2(x)dx$, $r(x) = x^{\alpha_1}(1-x)^{\beta_1}$.

Proof. By virtue of Lemma 4, Lemma 5 and the inequality $|u_k(t)| \le |\varphi_k|$, the following inequalities hold:

$$|u(x,t)| = \left|\sum_{k=1}^{+\infty} u_k(t) v_k(x)\right| \le \left|\sum_{k=1}^{+\infty} |\varphi_k| |v_k(x)|\right| = \left|\sum_{k=1}^{+\infty} \sqrt{\lambda_k}\varphi_k\right| \left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le$$

$$\le \left|\sum_{k=1}^{+\infty} \lambda_k \varphi_k^2 \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}\right|^{1/2} \le \left[G(x,x)\int_0^1 x^{\alpha}(1-x)^{\beta}\left[\varphi'(x)\right]^2 dx\right]^{1/2} \le C_1 \|\varphi'(x)\|_{L_{2,\rho}(0,1)}$$

From this, inequality (27) follows. Theorem 3 is proven.

Conclusion

In this paper, a non-local initial-boundary value problem for a degenerate secondorder equation with a fractional Caputo derivative and a Riemann–Liouville integral is considered in a rectangular domain. Using the method of separation of variables, a solution to the problem is obtained in the form of a series that converges absolutely and uniformly in the closure of the domain under consideration. In addition, the uniqueness of the solution and its continuous dependence on the given functions are established.

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