Rizaqulov Shakhzod Erkin o‘g‘li
Muhammad al-Khwarizmi Tashkent University
of Information Technologies assistant
E-mail: shaxzodrizaqulov2630@gmail.com
Olimov Umrbek Rashidovich
Muhammad al-Khwarizmi Tashkent University
of Information Technologies Associate Professor
E-mail: umrbek.olimov.92@mail.ru

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STRONG LAWS OF LARGE NUMBERS FOR ASSOCIATED RANDOM VARIABLES IN c_0 AND l_p SPACES

Abstract: *This paper investigates strong laws of large numbers for dependent random elements in infinite-dimensional Banach spaces. Positively and negatively associated random variables in Hilbert spaces, c_0 and l_p spaces are studied. Conditions for almost sure convergence are obtained using covariance inequalities and probabilistic techniques.*

Keywords: *associated random variables, strong law of large numbers, Hilbert space, Banach space, dependence.*

Ризакулов Шахзод Эркин ўғли
Ташкентский университет информационных технологий имени
Мухаммада аль-Хоразмий ассистент
Олимов Умрбек Рашидович
Ташкентский университет информационных технологий имени
Мухаммада аль-Хоразмий доцент

УСИЛЕННЫЙ ЗАКОНЫ БОЛЬШИХ ЧИСЕЛ ДЛЯ АССОЦИИРОВАННЫХ СЛУЧАЙНЫХ ВЕЛИЧИН В ПРОСТРАНСТВАХ c_0 И l_p

Abstract: В данной работе исследуются усиленный закон больших чисел для зависимых случайных элементов со значениями в бесконечномерных банаховых пространствах. Рассматриваются положительно и отрицательно ассоциированные случайные величины в пространствах Гильберта, а также в пространствах c_0 и l_p. Получены условия почти наверное сходимости с использованием ковариационных неравенств и вероятностных методов.

Ключевые слова: ассоциированные случайные величины, усиленный закон больших чисел, пространство Гильберта, банахово пространство,зависимость.

Rizaqulov Shaxzod Erkin o‘g‘li

Muhammad al-Xorazmiy nomidagi Toshkent axborot texnologiyalari universiteti assistenti

Olimov Umrbek Rashidovich

Muhammad al-Xorazmiy nomidagi Toshkent axborot texnologiyalari universiteti dotsenti

c_0 VA l_p VAZOLARIDA ASSOTSIRLANGAN TASODIFIY MIQDORLAR UCHUN KUCHAYTIRILGAN KATTA SONLAR QONUNI

Annotatsiya: Ushbu maqolada cheksiz o‘lchamli Banax fazolarida qiymat qabul qiluvchi bog‘liq tasodifiy elementlar uchun kuchaytirilgan katta sonlar qonunlari o‘rganiladi. Gilbert fazolari, shuningdek c_0 va l_p fazolaridagi musbat va manfiy assotsiatsiyalangan tasodifiy miqdorlar ko‘rib chiqiladi. Deyarli yaqinlashish shartlari kovariatsion tengsizliklar va ehtimollik nazariyasi usullari yordamida keltirib chiqariladi.

Kalit so'zlar: assotsirlangan tasodifiy miqdorlar, kuchaytirilgan katta sonlar qonuni, Gilbert fazosi, Banax fazosi, bog‘liqlik.

1. INTRODUCTION

The asymptotic behavior of associated random variables has been widely investigated in probability theory, and numerous researchers have studied various limit theorems related to such dependent sequences (see [1,2] and the references cited therein). In particular, several results concerning laws of large numbers for associated random variables have been obtained in previous studies. In this paper we will consider a separable Hilbert space H (with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|=\sqrt{(\cdot,\cdot)}$), c_0 space (a space of all sequences $x=\left(x^{(1)},x^{(2)},\ldots\right)$ of real numbers tending to zero, with a norm $\|x\|=¿1\leq i\leq\infty\left|x^{(i)}\right|$ and l_p ($1\leq p\leq 2$) spaces (a space of all sequences $x=\left(x^{(1)},x^{(2)},\ldots\right)$ of real numbers such that $\sum_{i=1}^{\infty}\left|x^{(i)}\right|^p<\infty$, with a norm $\|x\|=\left(\sum_{i=1}^{\infty}\left|x^{(i)}\right|^p\right)^{1/p}$).

We now present the definitions of positively and negatively associated sequences of random variables.

Definiton 1 We say that a sequence of random variables $\{X_1,\ldots,X_n\}$ is positively associated if for any coordinatewise nondecreasing functions $f,g:R^n\rightarrow R$ the following inequality holds

$$cov\left(f\left(X_1,\ldots,X_n\right),g\left(X_1,\ldots,X_n\right)\right)\geq 0$$

when this covariance exists.

Furthermore, an infinite sequence $\{X_n,n\geq 1\}$ is said to be positively associated if every finite subcollection of this sequence possesses the above property.

Definition 2 A finite family of random variables $\{X_1,\ldots,X_n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1,2,\ldots,n\}$ and any coordinatewise nondecreasing functions f on R^A and g on R^B

$$cov\left(f\left(X_j,j\in A\right),g\left(X_l,l\in B\right)\right)\leq 0\ ,$$

whenever f and g are such that the covariance exists.

The sequence of random variables $\{X_n,n\geq 1\}$ is negatively associated if any finite subsequence is negatively associted.

Our aim is to prove strong laws of large numbers for associated random variables with values in above mentioned infinite dimensional spaces.

In [3-6], authors consider coordinatewise negatively associated random variables with values in H . For the strong law of large numbers for coordinatewise mixing random variables with values in l_p $(1 \le p \le 2)$ see [7].

Denote by $\{e_i, i \ge 1\}$ a basis of all spaces H, c_0, l_p $(1 \le p \le 2)$. In the case of H $\{e_i, i \ge 1\}$ means an orthonormal basis, in the cases of c_0 and l_p $(1 \le p \le 2)$, $\{e_i, i \ge 1\}$ is standard basis i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots)$ where the $i-¿$ th component is equal to 1 and others are zero.

In all above spaces we can represent a random variable X as

$$X = \sum_{i=1}^{\infty} X^{(i)} e_i$$

In next two definitions by $\{X_n, n \ge 1\}$, we denote a sequence of random variables with values in one of the three above mentioned spaces.

Definition 3 We say that a sequence $\{X_n, n \ge 1\}$ is coordinatewise negatively associated (CNA) if for any i, a sequence $\{X_n^i, n \ge 1\}$ is negatively associated.

Definition 4 We say that a sequence $\{X_n, n \ge 1\}$ is coordinatewise positively associated (CPA) if for any i, a sequence $\{X_n^i, n \ge 1\}$ is positively associated.

2. MAIN RESULTS

In that follows we assume that $E X_k = 0, k = 1, 2, \ldots$ and denote $S_n = \sum_{i=1}^{n} X_i$.

The following theorems are our main results.

Theorem 1 Let $\{X_n, n \ge 1\}$ be a CPA sequence of random variables with values in c_0 satisfiying

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{n^2} cov\left(X_n^{(i)}, S_n^{(i)}\right).$$

Then as $n \to \infty$

$$\frac{S_n}{n} \to 0 \text{ a.s.}$$

The geometrical structure of c_0 space differs from, for instance, Hilbert space and there are some unusual examples concerning law of large numbers in this space, see [2] Example in 7.11.

Theorem 2 Let $\{X_n, n \geq 1\}$ be a CAN sequence of random variables with values in l_p satisfying

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \left(\frac{1}{n^2} \sum_{j=1}^{n} E \left| X_j^{(i)} \right|^2 \right)^{\frac{p}{2}} < \infty .$$

Then as $n \to \infty$

$$\frac{S_n}{n} \to 0 \text{ a.s.}$$

3. PROOFS

Proof of Theorem 1. We will prove the theorem in two steps.

Step1. We will show that as $n \to \infty$

$$\frac{S_{2^n}}{2^n} \to 0 \text{ a.s.} \tag{1}$$

We use the following inequality

$$E(S_{2^n}^k)^2 = \sum_{i,j=1}^{2^n} cov(X_i^{(k)}, X_j^{(k)}) \leq 2 \sum_{i=1}^{2^n} cov(X_i^{(k)}, S_i^{(k)}), \text{ see [1]} \tag{2}$$

From Chebyshev's inequality we get the following:

$$\sum_{n=1}^{\infty} P\left(\|S_{2^n}\| > 2^n \varepsilon \right) \leq \sum_{n=1}^{\infty} \frac{1}{4^n \varepsilon^2} E \|S_{2^n}\|^2 \leq \sum_{n=1}^{\infty} \frac{1}{4^n \varepsilon^2} E \left(\underset{k \geq 1}{¿} \left| S_{2^n}^{(k)} \right| \right)^2$$

$$\leq \frac{2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{4^n} \sum_{k=1}^{\infty} \sum_{i=1}^{2^n} cov\left(X_i^{(k)}, S_i^{(k)} \right) \leq \frac{2}{\varepsilon^2} \sum_{k=1}^{\infty} \sum_{i=1}^{2^n} \left(\sum_{n \geq \log_2 i} \frac{1}{4^n} \right) cov\left(X_i^{(k)}, S_i^{(k)} \right) \leq \frac{8}{3\varepsilon^2} \sum_{k=1}^{\infty} \sum_{i=1}^{2^n} cov\left(X_i^{(k)}, S_i^{(k)} \right).$$

Now Borel-Cantelli lemma implies (1).

Step 2. It is easy to see that for $2^n < m \leq 2^{n+1}$

$$\left\| \frac{S_m}{m} - \frac{S_{2^n}}{2^n} \right\| \leq \frac{\|S_m - S_{2^n}\|}{2^n} + \frac{\|S_{2^n}\|}{2^n} \quad \text{a.s.} \tag{3}$$

It suffices to prove that as $n \to \infty$

$$2^{-n} \max_{2^n < m \leq 2^{n+1}} \|S_m - S_{2^n}\| \to 0 \text{ a.s.}$$

We now use the following lemma.

Lemma 1.[1] Let $X_n, n \in N$ be centered, square- integrable and positively associated real-valued random variables.

Then $E M_n^2 \leq var\, S_n = E S_n^2$ for every $n \in N$ here $M_n = max(S_1, S_2, \ldots, S_{1n})$.

Using Chebyshev's inequality and Lemma 1 we obtain

$$\sum_{n=1}^{\infty} P\left(\max_{2^n<m\le 2^{n+1}} \|S_m - S_{2^n}\| > 2^n \varepsilon\right) \le \sum_{n=1}^{\infty} \frac{1}{4^n \varepsilon^2} E\left(\max_{2^n<m\le 2^{n+1}} \|S_m - S_{2^n}\|\right)^2 = \text{¿¿}$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n \varepsilon^2} E \max_{2^n<m\le 2^{n+1}} \left(\text{¿}_{k\ge 1} \left|S_{2^{n+1}}^{(k)} - S_{2^{n+1}}^{(k)}\right|\right)^2 \le \sum_{n=1}^{\infty} \frac{1}{4^n \varepsilon^2} \sum_{k=1}^{\infty} E \max_{2^n<m\le 2^{n+1}} \left|S_{2^{n+1}}^{(k)} - S_{2^{n+1}}^{(k)}\right|^2$$

$$\le \sum_{n=1}^{\infty} \frac{1}{4^n \varepsilon^2} \sum_{k=1}^{\infty} E\left(S_{2^n}^{(k)}\right)^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{4^n \varepsilon^2} E\left(S_{2^n}^{(k)}\right)^2 < \infty .$$

According to the Borel-Cantelli lemma as $n \to \infty$

$$2^{-n} \max_{2^n<m\le 2^{n+1}} \|S_m - S_{2^n}\| \to 0 \text{ a.s.} \tag{4}$$

(1),(3),(4) imply the statement of Theorem 1. Theorem 1 is proved.

Proof of Theorem 2. To prove this theorem we show that

$$\sum_{n=1}^{\infty} P\left(\|S_n\| > n\varepsilon\right) < \infty .$$

Using Chebyshev's inequality we have

$$\sum_{n=1}^{\infty} P\left(\|S_n\| > n\varepsilon\right) \le \sum_{n=1}^{\infty} \frac{E\|S_n\|^p}{(n\varepsilon)^p} = \text{¿} \sum_{n=1}^{\infty} \frac{1}{(n\varepsilon)^p} \sum_{i=1}^{\infty} E\left|S_n^{(i)}\right|^p \le \sum_{n=1}^{\infty} \frac{1}{(n\varepsilon)^p} \sum_{i=1}^{\infty} \left(E\left|S_n^{(i)}\right|^2\right)^{p/2} \le \frac{1}{\varepsilon^p} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} \frac{1}{n^2} E\left|X_j^{(i)}\right|^2\right)^{p/2} <$$

Now Borel-Cantelli lemma implies as $n \to \infty$

$$\frac{S_n}{n} \to 0 \text{ a.s.}$$

The theorem is proved.

In this work, strong laws of large numbers were obtained for positively and negatively associated random variables in separable Hilbert spaces and in the spaces c_0 and l_p ($1 \le p \le 2$). Therefore, the strong law continues to hold for CPA random variables in several infinite-dimensional settings.

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