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ANTI-PERIODIC BOUNDARY VALUE PROBLEM FOR A SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION WITH DEGENERACY

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Annotation. *In this article, we formulate and investigate an anti-periodic boundary value problem for a degenerate second-order partial differential equation with a Caputo fractional time-derivative. The spatial operator has a diffusion coefficient $x^{\alpha}(1-x)^{\beta}$ that vanishes at both endpoints of the interval, leading to a singular spectral problem. By constructing the Green's function, we reduce the spectral problem to a Fredholm integral equation of the second kind, and we establish the existence of a countable set of eigenvalues and eigenfunctions. The uniqueness of the boundary value problem's solution is proved via an energy-integral method. Existence is then shown by expanding the solution in the eigenfunction basis and justifying the uniform convergence of this series.*

Key words: *Degenerate partial differential equation; Caputo fractional derivative; anti-periodic boundary condition; spectral problem; Green's function; Fredholm integral equation; Mittag-Leffler function; method of energy integrals; Fourier method.*

IKKINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMA UCHUN DEGENERATSIYALI ANTI-DAVRIY CHEGARAVIY MASALA

Annotatsiya. *Ushbu maqolada Kaputo kasr tartibli vaqt hosilasi bilan berilgan, degeneratsiyali ikkinchi tartibli xususiy hosilali differensial tenglama uchun anti-davriy chegaraviy masala shakllantiriladi va tadqiq etiladi. Fazoviy operator diffuziya koeffitsientiga ega bo‘ lib, ushbu koeffitsient interval ikkala uchida nolga aylanadi, bu esa singular spektral masalaning kelib chiqishiga olib keladi. Green funksiyasini qurib, spektral masalani ikkinchi tur Fredholm integral tenglamasiga keltiramiz hamda xususiy qiymatlar va xususiy funksiyalarning sanab chiqiladigan to‘ plamining mavjudligini asoslaymiz. Chegaraviy masala yechimining yagonaligi energetik-integral usuli yordamida isbotlanadi. Yechimning mavjudligi esa xususiy funksiyalar bazisi bo‘ yicha yoyilma orqali ko‘ rsatiladi va ushbu qatorning bir tekis yaqinlashishi asoslanadi.*

Kalit so‘ zlar: *Degeneratsiyali xususiy hosilali differensial tenglama; Kaputo kasr tartibli hosilasi; anti-davriy chegaraviy shart; spektral masala; Green funksiyasi; Fredholm integral tenglamasi; Mittag-Leffler funksiyasi; energetik integrallar usuli.*

АНТИПЕРИОДИЧЕСКАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ ВЫРОЖДАЮЩЕГОСЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ВТОРОГО ПОРЯДКА

Аннотация. *В настоящей статье формулируется и исследуется антипериодическая краевая задача для вырождающегося дифференциального уравнения в частных производных второго порядка с дробной производной по времени в смысле Капуто. Пространственный оператор содержит коэффициент диффузии, обращающийся в нуль на обоих концах рассматриваемого интервала, что порождает сингулярную спектральную задачу. Посредством построения функции Грина спектральная задача сводится к интегральному уравнению Фредгольма второго рода, и устанавливается существование счётного множества собственных значений и собственных функций. Единственность решения краевой задачи доказывается методом энергетических интегралов. Существование решения обосновывается путём разложения по системе собственных функций с доказательством равномерной сходимости соответствующего ряда.*

Ключевые слова: *вырождающееся уравнение в частных производных; дробная производная Капуто; антипериодическое краевое условие; спектральная задача; функция Грина; интегральное уравнение Фредгольма; функция Миттаг-Леффлера; метод энергетических интегралов; метод Фурье.*

1. Introduction

Partial differential equations play a fundamental role in the mathematical modeling of various physical, biological, and engineering processes. Among them, boundary value problems for degenerate second-order PDEs have attracted considerable research attention since the last century and remain an active area of investigation. Such equations arise

naturally whenever the principal coefficients of the differential operator vanish on part of the domain or its boundary, leading to qualitatively different mathematical behaviour compared to the non-degenerate case.

More recently, increasing interest has been directed toward boundary value problems for differential equations involving fractional-order operators with respect to time. Such operators, in particular the Caputo derivative [1], capture memory and hereditary effects in complex media and provide more realistic models for anomalous diffusion processes. A systematic mathematical analysis of initial-boundary value problems for time-fractional diffusion equations was carried out by Gorenflo, Luchko, and Yamamoto [7], who studied these problems in fractional Sobolev spaces and established key regularity properties. Kubica and Ryszewska [8] proved the unique existence of weak solutions for the diffusion-wave equation with non-homogeneous Dirichlet boundary data and analysed the long-time behaviour of solutions.

Alongside classical boundary conditions, problems with non-local conditions have drawn growing attention. Karimov, Mamchuev, and Ruzhansky [9] studied a non-local initial problem for a second-order time-fractional equation with a Bessel operator in the space variable and obtained an explicit solution represented via Fourier-Bessel series in terms of multinomial Mittag-Leffler functions. In [10], Ruzhansky, Tokmagambetov, and Torebek studied a Bitsadze-Samarskii-type non-local problem for an integro-differential diffusion-wave equation on the Heisenberg group, establishing existence, uniqueness, and explicit solution representations. Non-local problems for linear parabolic equations were investigated by Chabrowski [11], who proved existence and uniqueness via maximum principles and Green' s function methods.

Anti-periodic boundary conditions form a special and important class of non-local conditions and have been studied for fractional-order ordinary differential equations; see, for example, [5] and the references therein. However, anti-periodic problems for degenerate partial differential equations with fractional-order time derivatives remain largely unexplored in the literature.

Related results on degenerate equations in rectangular domains include the work [3] of Tersenov and the work [4] of Urinov and Usmonov, where modified Cauchy-type problems were studied for degenerate hyperbolic equations of the second kind. However, to the best of our knowledge, the combination of fractional-order time evolution, spatial degeneracy, and anti-periodic boundary conditions has not been treated before in the literature, and the present paper is devoted to filling this gap.

In the present work, we investigate an anti-periodic boundary value problem for the following time-fractional degenerate equation in the rectangular domain $\Omega = \{(x,t): 0 < x < 1, 0 < t < T\}$:

$$CD_{0t}^{\gamma} u - \left[x^{\alpha}(1-x)^{\beta} u_x\right]_x = f(x,t), \qquad (1)$$

where $u(x,t)$ is the unknown function, $f(x,t)$ is a given source function, and CD_{0t}^{γ} denotes the Caputo fractional derivative of order γ with respect to t [1]. The real parameters α, β, γ satisfy $0 < \gamma < 1$, $0 \le \alpha, \beta < 1$.

Equation (1) is degenerate along two boundary lines at $x = 0$ and $x = 1$, where the diffusion coefficient $x^{\alpha}(1-x)^{\beta}$ vanishes. This spatial degeneracy, combined with the fractional-order time derivative, produces analytical difficulties that go beyond the classical setting.

The problem is supplemented with anti-periodic boundary conditions of the form $u(0,t) = -u(1,t)$, together with initial conditions at $t = 0$ Existence and uniqueness of a solution to this problem are established by combining the Fourier method with Green' s function theory for the associated spectral problem and energy integral estimates.

The remainder of the paper is organised as follows. Section 2 recalls the definition and key properties of the Mittag-Leffler function and the Caputo derivative. Section 3 formulates this problem , analyses the associated spectral problem via a Fredholm integral equation, proves auxiliary convergence lemmas, and establishes the existence and uniqueness of the solution. Concluding remarks are given at the end of Section 3.

2 Preliminaries

In this section, we will recall some definitions and basic concepts, which will be used throughout the paper.

Definition 1. [1] *For a function $u(t)$, the Caputo fractional derivative of order α with respect to t is defined as follows:*

$$CD_{0t}^{\gamma} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds,$$

where $t > 0, 0 < \gamma \le 1$, and $\Gamma(z)$ s Euler' s gamma-function[12].

Mittag-Leffler function: A two-parameter Mittag-Leffler function is defined by [6] as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+1)}, Re(\alpha) > 0, Re(\beta) > 0, z, \alpha, \beta \in C.$$

Lemma 1. *For every $\alpha \in (0,2)$ and β is constant number. One has*

$$\left|E_{\alpha,\beta}(-z)\right| < C, \ z \ge 0.$$

3. Formulation of a problem

We study the following initial-boundary value problem for equation (1):

Problem C. Find $u(x,t)$ function such that it has the following properties:

1) $u(x,t)$, $x^{\beta} u_x(x,t), u_t(x,t) \in C(\overline{\Omega})$; ${}_C D_{0t}^{\alpha} u\left(x,t\right), \left[x^{\beta} u_x(x,t)\right]_x \in C(\Omega)$;

2) Ω satisfies equation (1) in the field;

3) $u(x,t)$ satisfies the following boundary

$$\left.\begin{array}{l} u(0,t) = -u(1,t), t \in [0,T]; \\ \lim\limits_{x\to 0} x^{\beta} u_x(x,t) = -u_x(1,t), t \in [0,T]; \end{array}\right\}, \tag{2}$$

and the initial conditions:

$$u(x,0) = \varphi_1(x),\ u_t(x,0) = \varphi_2(x),\ x \in [0,1]; \tag{3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are the given functions.

3.1 Spectral Problem

By applying Fourier' s method to the problem C, we will have the following spectral problem: find the values of the parameter λ for which there exist nontrivial solutions to the equation

$$MX(x) \equiv -\left[x^{\beta} X'(x)\right]' = \lambda X(x), \tag{4}$$

satisfying the following conditions:

$$\left.\begin{array}{l} X(0) = -X(1) \\ \lim\limits_{x\to 0} x^{\beta} X'(x) = -X'(1) \end{array}\right\}. \tag{5}$$

Assume that there exist eigenvalues of the spectral problem {(4), (5)}. The spectral problem {(4), (5)} may have nontrivial solutions only when $\lambda > 0$.

To establish the existence of the eigenvaluens and eigenfunctions of the spectral problem {(4),(5)}, we equivalently reduce it to a second kind homogeneons. Fredholm integral equation with symmetric kernels. For this purpose, we will construct a Green' s function$G(x,s)$. It satisfies the following conditions:

1) $G(x,s)$ function is continuous in $x, s \in [0,1]$.

2) In the intervals $0,s)$ and $s,1$ it has a continuous derivative $(\partial/\partial x)G(x,s)$ and it has a jump $\left(-s^{-\beta}\right)$ when $x = s$, that is

$$(\partial/\partial x)G(x,s)|_{x=s+0} - (\partial/\partial x)G(x,s)|_{x=s-0} = -s^{-\beta}.$$

3) $G(x,s)$ function with respect argument to x in the intervals $(0,s)$ and $(s,1)$ satisfies equation $MG(x,s) = 0$.

4) $G(x,s)$ satisifies boundary conditions $G(0,s) = -G(1,s)$, $\lim\limits_{x\to 0} x^{\beta} G_x(x,s) = -G_x(1,s)$ for $s \in (0,1)$.

Using the general solution of equations $MG(x,s) = 0$, $x \in (0,s) \cup (s,1)$ and thee conditions $\{(1),(2),(3)\}$, it can be shown that $G(x,s)$ Green' s function exists and is unique, and has the following form

$$G(x,s) = \begin{cases} \frac{1}{2(1-\beta)}\left(x^{1-\beta} - s^{1-\beta} + \frac{1}{2}\right), 0 < x < s, \\ \frac{1}{2(1-\beta)}\left(-x^{1-\beta} + s^{1-\beta} + \frac{1}{2}\right), s < x < 1. \end{cases} \tag{7}$$

Then the problem {(4), (5)} is equivalent to the following integral equation with the symmetric kernel:

$$X(x) = \lambda \int_0^1 G(x,s)X(s)ds. \tag{8}$$

Since the kernel $G(x,s)$ is continuous, symmetric, and positive (i.e., $\lambda > 0$), the integral equation, and hence the problem {(4), (5)}, has a countable set of eigenvalues:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le +\infty,$$

and the corresponding system of eigenfunctions $v_1(x), v_2(x), v_3(x)\ldots..$ form an orthonormal system in $L_2[0,1]$. Furthermore, any function that can be repre- sentable in terms of the kernel $G(x,s)$ can be expanded into a Fourier series with respect to these eigenfunctions [2].

Let $\{X_k(x)\}|_{k=1}^{+\infty}$ and $\{\lambda_k\}|_{k=1}^{+\infty}$ be eigenfunctions and eigenvalues of the spectral

problem {(4), (5)} respectively.

Lemma 2: *If the function $g(x)$ satisfies the conditions $g(x)$, $x^{\beta} g'(x) \in C[0,1]$, $Mg'(x) \in L_2(0,1)$; $g(0) = -g(1)$, $\lim\limits_{x\to 0} x^{\beta} g'(x) = -g'(1)$, then the function $g(x)$ expands on the system $\{X_k(x)\}_{k=1}^{+\infty}$ in the interval $[0,1]$ in an absolutely and uniformly convergent series.*

3.2 Auxiliary Lemmas

In this section, we will prove the convergence of some series which will be used in the proof of the existence of a solution to the main problem. Let $\{\lambda_k\}|_{k=1}^{+\infty}$ and $\{X_k(x)\}|_{k=1}^{+\infty}$ be the eigenvalues and eigenfunctions of the problem {(4), (5)}, respectively. g_k represents the Fourier coefficient of the function $g(x)$ by the system of the eigenfunctions$\{X_k(x)\}|_{k=1}^{+\infty}$, i.e.

$$g_k = \int_0^1 g(x)X_k(x)dx,\ k = 1,2,3,\ldots$$

Lemma 3:[2] *In the interval $[0,1]$, the series*

$$\sum_{k=1}^{+\infty} \frac{X_k^2(x)}{\lambda_k},\ \sum_{k=1}^{+\infty} \frac{\left[x^{\beta} X_k'(x)\right]^2}{{\lambda_k}^2},$$

are uniformly convergent.

Lemma 4. *Let $f(x) \in C[0,1]$, $x^{\beta/2} f'(x) \in L_2(0,1)$ and $f(0) = -f(1)$, Then, the following inequality holds true:*

$$\sum_{k=1}^{+\infty}\lambda_k f_k^2 \le \int_0^1 x^\beta \left[f'(x)\right]^2 dx, \quad (9)$$

where f_k is the Fourier coefficient of the function $f(x)$ by the system eigenfunctions $\{X_k(x)\}|_{k=1}^{+\infty}$ defined by

$$f_k = \int_0^1 f(x)X_k(x)dx.$$

Lemma 5. *Let the functions $f(x)$, $f'(x)$, $x^\alpha f'(x)$ be continuous on $[0,1]$, and let $f(0) = -f(1)$, $\lim_{x\to 0} x^\beta f'(x) = -f'(1)$, $\left[x^\beta f'(x)\right]' \in L_2(0,1)$.*

Then, the following inequality holds true

$$\sum_{k=1}^{+\infty}\lambda_k^{\,2} f_k^{\,2} \le \int_0^1 \left[x^\beta f'(x)\right]'^2 dx. \quad (10)$$

Lemma 6: *Let the functions $f(x)$, $f'(x)$, $x^\alpha f'(x)$, $\left[x^\alpha f'(x)\right]'$ be continuous on $[0,1]$, and let the following conditions be fulfiled*

$$f(0) = -f(1),\ \lim_{x\to 0} x^\beta f'(x) = -f'(1),\ \lim_{x\to 0}\left[x^\beta f'(x)\right]' = -f''(1),\ x^{\beta/2}\left[x^\beta f'(x)\right]'' \in L_2(0,1).$$

Then, the following inequality holds true

$$\sum_{k=1}^{\infty}\lambda_k^{\,3} f_k^2 \le \int_0^1 x^\beta \left[x^\beta f'(x)\right]''^2 dx. \quad (11)$$

3.3 The existence of the solution of the problem C

In this section ,we prove the existence of the solution to the direct problem. To do this, first, we will expand the functions $\varphi_1(x)$ and $\varphi_2(x)$ by the series with respect to the system of functions $\{X_k(x)\}|_{k=1}^{+\infty}$ as follows

$$\varphi_j(x) = \sum_{k=1}^{+\infty}\varphi_{jk}X_k(x), f(x,t) = \sum_{k=1}^{+\infty} f_k(t)X_k(x),\ j = \overline{1,2} \quad (12)$$

where φ_{jk} and $f_k(t)$ are the Fourier coefficients of the functions $\varphi_{jk}(x)$ and $f(x,t)$ respectively, i.e.

$$f_k = \int_0^1 f_k(x)X_k(x)dx.$$

$$\varphi_{jk} = \int_0^1 \varphi_{jk}(x)X_k(x)dx, j = \overline{1,2}.$$

We will seek a solution of the problem C in the following form

$$u(x,t) = \sum_{k=1}^{+\infty} T_k(t)X_k(x), \quad (13)$$

where $T_k(t)$ is an unknown function.

To find this unknown function substituting (12) and (13) into the equation (1) and the initial condition (3), we have the following initial boundary value problem for the function $T_k(t)$:

$$\begin{cases} {}_C D_{0t}^{\alpha} T_k(t) + \lambda_k T_k(t) = f_k(t), \\ T_k(0) = \varphi_{1k}, T_k'(0) = \varphi_{2k}. \end{cases} \quad (14)$$

It is known that the unique solution to the problem (14) has the following form[1]

$$T_k(t) = \varphi_{1k}E_{\alpha,1}[-\lambda_k t^\alpha] + \varphi_{2k}tE_{\alpha,2}[-\lambda_k t^\alpha] + \\ + \int_0^t (t-z)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_k(t-z)^\alpha]f_k(z)dz, \quad (15)$$

where $E_{\alpha,\beta}(z)$is the Mittag-Leffler function which its definition is given in Section 2. Substituting (14) into (15), we will have the following representation of the solution of the problem C:

$$u(x,t) = \sum_{k=1}^{+\infty}\Big\{\varphi_{1k}E_{\alpha,1}[-\lambda_k t^\alpha] + \varphi_{2k}tE_{\alpha,2}[-\lambda_k t^\alpha] + \\ + \int_0^t (t-z)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_k(t-z)^\alpha]f_k(z)dz\Big\}X_k(x). \quad (16)$$

Theorem 1. If $\varphi_j(x)$, $x^\beta\varphi_j'(x)$, $\left[x^\beta\varphi_j'(x)\right]' \in C[0,1]$, $x^{\beta/2}\left[x^\beta\varphi_j'(x)\right]' \in L_2(0,1)$, $\varphi_j(0) = -\varphi_j(1)$, $x^\beta\varphi_j'(x)\big|_{x=0} = -\varphi_j'(1)$, $\left[x^\beta\varphi_j'(x)\right]'\Big|_{x=0} = -\left[x^\beta\varphi_j'(x)\right]'\Big|_{x=1}$. $f(x,t)$, $x^\beta f_x(x,t)$, $\left[x^\beta f_x(x,t)\right]_x \in C[0,1]$, $x^{\beta/2}\left[x^\beta f_x(x,t)\right]_{xx} \in L_2(0,1)$, $f(0,t) = -f(1,t)$, $x^\beta f_x(x,t)\big|_{x=0} = -\left[x^\beta f_x(1,t)\right]_x\Big|_{x=1}$ *Then the solution of the problem C exists and is unique.*

Proof. To prove the Theorem 1, it is sufficient to show that the series (16) and the series corresponding to the functions $x^{\beta}u_x(x,t)$ converge uniformly in $\bar{\Omega}$, and that the series corresponding to the functions $\left[x^{\beta}u_x(x,t)\right]_x$, $CD_{0t}^{\alpha}u(x,t)$ converge uniformly on any compact set $D \subset \Omega$.

First, we consider the series (16). Using Lemma 1, one can easily verify that $E_{\alpha,\beta}(-\lambda_k t^{\alpha}) < C$, $t \in [0.1]$. We have

$$|u(x,t)| = \left|\sum_{k=1}^{+\infty}\left[\varphi_{1k}E_{\alpha,1}[-\lambda_k t^{\alpha}] + \varphi_{2k}tE_{\alpha,2}[-\lambda_k t^{\alpha}] + \right.\right.$$
$$\left.\left. + \int_0^t (t-z)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_k(t-z)^{\alpha}]f_k(z)dz\right]X_k(x)\right| \le \sum_{k=1}^{+\infty}|C_1\varphi_{1k} + C_2T\varphi_{2k}||X_k(x)| +$$
$$+C_3\sum_{k=1}^{+\infty}\left|\left\{\int_0^1 (t-z)^{\alpha-1}f_k(z)dz\right\}X_k(x)\right| \le C_1\sum_{k=1}^{+\infty}|\sqrt{\lambda_k}\varphi_{1k}|\left|\frac{X_k(x)}{\sqrt{\lambda_k}}\right| +$$
$$+C_3\sum_{k=1}^{+\infty}|\sqrt{\lambda_k}\varphi_{2k}|\left|\frac{X_k(x)}{\sqrt{\lambda_k}}\right| + C_4\left[\int_0^T\sum_{k=1}^{+\infty}f_k(z)\sqrt{\lambda_k}dz\frac{X_k(x)}{\sqrt{\lambda_k}}\right]. \quad (17)$$

From the last, applying the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{+\infty}|\sqrt{\lambda_k}\varphi_{jk}|\left|\frac{X_k(x)}{\sqrt{\lambda_k}}\right| \le \left[\sum_{k=1}^{+\infty}\lambda_k\varphi_{jk}^2\sum_{k=1}^{+\infty}\frac{X_k^2(x)}{\lambda_k}\right]^{1/2}, j = \overline{1,2},$$
$$\int_0^T\sum_{k=1}^{+\infty}f_k(z)\sqrt{\lambda_k}dz\frac{X_k(x)}{\sqrt{\lambda_k}} \le \left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k{f_k}^2(x)dz\sum_{k=1}^{+\infty}\frac{{X_k}^2(x)}{\lambda_k}\right]^{1/2}.$$

The series on the right-hand side of these inequalities converge uniformly in $x \in [0,1]$ due to Lemmas 3 and 4. Therefore, the series on the left-hand side also converges uniformly in x on $[0,1]$. Hence, from (17), it follows that the series (16) converges absolutely and uniformly in Ω.

Now, we consider the series corresponding to the function $\left[x^{\beta}u_x(x,t)\right]_x$. From (13), considering (4), we obtain

$$\left|\left[x^{\beta}u_x(x,t)\right]_x\right| \le \left|\sum_{k=1}^{+\infty}\left[\varphi_{1k}E_{\alpha,1}[-\lambda_k t^{\alpha}]x^{\beta}{X_k}'(x)\right] + \sum_{k=1}^{+\infty}\left[\varphi_{2k}tE_{\alpha,2}[-\lambda_k t^{\alpha}]x^{\beta}{X_k}'(x)\right]\right| +$$
$$+\left|\sum_{k=1}^{+\infty}\left[\int_0^t(t-z)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_k(t-z)^{\alpha}]f_k(z)dz\right]x^{\beta}{X_k}'(x)\right|. \quad (18)$$

Using Lemma 1, one can easily verify that $E_{\alpha,\beta}(-\lambda_k t^{\alpha}) < C$, $t \in [0.1]$ and $t < T$. Considering this, we have

$$\left|\left[x^{\beta}u_x(x,t)\right]_x\right| \le C_5\sum_{k=1}^{+\infty}\left|\varphi_{1k}x^{\beta}{X_k}'(x)\right| + C_6\sum_{k=1}^{+\infty}\left|\varphi_{2k}Tx^{\beta}{X_k}'(x)\right| +$$
$$+C_7\sum_{k=1}^{+\infty}\left|\left[\int_0^T(t-z)^{\alpha}f_k(z)dz\right]x^{\beta}{X_k}'(x)\right|. \quad (19)$$

We rewrite (19) in the following form

$$\left|\left[x^{\beta}u_x(x,t)\right]_x\right| \le C_5\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_{1k}\frac{x^{\beta}{X_k}'(x)}{\lambda_k}\right| + C_8\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_{2k}\frac{x^{\beta}{X_k}'(x)}{\lambda_k}\right| +$$
$$+C_9\left|\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k(z)dz\left\{\frac{x^{\beta}{X_k}'(x)}{\lambda_k}\right\}\right|. \quad (20)$$

From the last, applying the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_{jk}\frac{x^{\beta}{X_k}'(x)}{\lambda_k}\right| \le \left[\sum_{k=1}^{+\infty}{\lambda_k}^2{\varphi_{jk}}^2\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}{X_k}'(x)\right]^2}{{\lambda_k}^2}\right]^{1/2}, j = \overline{1,2},$$

$$\left|\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k(z)dz\left\{\frac{x^{\beta}{X_k}'(x)}{\lambda_k}\right\}\right| \le \left[\int_0^T\sum_{k=1}^{+\infty}{\lambda_k}^2{f_k}^2(z)\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}{X_k}'(x)\right]^2}{{\lambda_k}^2}\right]^{1/2}.$$

The series on the right-hand side of these inequalities converge uniformly in $x \in [0,1]$ due to Lemmas 3 and 5. Therefore, the series on the left-hand side also onverges uniformly in x on $[0,1]$. Hence, from (20), it follows that the series (18) converges absolutely and uniformly in Ω.

Then, We consider the uniform convergence of the series corresponding to $CD_{ot}^{\alpha}u(x,t)$ functions. To this end, it is sufficient to show the uniform convergence of the series corresponding to $f(x,t)$ functions.

$$|f(x,t)| \le \sum_{k=1}^{+\infty}|f_k(t)X_k(x)| \le \sum_{k=1}^{+\infty}\left|f_k(t)\sqrt{\lambda_k}\frac{X_k(x)}{\sqrt{\lambda_k}}\right|, \quad (21)$$

From the last, applying the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{+\infty}\left|f_k(t)\sqrt{\lambda_k}\frac{X_k(x)}{\sqrt{\lambda_k}}\right| \le \left[\sum_{k=1}^{\infty}\lambda_k {f_k}^2(t)\sum_{k=1}^{\infty}\frac{{X_k}^2(x)}{\lambda_k}\right]^{1/2}.$$

The series on the right-hand side of these inequalities converge uniformly in $x \in [0,1]$ due to Lemmas 3 and 4. Therefore, the series on the left-hand side also onverges uniformly in x on [0,1]. Hence, from (21), it follows that the series corresponding to the function $f(x,t)$ converges absolutely and uniformly in Ω.

3.4 The Uniqueness of the Solution to problem C

Now we show that the solution of the problem is unique.

Assume that there exists two solutions $u_1(x,t)$ and $u_2(x,t)$. Let us denote their difference by $u(x,t) = u_1(x,t) - u_2(x,t)$. Then the pair $u(x,t)$ should satisfy the following equation

$$CD_{0t}^{\alpha}u(x,t) = \left[x^{\beta}u_x(x,t)\right]_{x'} \tag{22}$$

supplemented with the following conditions

$$\left.\begin{array}{l} u(0,t) = -u(1,t), t \in [0,T]; \\ \lim\limits_{x\to 0} x^{\beta}u_x(x,t) = -u_x(1,t), t \in [0,T]; \end{array}\right\}, \tag{23}$$

$$u(x,0) = 0,\ u_t(x,0) = 0,\ x \in [0,1]; \tag{24}$$

To establish the uniqueness of the solution, we consider following equality

$$T_k(t) = \int_0^1 u(x,t)\,X_k(x)dx, k \in \mathbb{N}. \tag{25}$$

Applying the operator CD_{0t}^{α} to both sides of the equation (25), we obtain

$$CD_{0t}^{\alpha}T_k(t) = \int_0^1 CD_{0t}^{\alpha}u(x,t)X_k(x)dx.$$

Using the equation (1), this expression simplifies to

$$CD_{0t}^{\alpha}T_k(t) = \int_0^1\left[x^{\beta}u_x(x,t)\right]_x X_k(x)dx.$$

Hence, considering the boundary conditions (23) and (24), we apply the integration rule twice and rewrite the expression as follows

$$CD_{0t}^{\alpha}T_k(t) = \int_0^1 u(x,t)\left[x^{\beta}X_k^{'}(x)\right]^{'}dx.$$

From the last, taking into account the equations (4) and (25), we have

$$CD_{0t}^{\alpha}T_k(t) + \lambda_k T_k(t) = 0, \tag{26}$$

$$\begin{cases} T_k(0) = 0, \\ T_k^{'}(0) = 0. \end{cases} \tag{27}$$

One can easily show that the solution to {(26),(27)} is given by $T_k(t) \equiv 0$. Since the spectral problem {(4),(5)} is self-adjoint, its set of eigenfunctions $\{X_k(x)\}|_{k=1}^{+\infty}$ is complete. Thus, $u(x,t) \equiv 0$. Therefore, we conclude that $u_1(x,t) = u_2(x,t)$, confirming the uniqueness of the solution to the problem C.

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