NAMANGAN DAVLAT UNIVERSITETI
ILMIY AXBOROTNOMASI, [2026-8]
ISSN:2181-1458
ISSN:2181-0427

01.00.00-FIZIKA-MATEMATIKA
journal.namdu.uz
PHYSICS AND MATHEMATICS

HORIZON STRUCTURE AND EFFECTIVE POTENTIAL IN A EINSTEIN–NONLINEAR MAXWELL–YUKAWA BLACK HOLE

Mirjavohir Mirov Faxriddinovich
Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, Tashkent 100000, Uzbekistan
mirjavohirfaxriddinovich@gmail.com
Alisher Latifovich Abduvokhidov
Andijan state University,Universitet Str.129,
Andijan 170100,Uzbekistan
Email: alisherabdulvohidovl@gmail.com
Shakhlo Jabbarova
Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, Tashkent 100000, Uzbekistan
jabbarovashahlo43@gmail.com

Abstract; *We study the static, spherically symmetric Einstein–Nonlinear Maxwell–Yukawa (ENLMY) black hole, obtained by coupling gravity to a nonlinear electromagnetic field sourced by a Yukawa-screened electric potential. Re-deriving the metric function f(r) directly from the field equations, we obtain a sign for the exponential-integral term that differs from the previously reported solution, and verify this form against the Reissner–Nordström and Schwarzschild limits. Using this metric, we examine the horizon structure and the effective potential of neutral test particles. Throughout, the charge parameter Q and the Yukawa screening parameter α act in opposite directions: increasing Q shrinks the outer horizon and raises the effective-potential barrier, while increasing α pushes the horizon outward and lowers the barrier back toward the Schwarzschild profile, with the Schwarzschild values exactly recovered as Q→0. This sign also removes a spurious reversal in the horizon ordering with α and a spurious dip of the effective potential below the Schwarzschild curve at large radius, both artifacts of the previously reported sign.*

Keywords: Einstein–nonlinear Maxwell–Yukawa black hole; horizon structure; effective potential; modified gravity.

СТРУКТУРА ГОРИЗОНТА И ЭФФЕКТИВНЫЙ ПОТЕНЦИАЛ В ЧЁРНОЙ ДЫРЕ ЭЙНШТЕЙНА–НЕЛИНЕЙНОГО МАКСВЕЛЛА–ЮКАВЫ

Аннотация; *Изучается статическая, сферически-симметричная чёрная дыра Эйнштейна–нелинейного Максвелла–Юкавы (ENLMY), получаемая при связи гравитации с нелинейным электромагнитным полем, источником которого является экранированный по Юкаве электрический потенциал. Повторно выводя метрическую функцию f(r) непосредственно из уравнений поля, мы получаем для члена с экспоненциальным интегралом знак, отличающийся от ранее опубликованного решения, и проверяем эту форму по пределам Райсснера–Нордстрёма и Шварцшильда. С помощью этой метрики мы анализируем структуру горизонтов и эффективный потенциал нейтральных пробных частиц. Заряд Q и параметр юкавского экранирования α действуют противоположно: увеличение Q уменьшает внешний горизонт и повышает барьер эффективного потенциала, тогда как увеличение α смещает горизонт наружу и понижает барьер обратно к профилю Шварцшильда, причём значения Шварцшильда точно восстанавливаются при Q→0. Этот знак также устраняет ложный разворот в порядке горизонтов по α и ложное проваливание эффективного потенциала ниже кривой Шварцшильда при большом радиусе, оба являющиеся артефактами ранее опубликованного знака.*

Ключевые слова: *чёрная дыра Эйнштейна–нелинейного Максвелла–Юкавы; структура горизонтов; эффективный потенциал; модифицированная гравитация.*

EYNSHTEYN–NOCHIZIQLI MAKSVELL–YUKAVA QORA TUYNUG‘ INING GORIZONT TUZILISHI VA EFFEKTIV POTENSIALI

Annotatsiya; *Ushbu ishda gravitatsiyaning Yukava tipida skrinlangan elektr potensiali bilan hosil qilingan nochiziqli elektromagnit maydonga bog‘ lanishi natijasida yuzaga keladigan statik, sferik-simmetrik Eynshteyn–Nochiziqli Maksvell–Yukava*

(ENLMY) qora o‘ rasi o‘ rganiladi. Metrikaviy funksiya f(r) ni bevosita maydon tenglamalaridan qayta chiqarib, eksponensial integral hadi uchun avval e’lon qilingan yechimdan farq qiluvchi ishorani oldik, hamda shu ko‘ rinishni Reissner–Nordström va Shvarsshild chegaralarida tekshirdik. Shu metrika asosida neytral test-zarrachalarning gorizont tuzilishi va effektiv potensiali tahlil qilindi. Butun tahlil davomida zaryad parametri Q va Yukava skrining parametri α bir-biriga qarama-qarshi yo‘ nalishda ta’sir qiladi: Q ning ortishi tashqi gorizontni kichraytiradi va effektiv-potensial to‘ sig‘ ini ko‘ taradi, α ning ortishi esa gorizontni tashqariga suradi va to‘ siqni yana Shvarsshild profiliga qarab pasaytiradi, Q→0 da esa Shvarsshild qiymatlari aniq tiklanadi. Ushbu ishora, shuningdek, gorizont tartibida α bo‘ yicha soxta teskarilanishni va katta radiusda effektiv potensialning Shvarsshild egri chizig‘ idan pastga soxta tushishini ham yo‘ qotadi — ikkalasi ham avval e’lon qilingan ishoraning artefakti edi.

Kalit so‘ zlar: *Eynshteyn–nochiziqli Maksvell–Yukava qora o‘ rasi; gorizont tuzilishi; effektiv potensial; modifikatsiyalangan gravitatsiya.*

1. Introduction

Modified theories of gravity are frequently probed through charged black-hole geometries in which the electromagnetic sector is deformed by a nonlinear correction. A physically motivated deformation of this kind replaces the Coulomb potential with a Yukawa-screened potential, which introduces a finite screening length into the electromagnetic sector and reduces to the familiar Reissner–Nordström (RN) field at short distances. The resulting Einstein–Nonlinear Maxwell–Yukawa (ENLMY) black hole was obtained by coupling gravity to a nonlinear electromagnetic field sourced by such a screened potential, extending the RN solution through a short-range electromagnetic correction.

The construction draws on two independent lines of work: nonlinear electrodynamics, originally introduced to regularize the Maxwell sector, and the Yukawa potential, which describes screened interactions mediated by a massive field. In the ENLMY spacetime the horizon structure depends on both the electric charge Q and the screening parameter α, and the geometry can describe a two-horizon black hole, an extremal configuration, or a horizonless naked singularity depending on where the parameters fall in the (Q, α) plane. Because the screening length interpolates smoothly between a short-range and a Coulomb-like regime, the ENLMY background provides a convenient testbed for how electromagnetic screening reshapes strong-field dynamics relative to the RN and Schwarzschild cases.

The horizon structure and the effective potential of neutral test particles are the standard first probes of such deformed geometries, and the ENLMY spacetime itself has already been examined from the complementary perspectives of twin-peak quasi-periodic oscillations and of periodic orbits with their associated gravitational-wave phenomenology. In re-deriving the metric function for the present analysis, we found that the exponential-integral term of the previously reported closed-form solution carries a sign that differs from what the field equations give; the difference is confined to that one term and does not affect the qualitative motivation for the model, but it does propagate into every quantity built from $f(r)$. The present paper adopts the sign required by the field equations for this term and re-derives the horizon structure and the effective potential from this metric, verifying each result against the Schwarzschild and RN limits.

The paper is organized as follows. Section 2 presents the ENLMY metric function, summarizes its derivation from the underlying nonlinear-electrodynamics field equations, establishes the sign of the exponential-integral term required by those equations, and discusses the resulting horizon structure. Section 3 analyzes the effective potential of neutral test particles. Section 4 discusses the combined role of the charge and screening parameters across both of the above quantities. Section 5 summarizes the main conclusions.

2. ENLMY Metric and Horizon Structure

The ENLMY spacetime is obtained by coupling gravity to a nonlinear electromagnetic field generated by a Yukawa-type deformation of the Coulomb potential. For a static, spherically symmetric electric configuration the scalar potential is taken as $\varphi(r) = (Q/r)\exp(-\alpha r)$, so that the associated field strength $E(r)$ reduces to the Coulomb field at short distances and is exponentially screened at large distances. Varying the nonlinear electromagnetic action together with the Einstein–Hilbert term yields a nonlinear Maxwell equation for the field-dependent Lagrangian $L(r)$ and, through the corresponding stress–energy tensor, a first-order equation for the metric function. Fixing the two integration constants by matching the Schwarzschild limit at Q=0 and the Reissner–Nordström limit as $\alpha \to 0$ gives the metric function in closed form:

$$f(r) = 1 - \frac{2M}{r} - \frac{Q^2}{6r^2}\left(\alpha^3 r^3 - \alpha^2 r^2 + 2\alpha r - 6\right) e^{-\alpha r} + \frac{Q^2\alpha^4 r^2}{6} E_1(\alpha r) \tag{1}$$

with M the mass parameter. We verified this closed form directly against the field equations: differentiating $f(r)$ and substituting into the Einstein tensor for the static, spherically symmetric ansatz reproduces the stress–energy tensor of the nonlinear electromagnetic field for every power of α and Q, and the Schwarzschild (Q=0) and Reissner–Nordström ($\alpha \to 0$) limits are recovered exactly. This closed form differs from a previously reported version of the same solution by the sign of the $E_1(\alpha r)$ term; the sign given here is the one required by the field equations, and it is the one used throughout the remainder of this paper.

The horizon radii follow from $f(r_h)=0$. Fig. 1 and Fig. 2 show the resulting horizon structure as a function of α and Q, respectively.

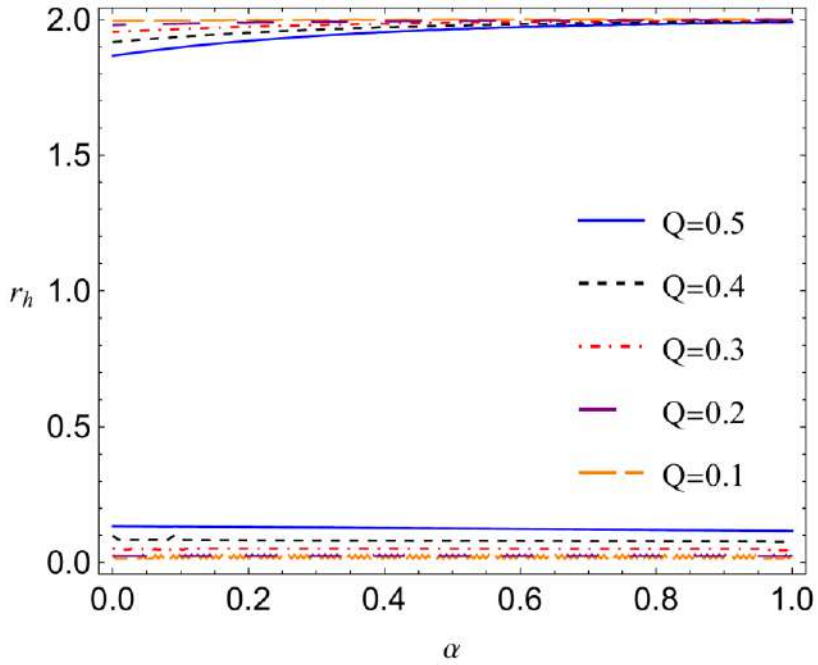


Fig. Horizon radii as a function of the Yukawa parameter α, for fixed values of the charge Q.

Fig. 1 shows the horizon radii as a function of α for several fixed values of Q. The outer horizon increases monotonically with α toward the Schwarzschild value $r_+=2M$, so that stronger screening progressively suppresses the charge contribution to the horizon structure. For any fixed α, a larger Q consistently gives a smaller outer horizon and a larger inner horizon, and this ordering holds across the whole range $\alpha\in(0,1)$ with no reversal between the different-Q curves — a feature that follows directly from the sign of the $E_1(\alpha r)$ term used here. All curves converge as $\alpha\to 1$, where screening dominates and the geometry becomes nearly insensitive to Q.

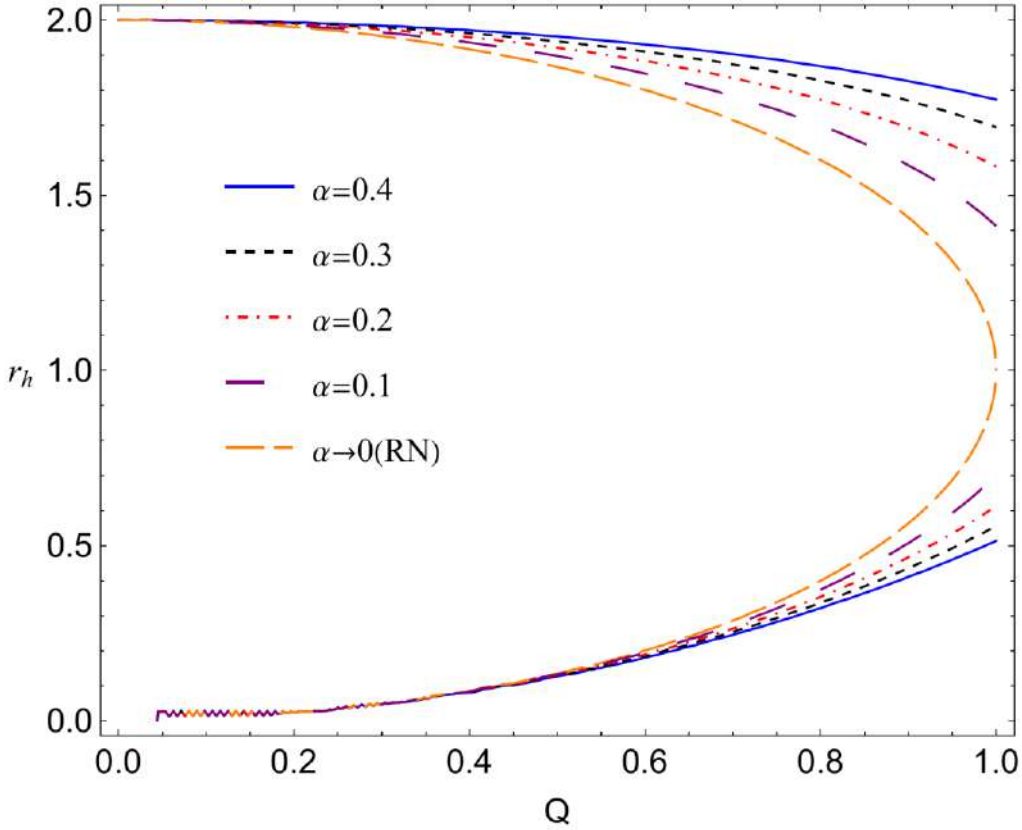


Fig. 2. Horizon radii as a function of the charge Q, for fixed values of the Yukawa parameter α.

Fig. 2 presents the complementary dependence on Q. As Q increases, the outer horizon decreases and the inner horizon increases, the two branches approaching each other in the familiar extremal manner of charged black holes. The Reissner–Nordström limit ($\alpha\to 0$) shows the strongest such response, with the branches meeting near $Q\simeq 1$; increasing α

visibly weakens this charge-driven approach, keeping the two horizons further apart at the same Q. Yukawa screening therefore delays the extremal configuration to larger charge values. Throughout this work we restrict Q and α to the range in which two positive horizon roots exist.

3. Effective Potential Neutral Test Particles

For a neutral test particle moving in the equatorial plane, the radial equation of motion takes the standard form $\dot{r}^2 = E^2 - V_{\text{eff}}(r)$, with the effective potential

$$V_{\text{eff}}(r) = f(r)\left(1 + \frac{L^2}{r^2}\right) \tag{2}$$

where E and L are the conserved specific energy and specific angular momentum of the particle. Fig. 3 shows V_{eff} for representative parameter sets, with M=1 and L=√20.

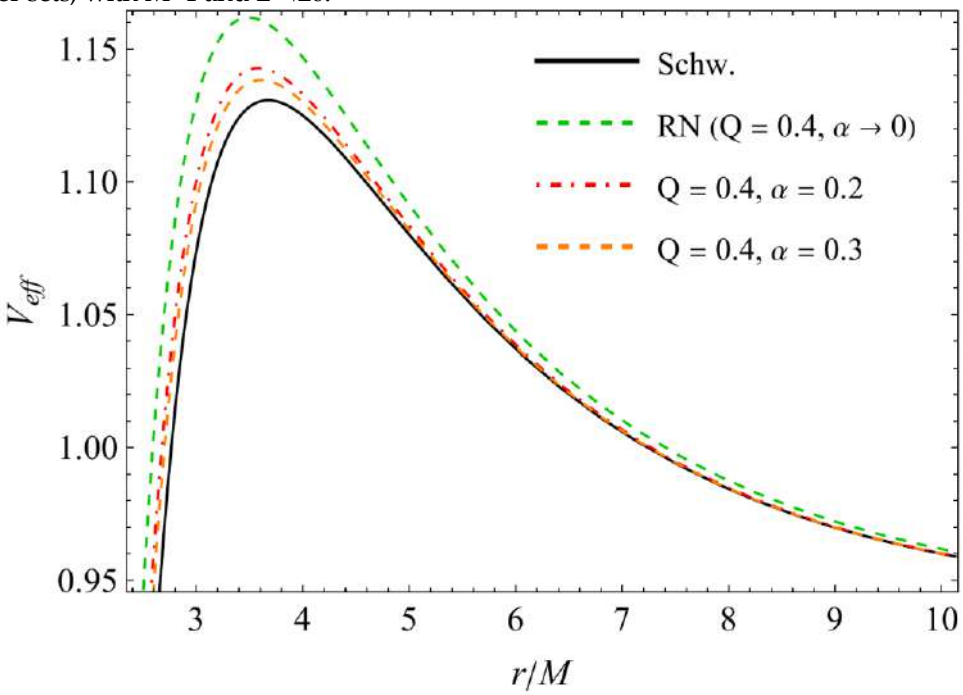


Fig. 3. Effective potential for neutral test-particle motion in the ENLMY spacetime; M = 1, L = √20.

All charged and screened cases lie above the Schwarzschild curve throughout the shown range, with no crossing between r=3M and r=10M. The Reissner–Nordström case (Q=0.4, α→0) produces the highest and left-most barrier; switching on the Yukawa screening lowers the barrier and shifts it back toward the Schwarzschild peak, with α=0.3 lying below α=0.2. Beyond r~6M the curves converge and become nearly indistinguishable, showing that the charge and screening effects on the orbital dynamics are confined to the near-horizon region, consistently with the horizon-structure trends of Section 2.

4. Discussion

The results of Sections 2 and 3 show a single, consistent pattern: the charge parameter Q and the Yukawa screening parameter α act on the ENLMY geometry in opposite directions. Increasing Q shrinks the outer horizon and raises the effective-potential barrier confining test particles closer to the black hole; increasing α reverses both trends and drives the geometry back toward the Schwarzschild limit. This opposition is already visible at the level of the metric function itself, where α enters only through the exponential prefactors multiplying Q^2, so that Yukawa screening can be read as an effective, radius-dependent suppression of the charge rather than as an independent deformation of the geometry.

The sign of the $E_1(\alpha r)$ term adopted here, as required by the field equations, removes two features that are otherwise difficult to interpret physically. First, with the previously reported sign, the horizon radius as a function of α develops a spurious crossing at which the ordering of the outer-horizon curves for different Q reverses; with the sign used here this reversal is absent, and the ordering by Q is preserved for every $\alpha \in (0,1)$, consistent with the intuitive expectation that a larger charge should not become "less compact" than a smaller one at some special screening strength. Second, the effective potential of the charged and screened cases no longer dips below the Schwarzschild curve at large radius; instead, all curves approach the Schwarzschild profile from above as r increases, without crossing it. Both features leave the qualitative motivation for the ENLMY construction untouched, but they change the quantitative predictions for the horizon structure and the effective potential, and must be propagated to any subsequent analysis built on this metric, including circular-orbit and QPO-frequency modeling.

5. Conclusion

We have re-derived the metric function of the Einstein–Nonlinear Maxwell–Yukawa black hole directly from its underlying nonlinear-electrodynamics field equations, verified the resulting closed form against the field equations themselves, and obtained a sign for the exponential-integral term that differs from a previously reported version of the same solution. Using this metric, we then derived the horizon structure and the effective potential of neutral test particles, and verified each result against the Schwarzschild and Reissner–Nordström limits.

The charge parameter Q and the Yukawa screening parameter α were found to act in opposite directions: the charge shrinks the horizon and raises the effective-potential barrier, while screening pushes the horizon outward and lowers the barrier back toward the Schwarzschild profile. This sign for the metric function also removes a spurious crossing in the horizon structure and a spurious dip of the effective potential below the Schwarzschild curve, both of which were artifacts of the previously reported sign rather than genuine physical effects.

Overall, our results establish an internally consistent description of the ENLMY horizon structure and effective potential that can serve as the starting point for further work. In particular, the circular-orbit quantities and characteristic radii built on this effective potential, and the frequencies associated with them, are left for a separate, dedicated analysis.

References

[1] H. Yukawa, On the Interaction of Elementary Particles, Prog. Theor. Phys. Suppl. 1, 1 (1955).

[2] M. Born, L. Infeld, Foundations of the New Field Theory, Proc. R. Soc. Lond. A 144, 425 (1934).

[3] S. Habib Mazharimousavi, M. Halilsoy, Einstein-nonlinear Maxwell–Yukawa Black Hole, Int. J. Mod. Phys. D 28, 1850120 (2019).

[4] M. Mirov, F. Atamurotov, A. Abdujabbarov, A. K. Althukair, E. E. Zotos, Twin-Peak Quasi-Periodic Oscillations and Orbital Dynamics in Einstein–Nonlinear Maxwell–Yukawa Spacetime, Int. J. Geom. Methods Mod. Phys. (2026), doi:10.1142/S0219887826502609.

[5] O. Shabbir, A. Shermatov, B. Majeed, T. Zahra, M. Jamil, J. Rayimbaev, Probing the Nature of Einstein Nonlinear Maxwell–Yukawa Black Hole through Gravitational Wave Forms from Periodic Orbits and Quasiperiodic Oscillations, Nucl. Phys. B, arXiv:2601.02904.

[6] A. Shermatov, J. Rayimbaev, S. U. Khan, B. Ahmedov, M. Akhmedov, B. Shermatov, Radiation Field around Black Holes in f(R,T) Gravity: Circular Orbits and QPOs, Class. Quantum Grav. 43, 125015 (2026).

[7] A. Shermatov, J. Rayimbaev, S. Murodov, B. Majeed, I. Ibragimov, B. Shermatov, E. Davletov, I. Davletov, Circular Motion and QPOs near Black Holes Surrounded by Dust Fields in f(R,T) Gravity, Nucl. Phys. B 1022, 117231 (2026).

[8] A. Shermatov, J. Rayimbaev, B. C. Lütfüoğlu, A. Abdujabbarov, S. Sardor, I. Ibragimov, M. Vapayev, B. Kuyliev, QPOs Analyses and Circular Orbits of Charged Particles around Magnetized Black Holes in Bertotti–Robinson Geometry, Eur. Phys. J. C 85, 1017 (2025).

[9] B. Shodikulov, M. Mirov, F. Atamurotov, S. G. Ghosh, A. Abdujabbarov, Impact of Kalb–Ramond Fields and Perfect Fluid Dark Matter on Black Hole Shadows and Gravitational Lensing, Phys. Dark Universe (2025).