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KEPLERIAN AND RADIAL EPICYCLIC FREQUENCIES IN A EINSTEIN–NONLINEAR MAXWELL–YUKAWA BLACK HOLE

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Abstract; *We study the Keplerian and radial epicyclic frequencies of neutral test particles in the static, spherically symmetric Einstein–Nonlinear Maxwell–Yukawa (ENLMY) black hole, using the metric function f(r) obtained in a companion analysis, where the sign of the exponential-integral term is established directly from the field equations and differs from a previously reported solution. Starting from the geodesic equations of circular motion, we derive closed-form expressions for the Keplerian angular frequency Ω_K and the radial epicyclic frequency Ω_r, and examine their dependence on the charge parameter Q and the Yukawa screening parameter α. The two parameters act in opposite directions throughout: increasing Q raises the peak of Ω_r and shifts it to smaller radii, while increasing α lowers the peak and shifts it outward; Ω_K decreases with Q and increases with α at fixed r. These trends mirror the opposite-direction pattern already found for the horizon structure and the effective potential, confirming that Yukawa screening consistently counteracts the compactifying effect of the charge across every dynamical quantity examined so far.*

Keywords: Einstein–nonlinear Maxwell–Yukawa black hole; Keplerian frequency; radial epicyclic frequency; modified gravity.

АНАЛИЗ КЕПЛЕРОВСКИХ И РАДИАЛЬНЫХ ЭПИЦИКЛИЧЕСКИХ ЧАСТОТ В ЧЁРНОЙ ДЫРЕ ЭЙНШТЕЙНА–НЕЛИНЕЙНОГО МАКСВЕЛЛА–ЮКАВЫ

Аннотация; *Изучаются кеплеровская и радиальная эпициклическая частоты нейтральных пробных частиц в статической, сферически-симметричной чёрной дыре Эйнштейна–нелинейного Максвелла–Юкавы (ENLMY), с использованием метрической функции f(r), полученной в сопутствующем исследовании, где знак члена с экспоненциальным интегралом устанавливается непосредственно из уравнений поля и отличается от ранее опубликованного решения. Исходя из геодезических уравнений кругового движения, мы получаем замкнутые выражения для кеплеровской угловой частоты Ω_K и радиальной эпициклической частоты Ω_r, и исследуем их зависимость от заряда Q и параметра юкавского экранирования α. Оба параметра действуют противоположно: увеличение Q повышает пик Ω_r и смещает его к меньшим радиусам, тогда как увеличение α понижает пик и смещает его наружу; Ω_K убывает с ростом Q и растёт с ростом α при фиксированном r.*

Ключевые слова: чёрная дыра Эйнштейна–нелинейного Максвелла–Юкавы; кеплеровская частота; радиальная эпициклическая частота; модифицированная гравитация.

EYNSHTEYN–NOCHIZIQLI MAKSVELL–YUKAVA QORA TUYNUG‘ IDA KEPLER VA RADIAL EPITSIKLIK CHASTOTALARNING TAHLILI

Annotatsiya; *Bu ishda statik va sferik-simmetrik Eynshteyn–Nochiziqli Maksvell–Yukava (ENLMY) qora o‘ rasida neytral test-zarrachalarning Kepler va radial epitsiklik chastotalari o‘ rganiladi; metrikaviy funksiya f(r) hamrohlik qiluvchi tahlilda olingan bo‘ lib, unda eksponensial integral hadining ishorasi bevosita maydon tenglamalaridan belgilanadi va avval e’lon qilingan yechimdan farq qiladi. Doiraviy harakat geodezik tenglamalaridan boshlab, Kepler burchak chastotasi Ω_K va radial epitsiklik chastota Ω_r uchun yopiq ko‘ rinishdagi ifodalar chiqarildi, va ularning zaryad parametri Q hamda Yukava skrining parametri α ga bog‘ liqligi tahlil qilindi. Ikkala parametr butun tahlil davomida qarama-qarshi yo‘ nalishda ta’sir qiladi: Q ning ortishi Ω_r cho‘ qqisini ko‘ taradi va*

uni kichikroq radiuslarga suradi, α ning ortishi esa cho‘ qqini pasaytiradi va tashqariga suradi; Ω_K esa belgilangan r da Q ortishi bilan kamayadi, α ortishi bilan ortadi. Bu tendensiyalar gorizont tuzilishi va effektiv potensial uchun avval topilgan qarama-qarshi yo‘ nalishdagi naqshni takrorlaydi, bu esa Yukava skriningning hozirgacha o‘ rganilgan har bir dinamik miqdorda zaryadning ixchamlashtiruvchi ta’sirini izchil ravishda qarshi ta’sir qilishini tasdiqlaydi.

Kalit so‘ zlar: Eynshteyn–nochiziqli Maksvell–Yukava qora o‘ rasi; Kepler chastotasi; radial epitsiklik chastota; modifikatsiyalangan gravitatsiya.

1. Introduction

The orbital and epicyclic frequencies of test particles are among the most direct dynamical probes of a black-hole spacetime: the Keplerian angular frequency Ω_K sets the orbital timescale of circular motion, while the radial epicyclic frequency Ω_r measures the restoring response to small radial perturbations and vanishes exactly at the innermost stable circular orbit. Together they underlie essentially every phenomenological model of the twin-peak quasi-periodic oscillations (QPO) observed in black-hole X-ray binaries, so any modification of the underlying metric leaves a direct, potentially observable imprint on these two frequencies.

The Einstein–Nonlinear Maxwell–Yukawa (ENLMY) black hole is obtained by coupling gravity to a nonlinear electromagnetic field sourced by a Yukawa-screened electric potential, extending the Reissner–Nordström (RN) solution through a short-range electromagnetic correction governed by the screening parameter α. In a companion analysis we re-derived the ENLMY metric function directly from the underlying nonlinear-electrodynamics field equations and obtained a sign for the exponential-integral term that differs from a previously reported version of the solution; using this metric, we found that the charge parameter Q and the screening parameter α act in opposite directions on the horizon structure and on the effective potential of neutral test particles, with Q compactifying the geometry and α counteracting this compactification.

In the present paper we extend this analysis to the Keplerian and radial epicyclic frequencies of neutral test particles, derived directly from this metric. We do not adopt any named twin-peak QPO prescription (relativistic precession, ER2–ER4, or otherwise) and do not attempt an observational or MCMC-based comparison here; the goal is to establish, in closed form and with full numerical verification, how Ω_K and Ω_r individually respond to Q and α, as the necessary basis for any subsequent frequency-model or observational analysis.

The paper is organized as follows. Section 2 recalls the ENLMY metric function. Section 3 derives the Keplerian and radial epicyclic frequencies in closed form and examines their dependence on Q and α. Section 4 discusses the combined pattern across both frequencies. Section 5 summarizes the main conclusions.

2. ENLMY Metric

The ENLMY spacetime is obtained by coupling gravity to a nonlinear electromagnetic field generated by a Yukawa-type deformation of the Coulomb potential, $\varphi(r) = (Q/r)\exp(-\alpha r)$. Re-deriving the metric function from the corresponding field equations and establishing the sign of the exponential-integral term as required by those equations gives the closed form

$$f(r) = 1 - \frac{2M}{r} - \frac{Q^2}{6r^2}\left(\alpha^3 r^3 - \alpha^2 r^2 + 2\alpha r - 6\right) e^{-\alpha r} + \frac{Q^2\alpha^4 r^2}{6} E_1(\alpha r) \tag{1}$$

with M the mass parameter; this form recovers the Schwarzschild metric at $Q=0$ and the Reissner–Nordström metric as $\alpha \to 0$, and was verified directly against the Einstein field equations in the companion analysis. The horizon structure that follows from $f(r_h)=0$ is discussed there in detail and is not repeated here; the present paper builds directly on this $f(r)$ to derive the orbital and epicyclic frequencies of neutral test particles.

3. Keplerian and Radial Epicyclic Frequencies

For a circular, equatorial geodesic, the Keplerian angular frequency measured by a static observer at infinity, $\Omega_K = d\varphi/dt$, follows from the circular-orbit condition $\partial V_{eff}/\partial r = 0$ and is given by the standard relation $\Omega_K^2 = f'(r)/(2r)$. It is convenient to introduce the auxiliary function

$$N(r) \equiv 6Mre^{\alpha r} + Q^2\alpha^4 r^4 e^{\alpha r} E_1(\alpha r) - Q^2\alpha^3 r^3 + Q^2\alpha^2 r^2 - 2Q^2\alpha r - 6Q^2 \tag{2}$$

in terms of which the Keplerian frequency takes the closed form

$$\Omega_K^2(r) = \frac{f'(r)}{2r} = \frac{N(r)\, e^{-\alpha r}}{6r^4} \tag{3}$$

which reduces to the standard Schwarzschild relation $\Omega_K^2 = M/r^3$ at $Q=0$. The radial epicyclic frequency, obtained from the second-order radial perturbation of a circular orbit, follows the general relation

$$\Omega_r^2(r) = \Omega_K^2(r)\left[3f(r) - 2rf'(r) + \frac{rf(r)f''(r)}{f'(r)}\right] \quad (4)$$

with f' the second radial derivative of f; this identity holds for any static, spherically symmetric metric of this form and reduces to the Schwarzschild relation $\Omega_r^2=\Omega_K^2(1-6M/r)$ at $Q=0$. Throughout, r is expressed in units of $M=1$. Fig. 1–4 show Ω_r and Ω_K as functions of r for representative values of Q and α.

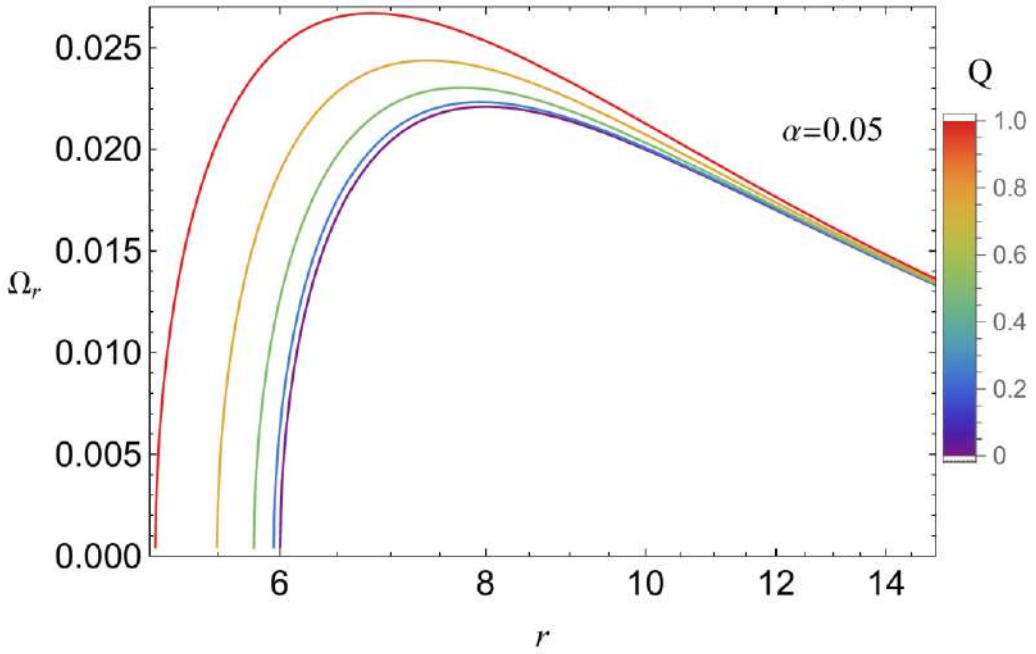


Fig. Radial epicyclic frequency Ω_r as a function of r, at fixed α=0.05, for Q from 0 to 1.

Fig. 1 shows $\Omega_r(r)$ at fixed α=0.05 for Q ranging from 0 to 1. Every curve vanishes at its left endpoint, where $\Omega_r=0$ by definition, rises to a single peak, and then decreases monotonically at large r, where all curves converge regardless of Q. Increasing Q raises the peak value and shifts it to smaller radii: from $\Omega_r=0.0221$ at $r=8.0001$ for $Q=0$, to $\Omega_r=0.0310$ at $r=5.9501$ for $Q=1$. The charge therefore both strengthens the radial restoring force near the black hole and pulls the region of maximal epicyclic response inward, toward the black hole.

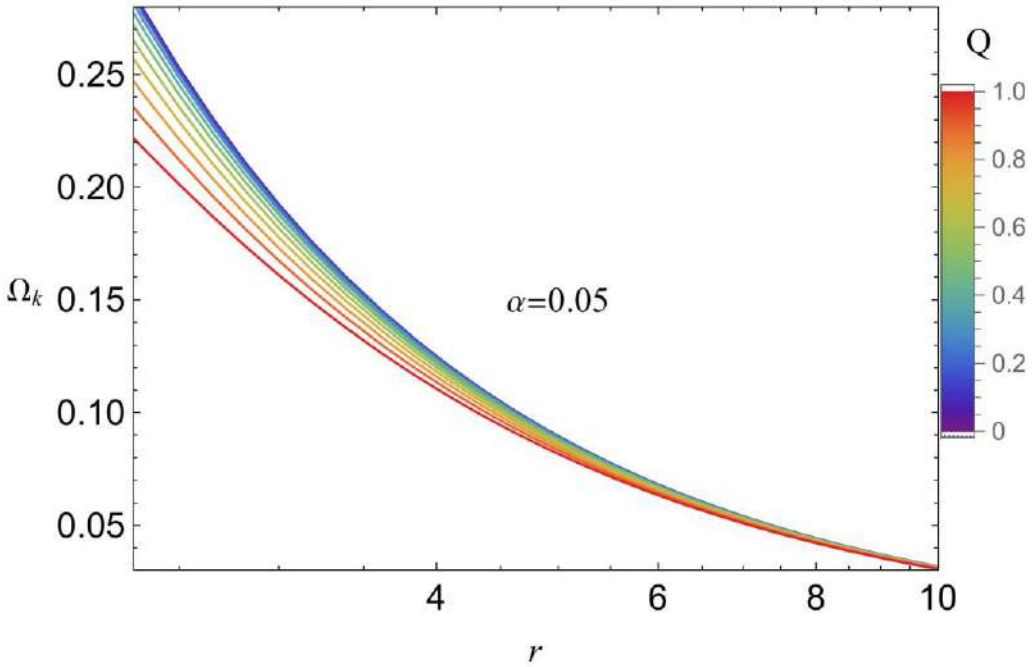


Fig. 2. Keplerian frequency Ω_K as a function of r, at fixed α=0.05, for Q from 0 to 1.

Fig. w shows the complementary behavior of $\Omega_K(r)$ at the same fixed α=0.05. Unlike Ω_r, Ω_K decreases monotonically with r and shows no extremum; increasing Q lowers Ω_K at every fixed r — for instance, from 0.0680 at $Q=0$ to 0.0633 at $Q=1$ at $r=6$ — with the separation between curves most visible at small r and vanishing as $r\to\infty$. A larger charge therefore slows the orbital angular frequency at a given radius, even as it raises the radial epicyclic response shown in Fig. 1.

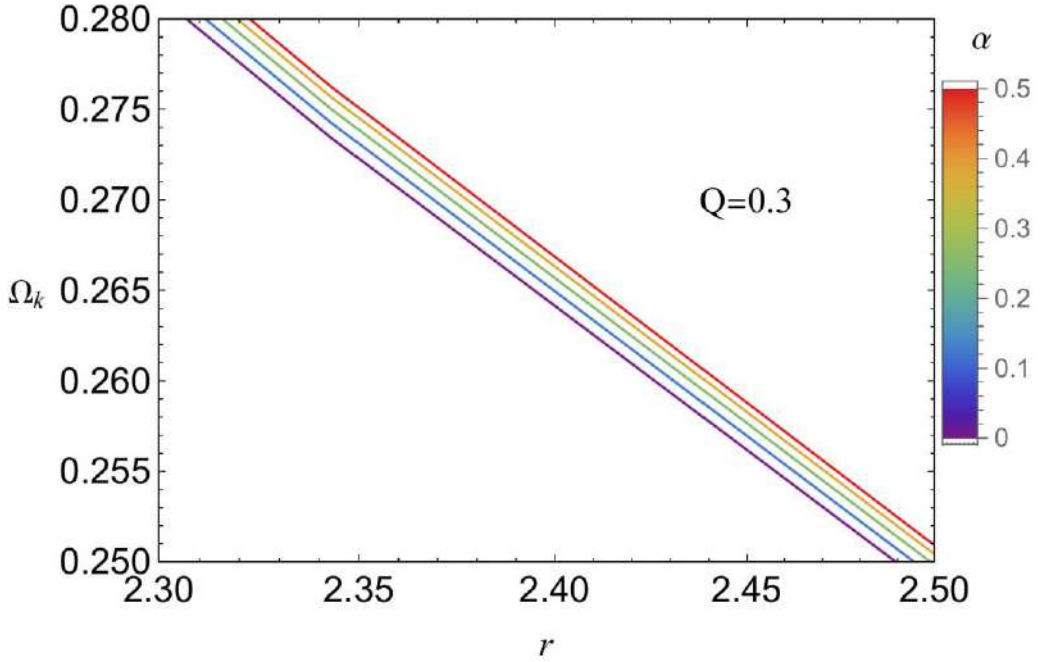


Fig. 3. Keplerian frequency Ω_K as a function of r, at fixed Q=0.3, for α from 0 to 0.5 (near-horizon detail).

Fig. 3 isolates the effect of α on Ω_K at fixed Q=0.3, in the near-horizon range $r\in$[2.30,2.50]. Here α acts in the opposite sense to Q: increasing α raises Ω_K at every fixed r — from 0.2639 at α=0 to 0.2670 at α=0.5 at r=2.40 — partially offsetting the charge-induced reduction seen in Fig. 2. The effect is modest in absolute size over this range but systematic and monotonic in α.

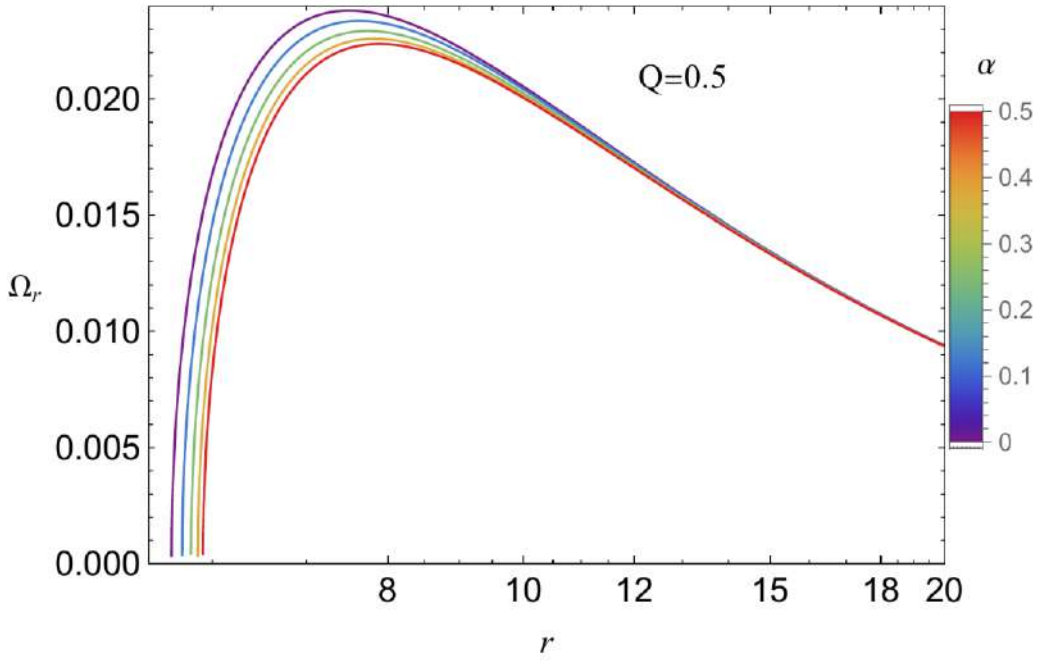


Fig. 4. Radial epicyclic frequency Ω_r as a function of r, at fixed Q=0.5, for α from 0 to 0.5.

Fig. 4 completes the picture, showing $\Omega_r(r)$ at fixed Q=0.5 for α from 0 to 0.5. Increasing α lowers the peak of Ω_r and shifts it outward — from Ω_r=0.0238 at r=7.5095 for α=0, to Ω_r=0.0223 at r=7.9333 for α=0.5 — the mirror image of the Q-driven trend in Fig. 1. As in Fig. 1, all curves converge at large r, confirming that the influence of both Q and α on the epicyclic response is confined to the near-horizon region.

4. Discussion

The results of Section 3 extend the opposite-direction pattern already established for the horizon structure and the effective potential to the Keplerian and radial epicyclic frequencies: at every fixed radius, increasing the charge Q raises Ω_r and lowers Ω_K, while increasing the screening parameter α does the reverse, driving both frequencies back toward their Schwarzschild values. Physically, the charge deepens and narrows the effective potential well near the black hole (as shown in the companion analysis), which simultaneously slows the mean orbital angular velocity and strengthens the radial

restoring force felt by a slightly perturbed circular orbit; Yukawa screening, entering only through exponential prefactors multiplying Q^2, systematically weakens both effects.

A further consequence, visible in Figs. 1 and 4, is that the peak of Ω_r shifts inward with Q and outward with α: the radius of maximal radial response is not fixed but moves together with Q and α themselves. Because twin-peak QPO models combine Ω_K and Ω_r at a common emission radius, this coupled shift means that any such combination inherits the same Q–α degeneracy already identified at the level of the individual frequencies, and must be treated with that degeneracy in mind.

5. Conclusion

Using the Einstein–Nonlinear Maxwell–Yukawa metric established in a companion analysis, we have derived closed-form expressions for the Keplerian angular frequency and the radial epicyclic frequency of neutral test particles, and verified both against the Schwarzschild limit.

The charge parameter Q and the Yukawa screening parameter α were found to act in opposite directions on both frequencies: the charge raises the peak of the radial epicyclic frequency and shifts it inward while lowering the Keplerian frequency at fixed radius, and screening reverses both trends, driving the geometry back toward the Schwarzschild profile. This pattern is the direct dynamical counterpart of the opposite-direction behavior already found for the horizon structure and the effective potential.

Overall, our results provide an internally consistent set of orbital and epicyclic frequencies for the ENLMY spacetime. The construction of twin-peak QPO frequency pairs from these results, and their comparison with observed microquasar and X-ray binary data through Bayesian parameter estimation, are left for a separate, dedicated analysis.

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