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DISCRETE RIGIDITY OF AUTOMORPHISMS INDUCED BY AN $\mathbf{A(z)}$-STRUCTURE

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Abstract: *This paper investigates the discrete rigidity of automorphisms of lemniscate-type domains determined by a fixed $A(z)-$analytic structure. The main objective is to distinguish the continuous symmetry typical of classical conformal and quasiconformal geometry from the analytic restrictions imposed by the exact functional invariance relation $F\circ f=F$. Comparative-analytic, structural, invariant-based, and infinitesimal deformation methods are employed. It is shown that admissible automorphisms preserve every fiber of F, the zero set, zero multiplicities, critical points, and the corresponding critical values. Differentiation of a hypothetical one-parameter family of admissible automorphisms yields the infinitesimal condition $dF_z\left(V(z)\right)=0$. Under a natural nondegeneracy assumption, this condition, together with preservation of a finite analytically distinguished skeleton, excludes nontrivial deformations through the identity. The action of the automorphism group on finite analytic data gives a homomorphism into a finite permutation group. The results clarify that discreteness and finiteness are logically distinct: global finiteness additionally requires triviality of the kernel of the permutation representation or an appropriate uniqueness principle. The proposed framework may be useful in the study of generalized analytic functions, anisotropic geometry, and inverse problems associated with $A(z)-$analytic systems.*

Keywords: *$A(z)$-analytic function, automorphism, discrete rigidity, lemniscate domain, fiber, invariance, critical point, permutation representation.*

$A(z)$-STRUKTURA TA'SIRIDA YUZAGA KELADIGAN AVTOMORFIZMLARNING DISKRET RIGIDLIGI

Annotatsiya: *Maqolada fiksirlangan $A(z)$-analitik struktura bilan aniqlangan lemniskata tipidagi sohalar avtomorfizmlarining diskret rigidligi o' rganiladi. Klassik konformal va kvazikonformal geometriyaga xos uzluksiz simmetriya $F\circ f=F$ aniq funksional invariantlik shartidan kelib chiqadigan analitik cheklovlar bilan qiyoslanadi. Ruxsat etilgan avtomorfizmlar F funksiyaning har bir tolasi, nol to' plami, nollar karraliligi, kritik nuqtalar va kritik qiymatlarni saqlashi ko' rsatiladi. Bir parametrli deformatsiyani differensiallash orqali $dF_z\left(V(z)\right)=0$ shart olinadi. Tabiiy nodegeneratsiya va yagonalik shartlari ostida avtomorfizmlar guruhining birlik komponenti trivial bo' lishi asoslanadi. Chekli analitik skeletga guruh ta'siri chekli permutatsion tasvirni hosil qiladi. Diskretlik va cheklilik alohida xulosalar ekani, global cheklilik uchun tasvir yadrosining trivialligi zarurligi aniqlashtiriladi.*

Kalit so' zlar: *$\mathbf{A(z)}$-analitik funksiya, avtomorfizm, diskret rigidlik, lemniskata sohasi, tola, invariantlik, kritik nuqta, permutatsion tasvir.*

ДИСКРЕТНАЯ РИГИДНОСТЬ АВТОМОРФИЗМОВ, ИНДУЦИРОВАННЫХ $\mathbf{A(z)}$-СТРУКТУРОЙ

Аннотация: *В статье исследуется дискретная ригидность автоморфизмов областей лемнискатного типа, определяемых фиксированной $A(z)$-аналитической структурой. Непрерывная симметрия классической конформной и квазиконформной геометрии сопоставляется с аналитическими ограничениями, возникающими из точного функционального условия $F\circ f=F$. Показано, что допустимые автоморфизмы сохраняют каждое волокно функции F, множество нулей, кратности нулей, критические точки и критические значения. Дифференцирование однопараметрической деформации приводит к условию $dF_z\left(V(z)\right)=0$. При естественных условиях невырожденности и*

единственности компонента единицы группы автоморфизмов тривиальна. Действие группы на конечном аналитическом скелете задаёт конечное перестановочное представление. Уточняется, что дискретность и конечность являются различными выводами, а для глобальной конечности требуется тривиальность ядра представления.

Ключевые слова: $\mathbf{A(z)}$*-аналитическая функция, автоморфизм, дискретная ригидность, лемнискатная область, волокно, инвариантность, критическая точка, перестановочное представление.*

1. Introduction

Automorphism groups are fundamental objects in complex analysis and geometric function theory because they describe symmetries compatible with a prescribed analytic or geometric category. In classical complex analysis, automorphisms of the unit disk provide the standard model. They depend continuously on parameters and form a non-discrete Lie group. Consequently, continuous deformations and local motions of points are natural features of classical conformal symmetry [3, 4].

The quasiconformal theory developed by Ahlfors, Lehto, and Virtanen retains a considerable degree of flexibility. A quasiconformal mapping is controlled by metric distortion, but it is not normally required to preserve the values of a predetermined analytic function. Hence the zero set, level sets, or critical configuration of an arbitrarily fixed function need not remain invariant under a general quasiconformal self-map [3, 4].

A substantially different situation occurs in generalized analytic function theory and in models involving anisotropic coefficient fields. In such settings, the coefficient structure becomes part of the analytic object itself. The theory of generalized and pseudoanalytic functions developed by Vekua and Bers shows that first-order elliptic systems can replace the classical Cauchy–Riemann equations. The prescribed coefficients then influence both local analytic behavior and global admissible symmetries [1, 2, 5].

Let F be a fixed $A(z)-$analytic function on a domain D. Consider the associated lemniscate-type domain

$$L = \{z \in D : |F(z)| < 1\}. \qquad (1)$$

The function F determines not only the boundary level $|F(z)|=1$ but also an internal decomposition of L into fibers $F(z)=c$. Therefore F acts as an analytic invariant encoding internal geometric states of the domain.

We study automorphisms $f: L \to L$ satisfying the exact functional invariance condition

$$F(f(z)) = F(z), \quad z \in L. \qquad (2)$$

Condition (2) is considerably stronger than the requirement $f(L)=L$. Domain invariance only guarantees that points remain inside L, whereas (2) requires the image of each point to have exactly the same F-value. Thus the freedom of an admissible automorphism is constrained by the full fiber structure of F.

The purpose of this paper is to identify the mechanism by which a fixed $A(z)-$structure suppresses continuous symmetry. We first derive invariant properties of fibers, zeros, multiplicities, and critical data. We then examine a hypothetical one-parameter family of automorphisms and obtain an infinitesimal constraint. Finally, the group action on a finite analytically distinguished set is used to formulate a permutation representation and a rigorous criterion for finiteness.

An important point of the present approach is the separation of discreteness from finiteness. The absence of nontrivial continuous deformations is related to the identity component of the group. Finiteness is a global conclusion and requires a faithful action on finite analytic data or an equivalent uniqueness theorem.

2. Literature Review and Theoretical Background

The foundations of generalized analytic function theory were established by Vekua and Bers. Their work introduced analytic frameworks in which first-order elliptic systems replace the classical Cauchy–Riemann equations. Bojarski further investigated generalized solutions of elliptic systems with discontinuous coefficients and clarified the structural role of coefficient fields [1, 2, 5].

For A-analytic functions, Sadullaev and Jabborov studied structural properties of the corresponding function class. Jabborov, Otaboyev, and Khursanov obtained analogues of the Schwarz inequality and the Schwarz formula. These results demonstrate that the A-structure enters directly into analytic identities. Related A-analytic systems also appear in two-dimensional tomography, as shown in the work of Arbuzov, Bukhgeim, and Kazantsev [6–8].

In classical quasiconformal theory, the basic restriction is expressed through metric distortion. In standard differential notation one considers [3, 4]

$$K_{\phi}(z)=\frac{|\partial\phi|+|\bar{\partial}\phi|}{|\partial\phi|-|\bar{\partial}\phi|}\le K<\infty. \qquad (3)$$

This inequality controls local eccentricity but does not impose $F\circ f=F$ for a fixed F. This distinction between a distortion inequality and an exact functional invariant is central to the rigidity problem considered here.

Most classical investigations of automorphism groups focus on the domain, a conformal class, or a hyperbolic metric as the primary object. The logical order in the present setting is different. First, the $A(z)$-analytic function F is fixed; then its fibers and distinguished local data determine an internal analytic structure; finally, only transformations preserving these data are admitted.

The existing literature provides the analytic background for the present study, but the mechanism by which exact functional invariance destroys continuous symmetry deserves a separate structural formulation. The aim here is to isolate this mechanism and identify the additional assumption required to pass from discreteness to finiteness.

3. Research Methodology

The research is theoretical and belongs to pure mathematical analysis. The principal objects are a fixed $A(z)$-analytic function F, the lemniscate-type domain L defined by (1), and automorphisms satisfying (2). Comparative-analytic, structural, invariant-based, local expansion, infinitesimal deformation, and group-action methods are used [6, 7].

At the first stage, the fibers of F are examined. The exact identity $F\circ f=F$ immediately gives invariance of every fiber and provides the basic structural restriction used throughout the paper.

At the second stage, zeros and their multiplicities are analyzed. The automorphism induces a permutation of the zero set. Local expansions show that, under local invertibility in the admissible analytic category, zeros of incompatible multiplicities cannot be freely exchanged.

At the third stage, critical points and critical values are considered. Differentiation of $F\circ f=F$ yields a covariance relation for the differential of. This identifies the critical set as invariant and provides additional labels for the finite analytic data.

The fourth stage is an infinitesimal deformation test. A differentiable one-parameter family $\{f_t\}$, with $f_0=id$, is substituted into the invariance relation and differentiated at $t=0$. The resulting condition $dF_z(V(z))=0$ restricts the infinitesimal velocity field.

Finally, the automorphism group is allowed to act on a finite analytically distinguished skeleton consisting of selected zeros and critical points with invariant labels. This action defines a homomorphism into a finite permutation group. The kernel of this homomorphism identifies precisely where an additional uniqueness principle is required.

4. Results and Discussion

4.1. Invariance of the Fibers of F

For $c\in F(L)$, define

$$L_c=\{z\in L: F(z)=c\}. \qquad (4)$$

Lemma 1. If $f:L\to L$ is a bijective admissible automorphism satisfying $F\circ f=F$, then $f(L_c)=L_c$ for every $c\in F(L)$

Proof. Let $z\in L_c$. Then $F(z)=c$ and, by (2), $F(f(z))=F(z)=c.$ Hence $f(z)\in L_c$ o-0=\ and $f(L_c)\subset L_c$. Since f is bijective, the same argument applied to f^{-1} gives the reverse inclusion. Therefore $f(L_c)=L_c$.

Lemma 1 shows that an admissible automorphism preserves not merely the collection of level sets but every fiber associated with a fixed value c. In particular, different F -values cannot be exchanged.

For $L=\{z:|F(z)|<1\}$, equation (2) also gives $|F(f(z))|=|F(z)|$. Thus every modulus level $|F(z)|=r$ is invariant. Under the required boundary regularity, preservation of $|F(z)|=1$ follows from the stronger internal functional identity.

4.2. Zero Set and Multiplicity

Let

$$Z(F)=\{a\in L: F(a)=0\}. \qquad (5)$$

Lemma 2. Every admissible automorphism satisfying (2) preserves the zero set: $f(Z(F))=Z(F)$.

Proof. If $a \in Z(F)$, then $F(a) = 0$. Equation (2) gives $F(f(a)) = 0$, so $f(a) \in Z(F)$. Hence $f(Z(F)) \subset Z(F)$. Applying the same argument to f^{-1} gives equality.

Suppose that a is a zero of order m. In a suitable local analytic coordinate one may write

$$F(z) = (z-a)^m u(z), u(a) \equiv 0. \qquad (6)$$

Let $b = f(a)$. Since an admissible automorphism is locally invertible, composition with f cannot arbitrarily change the local vanishing order. Comparing the orders of the two sides of $F \circ f = F$ shows that the zero at b must have a compatible local multiplicity.

Consequently, the zero set decomposes into multiplicity classes and the automorphism acts by permutations within analytically compatible classes. When $Z(F)$) is finite, this provides the first finite constraint on the automorphism group.

4.3. Critical Points and Critical Values

Differentiating the exact invariance relation gives the formal covariance identity

$$dF_{f(z)}\, dfz \;=\; dF_z \qquad (7)$$

Because df_z is invertible, degeneration of dF is preserved. If $dF_z = 0$, then $dF_{\{f(z)\}} = 0$. Hence the critical set $Crit(F)$ is invariant.

$F(f(z)) = F(z)$ Moreover, fixes the critical value attached to each critical point. Critical points may therefore be labeled by their F -values and, when available, by their local critical types.

Zeros with multiplicities and critical points with invariant labels form a finite analytic skeleton whenever the relevant sets are finite. This skeleton is intrinsic: it is generated by F itself rather than selected externally.

4.4. One-Parameter Deformations and Discrete Rigidity

Assume that {f_t}_{|t|<ε} is a differen$t = 0$tiable one-parameter family of admissible automorphisms such that $f_0 = id$ and

$$F(f_t(z)) \;=\; F(z). \qquad (8)$$

Define the infinitesimal velocity field by

$$V(z) \;=\; (d/dt)\, f_t(z)|_{t=0}. \qquad (9)$$

Differentiating (8) at yields

$$dF_z(V(z)) \;=\; 0. \qquad (10)$$

At regular points of F, condition (10) restricts V to directions compatible with the fiber through z. This local equation alone does not always imply $V \equiv 0$; the global distinguished data must also be used.

If the zero set and the selected critical configuration are finite, the family f_t induces a continuous map from a connected parameter interval into a finite permutation group. Such a map is locally constant. Since $f_0 = id$, all labeled distinguishe $f_t = id$ d points are fixed individually for sufficiently small t.

Theorem 1. Let S be a finite analytic skeleton associated with F. Assume that every admissible automorphism preserves S and that A an automorphism satisfying $F \circ f = F$ and fixing S pointwise must be the identity. Then the identity component of the admissible automorphism group is trivial.

Proof. Let $G = Aut_{A(z)}(L)$ denote the group of admissible automorphisms of L, and let G^0 be the identity component of G. We prove that $G^0 \;=\; \{id\}$. Suppose, to the contrary, that G^0 contains a non-identity element. Then there exists a nontrivial continuous path $\{ f(t)\{t \in [0,1]\}$ in G^0 such that $f_0 \;=\; id.$ Fix an arbitrary point $p \in S$. Since every admissible automorphism preserves the analytic skeleton S, for every $t \in [0,1]$ we have $f_t(p) \in S.$ Consider the orbit map $\gamma_p : [0,1] \to S$ defined by $\gamma_p(t) = f_t(p)$ The path $t \mapsto f_t$ is continuous in the automorphism group, and evaluation at a fixed point p is continuous. Hence γ_p is a continuous map.

The interval $[0,1]$ is connected. Therefore its continuous image $\gamma_p([0,1])$ must also be connected. On the other hand, S is finite and thus carries the discrete topology inherited from the ambient domain. The only connected subsets of a finite discrete set are singletons. Consequently, $\gamma_p([0,1])$ consists of one point, and $f_t(p)$ is independent of t.

Since $f_0 = id$, we obtain

$f_t(p) = f_0(p) = p$ *for every* $t \in [0,1]$. *(11)*

Because $p \in S$ was arbitrary, every automorphism along the path fixes the whole analytic skeleton pointwise:

$$f_t|_S = id_S \quad (12)$$

By the uniqueness assumption of the theorem, an admissible automorphism satisfying $F \circ f = F$ and fixing S pointwise must be the identity on L. Therefore,

$f_t = id_L$ *for every* $t \in [0,1]$. *(13)*

Thus the path $\{f_t\}$ is constant, which contradicts the assumption that it is nontrivial. Hence no nontrivial continuous path through the identity exists in the admissible automorphism group. Therefore,

$G^0 = \{id\}$. *(14)*

This proves that the identity component of the admissible automorphism group is trivial.

4.5. Finite Permutation Representation

Let S be a finite analytic skeleton refined by multiplicity, critical value, and other invariant local labels. Every admissible automorphism induces a label-preserving permutation of S. Therefore there is a group homomorphism

$$\rho : Aut_{A(L,F)} \to Perm(S). \quad (15)$$

Since S is finite, the image of ρ is finite. However, this fact alone does not imply that $G = Aut_{A(L,F)}(L)$ is finite, because the kernel may contain automorphisms acting trivially on S.

Theorem 2. If

$$\ker \rho = \{\mathrm{id}\}, \quad (16)$$

then the admissible automorphism group G is finite.

Proof. Since every admissible automorphism preserves the finite analytic skeleton S, each element $f \in G$ induces a permutation of S. Therefore, the action of G on S defines a group homomorphism

$$\rho : G \to \mathrm{Sym}(S).$$

Condition (16) means that the only automorphism acting trivially on every point of S is the identity. Hence ϱ has trivial kernel and is therefore injective. Consequently, G is isomorphic to its image:

$G \cong \rho(G)$. Since the group G can be identified with a subgroup of the symmetric group of the finite set S. If |S| = N, then $|\mathrm{Sym}(S)| = N!$. Thus,

$$|G| = |\rho(G)| \le N! < \infty, \quad (17)$$

which proves the finiteness of G. The analytic structure imposes additional restrictions on the possible permutations. Indeed, if an admissible automorphism satisfies

$F \circ f = F$, then for every w and every $z \in F^{-1}(w)$, $F(f(z)) = F(z) = w$. Hence

$f(F^{-1}(w)) \subseteq F^{-1}(w)$. Since f is bijective, we obtain

$$f(F^{-1}(w)) = F^{-1}(w). \quad (18)$$

Therefore, the induced permutations of the points of S must also be compatible with the full fiber structure of F. In particular, not every element of $Sym(S)$ is realized by an admissible automorphism. Thus the actual group G is generally much smaller than the full symmetric group.

Hence G is finite. □

5. Illustrative Example and Geometric Interpretation

Consider the classical analytic model $F(z) = z^n - 1,\ n \ge 2$, and the lemniscate-type domain

$$L = \{z \in \mathbb{C} : |z^n - 1| < 1\}. \quad (19)$$

The zeros of F are

$$z_k = exp(2\pi ik/n), \quad k = 0,1,\ldots,n-1. \quad (20)$$

All zeros are simple and $Z(F)$ consists of n points. By Lemmas 1 and 2, every automorphism satisfying $F \circ f = F$ preserves every fiber and induces a permutation of $Z(F)$. Hence

$$\rho : Aut_{A(L,F)} \to Perm(S). \qquad (21)$$

is defined. If an admissible automorphism is uniquely determined by its action on $Z(F)$, then Theorem 2 implies finiteness of the group.

The example illustrates an important geometric feature. Even when the zero configuration is highly symmetric, the condition $F \circ f = F$ does not permit a free continuous motion of zeros. An admissible transformation must be compatible with both the finite zero configuration and the entire fiber decomposition.

From a geometric viewpoint, the $A(z)$-structure can be interpreted as symmetry-breaking internal data. A coordinate change may preserve the outer shape of a domain while modifying its internal anisotropy. Exact preservation of the $A(z)$-analytic invariant restricts the admissible class to transformations that retain the encoded internal states.

This interpretation is also relevant to inverse problems. Continuous symmetries often generate continuous nonuniqueness or gauge freedom. If the admissible symmetry class collapses to a discrete or finite residual group, the structural ambiguity in reconstruction may be reduced. The present paper does not solve a specific tomography problem, but it identifies a rigidity mechanism that may enter future uniqueness arguments [8].

6. Discussion

The obtained results reveal three levels of restriction produced by $F \circ f = F$ First, the functional value is preserved and every fiber is invariant. Second, local analytic data such as zero multiplicities and critical labels are retained. Third, the action on finite distinguished data allows the automorphism group to be compared with a finite permutation group.

A conceptual argument may be tempted to conclude that the absence of continuous families automatically implies finiteness. The present formulation avoids this overstatement. Triviality of the identity component concerns local group structure and supports discreteness. Finiteness is global and depends on the absence of an infinite residual kernel.

The kernel of the permutation representation identifies the exact point at which a separate uniqueness theorem is needed. In a concrete $A(z)$-analytic algebra, one may have to prove that an automorphism fixing all selected zeros, critical points, and relevant fiber data is necessarily the identity.

Highly symmetric or degenerate choices of F may retain nontrivial finite symmetries. Conversely, a sufficiently asymmetric configuration of multiplicities and critical values may force the group to be trivial. Thus the analytic configuration of F is a natural predictor of the size of the admissible automorphism group.

Future work should formulate verifiable nondegeneracy conditions for specific $A(z)$-analytic algebras, obtain effective upper bounds for the order of the automorphism group, and investigate the interaction of this rigidity mechanism with integral operators and inverse problems.

7. Conclusion

This paper has investigated the rigidity mechanism of automorphisms associated with a fixed $A(z)$-analytic structure. The exact relation $F \circ f = F$ was shown to be substantially stronger than ordinary domain invariance.

Every fiber of F is preserved. The zero set is invariant, and local invertibility forces compatibility of zero multiplicities. Critical points and their critical values provide further analytically distinguished data.

Differentiation of a one-parameter family gives the infinitesimal condition $dF_z(V(z)) = 0.$ Combined with a finite analytic skeleton and a natural uniqueness assumption, this eliminates nontrivial deformations through the identity and explains discrete rigidity.

The action on a finite labeled analytic skeleton gives a finite permutation representation. If the kernel is trivial, the entire automorphism group is finite. Therefore discreteness and finiteness are related but logically distinct conclusions.

The proposed framework provides a conceptual basis for further studies of $A(z)$-analytic geometry, anisotropic structures, and symmetry or uniqueness problems associated with generalized analytic systems.

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