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FAZODA LAPLAS TENGLAMASI UCHUN KOSHI MASALASI

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Annotatsiya: Maqolada fazodagi Laplas tenglamasi uchun Koshi masalasi qaralgan. Koshi masalasining chegaraning bir qismida berilgan ma'lumotlar orqali yechimni sohaning ichki qismiga davom ettirish masalasini tadqiq qilish imkonini berilgan. Olingan natijalar elliptik tipdagi tenglamalar uchun nokorrekt masalalarni o'rganishda va ularning yechimlarini davom ettirish masalalarida foydalanish mumkin.

Kalit so'zlar: Koshi masalasi, nokorrekt masalalar, Karleman funksiyasi, davom ettirish formulalari.

ЗАДАЧА КОШИ ДЛЯ УРАВНЕНИЯ ЛАПЛАСА В ПРОСТРАНСТВЕ

Аннотация: В статье рассматривается задача Коши для уравнения Лапласа в пространстве. Исследуется задача продолжения решения во внутреннюю часть области по данным, заданным на части границы. Полученные результаты могут быть использованы при исследовании некорректных задач для уравнений эллиптического типа и задач продолжения их решений.

Ключевые слова: Задача Коши, некорректные задачи, функция Карлемана, формулы продолжения.

THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN SPACE

Abstract: The paper considers the Cauchy problem for the Laplace equation in space. The problem of continuation of a solution into the interior of the domain from data given on a part of the boundary is investigated. The obtained results can be used in the study of ill-posed problems for elliptic-type equations and problems of continuation of their solutions.

Keywords: Cauchy problem, ill-posed problems, Carleman function, continuation formulas.

Nokorrekt masalalarni yechimlar sinfini kompaktga qadar toraytirilsa, bu masala turg' un bo' lishiga doir birinchi natijalar A.N.Tixonov ishlarida keltirilgan. Laplas tenglamasi uchun Koshi masalasining Karleman funksiyasi tushunchasi M.M.Lavrent'yev[2] tomonidan kiritilgan. Tekislikdagi yo' lak ko' rinishdagi cheksiz sohada Laplas tenglamasi uchun Koshi masalasining regulyarizatsiyasi V.K.Ivanov[3] tomonidan o' rganilgan. Ko' p o' lchamli fazoda soha chegarasining bir qismi konus bo' lganda Laplas va Gelmgols tenglamalari uchun Koshi masalasining yechimi Sh.Yarmuxamedov[4] tomonidan Karleman funksiyasini qurish asosida olingan. Tekislikda Laplas tenglamasi uchun Koshi masalasining Karleman funksiyasi [5] da oysimon soha uchun o' rganilgan.Bu maqolada fazoda Laplas tenglamasi uchun Koshi masalasining yangi ko' rinishdagi Karleman funksiyasi asosida yechimi berilgan.

1. Masalaning qo' yilishi. fazoda birlik sfera qismi hamda silliq sirt bilan chegaralangan sohani qaraymiz. Bunda koordinata boshi sohadan tashqarida joylashgan deb olamiz. R^3SD

$x = (x_1, x_2, x_3) \in D$ va bo' lsin. Qaralayotgan sohada $y = (y_1, y_2, y_3) \in \bar{D}D$

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = 0 \quad (1)$$

Laplas tenglamasi yechimi garmonik funksiyaning Koshi berilganlari da berilgan:

$$U(x) = U(x_1, x_2, x_3) \in C^2(D) \cap C^1(\bar{D})S$$

$$U(x)|_S = f_0(x), \quad \frac{\partial U(x)}{\partial n} \Big|_S = f_1(x), \quad (2)$$

bu yerda va - berilgan funksiyalar. (2) shartlarga ko' ra funksiyani sohada topish masalasi qaraladi. $f_0(x)f_1(x)U(x)$

2. Karleman funksiyasi qurish sxemasi. (1) – (2) masala yechimini topishda Karleman funksiyasi metodidan foydalanamiz. [2] ga ko' ra Laplas tenglamasining Karleman funksiyasi ta'rifini keltiramiz:

Ta'rif. to' plamning sohaga nisbatan Karleman funksiyasi deb ikkita va nuqtalarga hamda musbat sonli parametrga bog' liq bo' lgan, quyidagi ikki shartlarni qanoatlantiruvchi funksiyaga aytiladi: $SDx y \sigma G(x, y, \sigma)$

$$1) G(x, y, \sigma) = \frac{1}{|x-y|} + g(x, y, \sigma), \quad (3)$$

bu yerda funksiya o' zgaruvchi bo' yicha sohada garmonik funksiya: $g(x, y, \sigma)yD$

$$2) \int_{\partial D \setminus S} \left(|G(x, y, \sigma)| + \left| \frac{\partial G(x, y, \sigma)}{\partial n} \right| \right) ds \leq \alpha(\sigma); \quad (4)$$

bunda funksiya da nolga intiladi. $\alpha(\sigma)\sigma \rightarrow \infty$

Bu ta'rifda funksiya Karleman funksiyasining regulyar qismi deyiladi. $g(x, y, \sigma)$

$$G(x, y, N) = \sum_{n=N+1}^{\infty} \frac{\rho^n}{\rho_1^{n+1}} P_n(\cos \gamma) = \frac{1}{r} - \sum_{n=0}^N \frac{\rho^n}{\rho_1^{n+1}} P_n(\cos \gamma)$$

deb belgilaymiz. Bu yerda Lejandr polinomlari, $P_n(\cos \gamma)$ –

$$\gamma = \angle(x, y), x = (\rho, \theta, \varphi), y = (\rho_1, \theta_1, \varphi_1),$$

$$|x| = \rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, |y| = \rho_1 = \sqrt{y_1^2 + y_2^2 + y_3^2},$$

$$r = |x - y| = \sqrt{\rho^2 + \rho_1^2 - 2\rho\rho_1 \cos \gamma}.$$

Yuqorida berilgan ta'rifdagi Karleman funksiyasining regulyar qismini

$$g(x, y, N) = - \sum_{n=0}^N \frac{\rho^n}{\rho_1^{n+1}} P_n(\cos \gamma) \quad (5)$$

ko' rinishda olamiz, bu yerda Karleman funksiyasining parametri sifatida natural son olamiz. σN

Teorema. funksiya qaralayotgan sohada yoyi uchun Laplas tenglamasining Karleman funksiyasi bo' ladi.

$$G(x, y, N) = \frac{1}{r} + g(x, y, N)DS$$

Teoremani isbotlashdan oldin quyidagi lemmani isbotlaymiz.

Lemma. (5) tenglik bilan aniqlangan funksiya sohada o' zgaruvchi bo' yicha garmonik funksiya bo' ladi.

$$g(x, y, N) Dy$$

Isbot. funksiyani o' zgaruvchi bo' yicha Laplas tenglamasini qanoatlantirishini ko' rsatamiz ya'ni: $g(x, y, N)y$

$$\Delta_y g(x, y, N) = - \sum_{n=0}^N \rho^n \Delta_y \left(\frac{P_n(\cos \gamma)}{\rho_1^{n+1}} \right).$$

Qatordagi laplasian ichidagi funksiyani belgilaymiz. $g^* = \rho_1^{-n-1} P_n(\cos \gamma)$

Δg^* tegishli hosilalarni hisoblaymiz:

$$\begin{aligned} \frac{\partial^2 g^*}{\partial y_i^2} &= \frac{\partial^2 (\rho_1^{-n-1})}{\partial y_i^2} P_n(\cos \gamma) + \frac{\partial (\rho_1^{-n-1})}{\partial y_i} P'_n(\cos \gamma) \frac{\partial \cos \gamma}{\partial y_i} + \\ &+ \frac{\partial (\rho_1^{-n-1})}{\partial y_i} P'_n(\cos \gamma) \frac{\partial \cos \gamma}{\partial y_i} + \rho_1^{-n-1} P''_n(\cos \gamma) \left(\frac{\partial \cos \gamma}{\partial y_i} \right)^2 + \\ &+ \rho_1^{-n-1} P'_n(\cos \gamma) \frac{\partial^2 \cos \gamma}{\partial y_i^2} \end{aligned}$$

$$\begin{aligned}\Delta_y g^* &= P_n(\cos \gamma) \sum_{i=1}^3 \frac{\partial^2 (\rho_1^{-n-1})}{\partial y_i^2} + 2P'_n(\cos \gamma) \sum_{i=1}^3 \frac{\partial (\rho_1^{-n-1})}{\partial y_i} \cdot \frac{\partial \cos \gamma}{\partial y_i} + \\ &+ P'_n(\cos \gamma) \rho_1^{-n-1} \sum_{i=1}^3 \frac{\partial^2 \cos \gamma}{\partial y_i^2} + P''_n(\cos \gamma) \rho_1^{-n-1} \sum_{i=1}^3 \left(\frac{\partial \cos \gamma}{\partial y_i} \right)^2 \\ \frac{\partial^2 (\rho_1^{-n-1})}{\partial y_i^2} &= (n+1)(n+3) y_i^2 \rho_1^{-n-5} - (n+1) \rho_1^{-n-3}; \\ \sum_{i=1}^3 \frac{\partial^2 \cos \gamma}{\partial y_i^2} &= \Delta_y \cos \gamma = -2\rho_1^{-2} \cos \gamma; \\ \sum_{i=1}^3 \left(\frac{\partial \cos \gamma}{\partial y_i} \right)^2 &= \rho_1^{-2} - 2\rho_1^{-2} \cos^2 \gamma + \rho_1^{-2} \cos^2 \gamma = \rho_1^{-2} (1 - \cos^2 \gamma)\end{aligned}$$

ifodalarni e'tiborga olib

$$\begin{aligned}\Delta_y g^* &= n(n+1) \rho_1^{-n-3} P_n(\cos \gamma) - \\ &- 2\rho_1^{-n-3} \cos \gamma P'_n(\cos \gamma) + (1 - \cos^2 \gamma) \rho_1^{-n-3} P''_n(\cos \gamma) \\ &\text{ekanligini hosil qilamiz va ni hosil qilamiz. Demak, } \Delta_y g^* = 0\end{aligned}$$

$$\Delta_y g(x, y, N) = - \sum_{n=0}^N \rho^n \Delta_y \left(\frac{P_n(\cos \gamma)}{\rho_1^{n+1}} \right) = 0$$

(5) funksiyaning o' zgaruvchi bo' yicha garmonikligi isbotlandi.

Isbotlangan lemmadan funksiya Laplas tenglamasining Karleman funksiyasining 1) shartini qanoatlantirishi kelib chiqadi.

Endi funksiya ta'rifni 2) shartini qanoatlantirishini ko' rsatish uchun $G(x, y, N) G(x, y, N)$

$$\int_{\partial D \setminus S} \left(|G(x, y, N)| + \left| \frac{\partial G(x, y, N)}{\partial n} \right| \right) ds_y, \quad x \in D$$

integralni baholaymiz.

Lejandr polinomi bahosi uchun quyidagi lemmani keltiramiz.

2-Lemma. Lejandr polinamlari uchun quyidagi baho o' rinli:

$$\left| \frac{dP_n(t)}{dt} \right| \leq n^2.$$

Lemma isboti Markov tengsizligidan [Natanson I.P., 174 bet] foydalaniladi.

$G(x, y, N)$ funksiyaning $x \in D$, $y \in \partial D / S$ uchun baholaymiz:

$$|G(x, y, N)|_{\partial D \setminus S} = \left| \sum_{n=N+1}^{\infty} \frac{\rho^n}{\rho_1^{n+1}} P_n(\cos \gamma) \right|_{\rho_1=1} \leq \frac{\rho^{N+1}}{1 - \rho}. \quad (10)$$

Xuddi shunga o' xshash funksiyaning nuqtadagi normal bo' yicha hosilasini baholaymiz: $G_y \in \partial D \setminus S$

$$\begin{aligned}\left| \frac{\partial G}{\partial n} \right|_{\partial D \setminus S} &= \left| \frac{\partial G}{\partial \rho_1} \right|_{\rho_1=1} = \\ &= \frac{\rho^{N+1}}{1 - \rho} + \frac{\rho^{N+1} (N+1 - N\rho)}{(1 - \rho)^2} + \frac{\rho^{N+1} ((1 - \rho)^2 N^2 + 2(1 - \rho)N + \rho + 1)}{(1 - \rho)^3} =\end{aligned}$$

$$= \frac{\rho^{N+1}((1-\rho)^2 N^2 + 2(\rho^2 - 4\rho + 3)N + \rho^2 - 2\rho + 2)}{(1-\rho)^3} \quad (11)$$

Shunday qilib,

$$\begin{aligned} \int_{\partial D \setminus S} \left(|G(x, y, N)| + \left| \frac{\partial G(x, y, N)}{\partial n} \right| \right) dS_y &\leq \\ &= \\ &\leq \frac{4\pi\rho^{N+1}((1-\rho)^2 N^2 + 2(\rho^2 - 4\rho + 3)N + 2\rho^2 - 4\rho + 3)}{(1-\rho)^3}, \end{aligned}$$

Demak,

$$\int_{\partial D \setminus S} \left(|G(x, y, N)| + \left| \frac{\partial G(x, y, N)}{\partial n} \right| \right) dS_y \leq \frac{4\pi\rho^{N+1}Q(N)}{(1-\rho)^3} \quad (12)$$

bu yerda ga nisbatan ikkinchi darajali ko' phad.

$$Q(N) = (1-\rho)^2 N^2 + 2(\rho^2 - 4\rho + 3)N + 2\rho^2 - 4\rho + 3N$$

(12) tengsizlikning o' ng tomonidagi ifoda bo' lgani uchun da nolga intiladi. $0 < \rho < 1 \quad N \rightarrow \infty \rightarrow \frac{4\pi\rho^{N+1}Q(N)}{(1-\rho)^3}$

Bu Karleman funksiyasining ikkinchi shartini isbotlaydi. Teorema isbotlandi.

Grin formulasiga ko' ra (1), (2) masala yechimi uchun

$$U(x) = \frac{1}{2\pi} \int_{\partial D} \left[\frac{1}{r} \cdot \frac{\partial U(y)}{\partial n} - U(y) \cdot \frac{\partial \frac{1}{r}}{\partial n} \right] dS \quad x \in D \quad (13)$$

integral tasvir o' rinli.

$g(x, y, N)$ funksiya o' zgaruvchi bo' yicha sohada garmonikligidan uchun Grin formulasiga ko' ray $Dx \in D$

$$\begin{aligned} \int_{\partial D} \left[g(x, y, N) \cdot \frac{\partial U(y)}{\partial n_y} - U(y) \cdot \frac{\partial g(x, y, N)}{\partial n_y} \right] dS_y &= \\ &= \int_D [g(x, y, N)\Delta_y U(y) - U(y)\Delta_y g(x, y, N)] dy = 0 \end{aligned} \quad (14)$$

tenglik o' rinli. Masala shartidagi berilganlarni hisobga olib (13) va (14) tengliklardan

$$\begin{aligned} U(x) &= \int_{\partial D \setminus S} \left[G(x, y, N) \cdot \frac{\partial U(y)}{\partial n} - U(y) \cdot \frac{\partial G(x, y, N)}{\partial n} \right] dS_y + \\ &+ \int_S \left[G(x, y, N) \cdot f_1(y) - f_0(y) \cdot \frac{\partial G(x, y, N)}{\partial n} \right] dS_y \end{aligned}$$

tenglikni hosil qilamiz. Oxirgi tenglikda hamda (12) tengsizlikni hisobga olgan holda limitga o' tib

$$U(x) = U(x_1, x_2, x_3) \in C^2(D) \cap C^1(\bar{D}) \quad N \rightarrow \infty$$

$$U(x) = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_S \left[G(x, y, N) \cdot f_1(y) - f_0(y) \cdot \frac{\partial G(x, y, N)}{\partial n} \right] dS_y$$

tenglikni olamiz. Teorema isbotlandi.

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