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НАМАНГАНСКИЙ
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UZBEKISTAN ACADEMY
OF SCIENCES
V.I.ROMANOVSKIY
INSTITUTE OF
MATHEMATICS

O'zR FA
V.I.ROMANOVSKIY
NOMIDAGI
MATEMATIKA
INSTITUTI

ИНСТИТУТ
МАТЕМАТИКИ им
В.И. РОМАНОВСКОГО
АН РУз



ABSTRACTS OF THE CONFERENCE “NEW THEOREMS OF YOUNG MATHEMATICIANS-2025”

«YOSH MATEMATIKLARNING YANGI
TEOREMALARI – 2025»
ILMIY ANJUMAN

TEZISLARI TO'PLAMI

ТЕЗИСЫ ДОКЛАДОВ

НАУЧНОЙ КОНФЕРЕНЦИИ
«НОВЫЕ ТЕОРЕМЫ МОЛОДЫХ
МАТЕМАТИКОВ – 2025»

Namangan, Uzbekistan,
20-21.10.2025

Namangan, O'zbekiston
20-21.10.2025 yil

Наманган, Узбекистан,
20-21.10.2025 год.

**O‘ZBEKISTON RESPUBLIKASI
OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI**

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ИНСТИТУТ МАТЕМАТИКИ им. В.И. РОМАНОВСКОГО АН РУз

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**Namangan, Uzbekistan,
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**Namangan, O‘zbekiston
20 – 21 Oktabr, 2025 yil**

**Наманган, Узбекистан,
20 – 21 октября 2022 год.**

To'plam O'zbekiston Respublikasi Prezidentining 2023-yil 4-iyuldagi "Ma'muriy islohotlar doirasida oliy ta'lim, fan va innovatsiyalar sohasida davlat boshqaruvini samarali tashkil qilish chora-tadbirlari to'g'risida" 2023-yil 3-iyundagi PQ-200 – son Qarori 9-bandida belgilangan vazifalar ijrosini ta'minlash maqsadida qabul qilingan Oliy ta'lim, fan va innovatsiyalar vazirining 2024-yil 27-dekabrda 490-sonli buyrug'i bilan tasdiqlangan anjumanlar ro'yxatiga asosan **2025 yilning 20-21 oktabr** kunlari Namangan davlat universitetida bo'ladigan **"Yosh matematiklarning yangi teoremlari - 2025"** mavzusidagi Respublika ilmiy anjumani ishtirokchilarining ilmiy ishlarini o'z ichiga olgan bo'lib, *Algebra, funksional analiz, geometriya, matematik analiz, differensial tenglamalar va matematik fizika tenglamalari, matematik modellash va dinamik sistemalar, ehtimollar nazariyasi va matematik statistika bilan shug'ullanuvchi talabalar, magistrlar, katta ilmiy xodim-tadqiqotchilar va oliygoh o'qituvchilari uchun mo'ljallangan.*

Eslatma: To'plamdagi tezislarning mazmuniga mualliflar javobgardirlar.

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«YOSH MATEMATIKLARNING YANGI TEOREMALARI – 2025»

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I.Sattarov	– O'zR FA Matematika instituti, PhD.
R.Polvanov	– Namangan davlat universiteti, dotsent.

SO‘Z BOSHI

Ushbu to‘plam O‘zbekiston Respublikasi Prezidentining 2023-yil 4-iyuldagi “Ma‘muriy islohotlar doirasida oliy ta‘lim, fan va innovatsiyalar sohasida davlat boshqaruvini samarali tashkil qilish chora-tadbirlari to‘g‘risida” 2023-yil 3-iyundagi PQ-200 – son Qarori 9-bandida belgilangan vazifalar ijrosini ta‘minlash maqsadida qabul qilingan Oliy ta‘lim, fan va innovatsiyalar vazirining 2024-yil 27-dekabrdagi 490-sonli buyrug‘i bilan tasdiqlangan anjumanlar ro‘yxatiga asosan **2025 yilning 20-21 oktabr** kunlari Namangan davlat universitetida bo‘ladigan **“Yosh matematiklarning yangi teoremlari - 2025”** mavzusidagi Respublika ilmiy anjumani ishtirokchilarining ilmiy ishlarini o‘z ichiga olgan.

Anjuman ishida O‘zbekistonning deyarli barcha oliygohlaridan 200 dan ortiq yetakchi olimlar va yosh mutaxassislar ishtirok etishlari kutilmoqda.

Anjumaning maqsadi Respublikamiz oliygohlarida keyingi yillarda matematikaning turli sohalari bo‘yicha tayyorlangan ilmiy natijalarni muhokama qilish va chop etishdan iborat bo‘ldi.

Anjuman dasturiga muvofiq anjuman qatnashchilari matematikaning zamonaviy muammolari quyidagi sho‘balarda ko‘rib chiqildi.

SHO‘BALAR

1. *Algebra, funksional analiz, geometriya.*
2. *Matematik analiz.*
3. *Differensial tenglamalar va matematik fizika tenglamalari.*
4. *Matematik modellashtirish va dinamik sistemalar.*
5. *Ehtimollar nazariyasi va matematik statistika.*

ANJUMAN OCHILISHI YALPI MAJLISI

2025 yil 20 oktabr kuni soat 10:00-12:30 da universitet asosiy binosining Anjumanlar zalida ochiladi.

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N.Xatamov, NamDU, matematika kafedrası mudiri, DSc, dotsent.

1-ma‘ruza

“Gibbs measures in machine learning”

U.A.Rozikov, O‘zR FA Matematika instituti direktor o‘rinbosari, f-m.f.d., akademik.

2-ma‘ruza

“To‘rtinchi tartibli Balka va plastinaning buziladigan tenglamalari uchun boshlang‘ich shartli masala”

Sh.Karimov Farg‘ona davlat universiteti, DSc, professor.

3-ma‘ruza

“F-kvadratik operatorlar”

U.Jamilov, O‘zR FA Matematika instituti, DSc, professor.

4-ma‘ruza

“Bo‘lajak olimlar uchun noassosiativ algebralar”

J.Adashev O‘zR FA Matematika instituti, DSc, professor.

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1-SHO‘BA:ALGEBRA,FUNKSIONALANALIZ,GEOMETRIYA.

Local derivations of some naturally graded associative algebras with characteristic sequence $(3, 2)$

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The concept of local derivations was introduced by R. Kadison in his paper [3]. In this paper R. Kadison proved that each continuous local derivation from a von Neumann algebra into its dual Banach bmodule is a derivation. In [2] B. Jonson extended the above result by proving that every local derivation from a C^* -algebra into its Banach bimodule is a derivation. Based on these results, many authors have studied local derivations on operator algebras.

In the present paper local derivations of five-dimensional naturally graded associative algebra with characteristic sequence $C(A) = (3, 2)$ are studied. Namely, a general form of the matrices of derivations and local derivations of these algebras is clarified. It turns out that the general form of the matrix of an derivation on these algebras does not coincide with the local derivation's matrix's general form on these algebras. Therefore, these associative algebras have local derivations that are not derivations.

For an algebra $\mathcal{A}$ of an arbitrary variety, we consider the series

$$\mathcal{A}^1, \quad \mathcal{A}^{i+1} = \sum_{k=1}^i \mathcal{A}^k \mathcal{A}^{i-k+1}, \quad i \geq 1.$$

We say that an algebra $\mathcal{A}$ is *nilpotent* if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathcal{A}^i = 0$ is called the index of nilpotency or nilindex of $\mathcal{A}$.

An algebra $\mathcal{A}$ is called *N-graded* if there is a decomposition $\mathcal{A} = \bigoplus_{i \in \mathbb{N}} \mathcal{A}_i$, $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$, for all $i, j \in \mathbb{N}$. Given an associative algebra $\mathcal{A}$, one can consider its associated graded associative algebra $gr\mathcal{A} = \bigoplus_{i \in \mathbb{N}} (\mathcal{A}_i / \mathcal{A}_{i+1})$ with the product $(x + \mathcal{A}_{i+1})(y + \mathcal{A}_{j+1}) = xy + \mathcal{A}_{i+j+1}$, $x \in \mathcal{A}_i, y \in \mathcal{A}_j$.

Definition 1. Given a nilpotent associative algebra $\mathcal{A}$, we have a natural grading on $\mathcal{A}$ induced by the series $\mathcal{A}_i$, $\mathcal{A}_i = \mathcal{A}_i / \mathcal{A}_{i+1}$, $1 \leq i \leq k-1$, and $gr\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots \oplus \mathcal{A}_k$. If $gr\mathcal{A}$ and $\mathcal{A}$ are isomorphic, denoted by $gr\mathcal{A} \cong \mathcal{A}$, we say that the algebra $\mathcal{A}$ is naturally graded. For any element x in $\mathcal{A}$, we define the left multiplication operator as: $L_x : \mathcal{A} \rightarrow \mathcal{A}, z \rightarrow xz, z \in \mathcal{A}$. For an element $x \in \mathcal{A} \setminus \mathcal{A}_2$, define the decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$ which consists of the dimensions of the Jordan blocks of the left multiplication operator L_x . Endow the set of these sequences with the lexicographic order, i.e., $C(x) = (n_1, n_2, \dots, n_k) \leq C(y) = (m_1, m_2, \dots, m_s)$, which means that there is an index $i \in \mathbb{N}$, such that $n_j = m_j$ for all $j < m_i$.

Definition 2. The sequence $C(\mathcal{A}) = \max_{x \in \mathcal{A} \setminus \mathcal{A}_2} C(x)$ is defined to be the characteristic sequence of the algebra $\mathcal{A}$.

We assume that $\mathbb{F}$ is an algebraically closed field of characteristic zero. Let $\mathcal{A}$ be a naturally graded n -dimensional quasi-filiform associative algebra. Then, there are two possibilities for the characteristic sequence, either $C(\mathcal{A}) = (n-2, 1, 1)$ or $C(\mathcal{A}) = (n-2, 2)$.

The associative algebras

$$\pi_2 : \begin{cases} e_1 e_1 = e_2, \\ e_1 e_2 = e_2 e_1 = e_3, \\ e_1 e_4 = e_4 e_1 = e_5, \\ e_4 e_4 = e_5 \end{cases}, \quad \pi_3 : \begin{cases} e_1 e_1 = e_2, \\ e_1 e_2 = e_2 e_1 = e_3, \\ e_1 e_4 = e_5, \\ e_4 e_4 = e_5 \end{cases}$$

are five-dimensional naturally graded associative algebras with characteristic sequence $C(\mathcal{A}) = (3, 2)$ [2].

Now we give a description of derivations and local derivations of the associative algebras π_2 and π_3 .

Theorem 1. A linear operator on the algebra π_2 is a derivation if and only if its matrix has the following form

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ a_{2,1} & 2a_{1,1} & 0 & 0 & 0 \\ a_{3,1} & 2a_{2,1} & 3a_{1,1} & a_{3,4} & 0 \\ a_{4,1} & 0 & 0 & a_{4,1} + a_{1,1} & 0 \\ a_{5,1} & 2a_{4,1} & 0 & a_{5,4} & 2(a_{4,1} + a_{1,1}) \end{pmatrix} \quad (1)$$

Theorem 2. A linear operator on the algebra π_2 is a local derivation if and only if its matrix has the following form

$$\begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 \\ b_{41} & 0 & 0 & b_{41} + b_{11} & 0 \\ b_{51} & b_{52} & 0 & b_{54} & b_{22} + b_{52} \end{pmatrix}.$$

Theorem 3. A linear operator on the algebra π_3 is a derivation if and only if its matrix has the following form

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ a_{2,1} & 2a_{1,1} & 0 & 0 & 0 \\ a_{3,1} & 2a_{2,1} & 3a_{1,1} & a_{3,4} & 0 \\ 0 & 0 & 0 & a_{1,1} & 0 \\ a_{5,1} & 0 & 0 & a_{5,4} & 2a_{1,1} \end{pmatrix}$$

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ABSOLUTE NILPOTENT ELEMENTS OF GENETIC ALGEBRAS

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Consider a population consisting of m species (or traits) which are denoted by $E = \{1, 2, \dots, m\}$. Assume that $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)})$ is a probability distribution of species at an initial state and $p_{ij,k}$ a probability that individuals in the i^{th} and j^{th} species interbreed to produce an individual from k^{th} species. Then a probability distribution $\mathbf{x}^{(1)} = (x_1^{(1)}, \dots, x_m^{(1)})$ of the species in the first generation can be found as a total probability, i.e., $x_k^{(1)} = \sum_{i,j=1}^m p_{ij,k} x_i^{(0)} x_j^{(0)}$, $k \in \{1, \dots, m\}$. It defines a correspondence $\mathbf{x}^{(0)} \rightarrow \mathbf{x}^{(1)}$ which is called *the evolution operator* or *quadratic stochastic operator* (QSO).

A map $V : S^{m-1} \rightarrow S^{m-1}$ is called a quadratic stochastic operator if

$$V : x'_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad k = 1, \dots, m \quad (1)$$

for any $\mathbf{x} \in S^{m-1}$, where

$$p_{ij,k} = p_{ji,k} \geq 0 \text{ for all } i, j, k \in E, \quad \sum_{k=1}^m p_{ij,k} = 1 \text{ for any } i, j \in E. \quad (2)$$

The quadratic stochastic operator V is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$.

Let us consider the following Volterra QSO defined on the S^2

$$V : \begin{cases} x'_1 = x_1^2 + (1-a)x_1x_2 + (1+c)x_1x_3, \\ x'_2 = x_2^2 + (1+a)x_1x_2 + (1-b)x_2x_3, \\ x'_3 = x_3^2 + (1-c)x_1x_3 + (1+b)x_2x_3, \end{cases} \quad (3)$$

where $-1 \leq a, b, c \leq 1$.

It is known that the set $E = \{1, 2, 3\}$ has the following permutations:

$$\pi_1 = \mathbf{e}, \quad \pi_2 = (12), \quad \pi_3 = (23), \quad \pi_4 = (13), \quad \pi_5 = (123), \quad \pi_6 = (132).$$

In [1] showed that the corresponding quadratic automorphisms of S^2 are defined by $V_i = T_{\pi_i} V$, $i = 1, \dots, 6$.

It is easy to see that for the permutation π_1 one has $T_{\pi_1} = Id$, and for π_2, π_3, π_4 , we have $T_{\pi_i}^2 = Id$, $i = 2, 3, 4$ and for the rest permutations π_5, π_6 one has $T_{\pi_i}^3 = Id$, $i = 5, 6$ where Id is the identity map.

Definition. Two quadratic stochastic operators V and W are called *conjugate* if there exists a permutation π such that $T_{\pi}^{-1} V T_{\pi} = W$, the last one is denoted by $V \sim W$.

Let V and W be QSOs defined on S^2 , and $\mathcal{A}_V, \mathcal{A}_W$ be the corresponding genetic algebras. These algebras are called *stochastically isomorphic* if there is a stochastic linear bijection $\psi : \mathcal{A}_V \rightarrow \mathcal{A}_W$ such that $\psi(\mathbf{x} \circ_V \mathbf{y}) = \psi(\mathbf{x}) \circ_W \psi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}_V$.

Let $\mathcal{A}_k$ be a three-dimensional genetic algebra associated with the quadratic automorphisms V_k , $k = 1, \dots, 6$ of S^2 . Then it is isomorphic to one of the following pairwise non-isomorphic algebras: $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_5$ associated with the operators V_1 , V_2 and V_5 , respectively.

We describe the absolute nilpotent elements of the genetic algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_5$. For the genetic algebra $\mathcal{A}_k$, $k \in \{1, 2, 5\}$ one can check that element $\mathbf{x} \in \mathcal{A}_k$, $k \in \{1, 2, 5\}$ is absolute nilpotent if it holds $V_k(\mathbf{x}) = 0$, $k \in \{1, 2, 5\}$. By $\mathcal{N}(\mathcal{A}_k)$, $k \in \{1, 2, 5\}$ we denote all absolute nilpotent elements of $\mathcal{A}_k$, $k \in \{1, 2, 5\}$.

Theorem. *Let $\mathcal{A}_k$, $k \in \{1, 2, 5\}$ be a genetic algebra. Then it holds*

$$\mathcal{N}(\mathcal{A}_k) = \{\mathbf{O}\} \cup \begin{cases} H_0, & \text{if } a = b = c = 0, \\ \{(0, \alpha, -\alpha), (-\alpha, \alpha, 0)\}, & \text{if } a = b = 0, \quad c \neq 0, \\ \{(\alpha, 0, -\alpha), (\alpha, -\alpha, 0)\}, & \text{if } a = c = 0, \quad b \neq 0, \\ \{(0, -\alpha, \alpha), (-\alpha, 0, \alpha)\}, & \text{if } b = c = 0, \quad a \neq 0, \\ \{(\alpha, 0, -\alpha)\}, & \text{if } c = 0, \quad ab \neq 0, \\ \{(0, \alpha, -\alpha)\}, & \text{if } b = 0, \quad ac \neq 0, \\ \{(\alpha, -\alpha, 0)\}, & \text{if } a = 0, \quad bc \neq 0, \\ \emptyset, & \text{if } abc \neq 0, \end{cases}$$

where $a, b, c \in [-1, 1]$, $\alpha \in \mathbb{R}$ and $\mathbf{O} = (0, 0, 0)$.

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Almost inner derivations of finite dimensional simple Leibniz Algebras

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Almost inner derivations of Leibniz algebras were studied in [1, 3-5]. In this note, almost inner derivations of semisimple Leibniz algebras $\mathcal{L} = (\mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2) \dot{+} \mathcal{I}$ are considered.

Definition. An algebra $\mathcal{L}$ over a field $\mathbb{F}$ is called a *Leibniz algebra* if for any $x, y, z \in \mathcal{L}$, the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

is satisfied, where $[-, -]$ is the multiplication in $\mathcal{L}$.

Let $\mathcal{L}$ be a finite-dimensional Leibniz algebra. For Leibniz algebra $\mathcal{L}$ we consider the following central and derived series:

$$\begin{aligned} \mathcal{L}^1 &= \mathcal{L}, & \mathcal{L}^i &= [\mathcal{L}^{i-1}, \mathcal{L}], \quad i \geq 1, \\ \mathcal{L}^{[1]} &= \mathcal{L}, & \mathcal{L}^{[k]} &= [\mathcal{L}^{[k-1]}, \mathcal{L}^{[k-1]}], \quad k \geq 1. \end{aligned}$$

A Leibniz algebra $\mathcal{L}$ is *nilpotent (solvable)* if there exists $m \geq 1$ such that $\mathcal{L}^m = 0$ ($\mathcal{L}^{[m]} = 0$).

Let $\mathfrak{g}$ a simple Lie algebra. For a Leibniz algebra $\mathcal{L}$, a subspace generated by its squares $\mathcal{I} = \text{span}\{[x, x] : x \in \mathcal{L}\}$ due to Leibniz identity becomes an ideal, and the quotient $\mathfrak{g}_{\mathcal{L}} = \mathcal{L}/\mathcal{I}$ is a Lie algebra called *liaization* of $\mathcal{L}$. Moreover, $[\mathcal{L}, \mathcal{I}] = 0$. In general, $[\mathcal{I}, \mathcal{L}] \neq 0$. Since we are interested in Leibniz algebras which are not Lie algebras, we will always assume that $\mathcal{I} \neq 0$. Then $[\mathcal{I}, \mathfrak{g}] = \mathcal{I}$.

A Leibniz algebra $\mathcal{L}$ is called *simple* if its ideals are only $\{0\}$, $\mathcal{I}$, $\mathcal{L}$ and $[\mathcal{L}, \mathcal{L}] \neq \mathcal{I}$.

Definition. A leibniz algebra $\mathcal{L}$ is called *semisimple* if its maximal solvable ideal is equal to $\mathcal{I}$.

A *derivation* of Leibniz algebra $\mathcal{L}$ is a linear map $D : \mathcal{L} \rightarrow \mathcal{L}$ which satisfies the Leibniz law, that is,

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in \mathcal{L}$.

The set of all derivations of $\mathcal{L}$ with respect to the commutation operation is a Lie algebra and it is denoted by $\text{Der}(\mathcal{L})$.

Let $\mathcal{L}$ be a Leibniz algebra. For each $a \in \mathcal{L}$, the operator $R_a : \mathcal{L} \rightarrow \mathcal{L}$ which is called the *right multiplication*, such that $R_a(y) = [y, a]$, $y \in \mathcal{L}$, is a derivation. This derivation is called an *inner derivation* of $\mathcal{L}$, and we denote the space of all inner derivations by $\text{InDer}(\mathcal{L})$.

Definition. The derivation $D \in \text{Der}(\mathcal{L})$ of the Leibniz algebra $\mathcal{L}$ is called *almost inner derivation*, if $D(x) \in [x, \mathcal{L}]$ holds for all $x \in \mathcal{L}$; in other words, there exists $a_x \in \mathcal{L}$ such that $D(x) = [x, a_x]$. The space of all almost inner derivations of $\mathcal{L}$ is denoted by $\text{AID}(\mathcal{L})$.

Consider semisimple Leibniz algebra $\mathcal{L} = (\mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2) \dot{+} \mathcal{I}$ with a basis

$$\{e_1, h_1, f_1, e_2, h_2, f_2, x_i \otimes y_j : i = 0, 1, \dots, m; j = 0, 1, \dots, n\}$$

and the product table [Example 3.6, 2]:

$$\begin{aligned} [e_i, h_i] &= -[h_i, e_i] = 2e_i, & [h_i, f_i] &= -[f_i, h_i] = 2f_i, \\ [e_i, f_i] &= -[f_i, e_i] = h_i, \\ [x_i \otimes y_j, h_1] &= (m - 2i)x_i \otimes y_j, & [x_i \otimes y_j, e_1] &= ix_{i+1} \otimes y_j, \\ [x_i \otimes y_j, h_2] &= (n - 2j)x_i \otimes y_j, & [x_i \otimes y_j, e_2] &= jx_i \otimes y_{j+1}, \\ [x_i \otimes y_j, f_1] &= (m - i)x_{i-1} \otimes y_j, & [x_i \otimes y_j, f_2] &= (n - j)x_i \otimes y_{j-1}, \end{aligned}$$

where $0 \leq i \leq n$ and $0 \leq j \leq m$. If V_1 is the $(m + 1)$ -dimensional irreducible $\mathfrak{sl}_2^1$ -module and V_2 is the $(n + 1)$ -dimensional irreducible $\mathfrak{sl}_2^2$ -module, then $\mathcal{L} = (\mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2) \ltimes (V_1 \otimes V_2)$.

The following theorem is the main result of this note.

Theorem. Any almost inner derivation of the semisimple Leibniz algebra $\mathcal{L} = (\mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2) \dot{+} \mathcal{I}$ is inner.

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Solvable Leibniz 4-algebra constructed by a given solvable Leibniz algebra

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Currently, one of the important areas of modern algebra is research devoted to the theory of Leibniz algebras. This class of algebras, which naturally generalizes Lie algebras and is defined by the Leibniz identity, was first proposed by Bloch in 1965 under the name D-algebras [4]. Much later, in the early 1990s, D-algebras arose in the work of J.-L. Lode and T. Pirashvili [3] under the modern terminology of Leibniz algebras.

In this paper we present a Leibniz 4-algebra constructed by a solvable Leibniz algebra with a non-Lie quasi-filiform nilradical of maximal length.

Definition 1. An algebra $(L, [-, -])$ over a field F is called a Leibniz algebra if for any elements $x, y, z \in L$ the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

For a given Leibniz algebra L , we define the lower central and derived series to the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1; \quad L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$$

Definition 2. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists an $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$). The minimal number n (respectively, m) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra L .

Definition 3. The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Definition 4. A Leibniz algebra L is called quasi-filiform if $L^{n-2} \neq 0$ and $L^{n-1} = 0$, where $n = \dim L$.

A Leibniz n -algebra L is a K -vector space equipped with an n -linear bracket $[-, -, \dots, -] : L^{\otimes n} \rightarrow L$ satisfying the following fundamental identity

$$[[x_1, \dots, x_n], y_1, \dots, y_{n-1}] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_1, \dots, y_{n-1}]]. \quad (1)$$

In case $n = 2$ the fundamental identity (1) becomes the Leibniz identity, so a Leibniz 2-algebra is just a Leibniz algebra [2].

Any Leibniz algebra g is also a Leibniz n -algebra under the following n -bracket:

$$[x_1, x_2, \dots, x_n] := [x_1, [x_2, \dots, [x_{n-1}, x_n] \dots]].$$

The following theorem gives the classification Leibniz algebras whose nilradical is quasi-filiform Leibniz maximal length and the dimension of the complementary space to the nilradical is equal two.

Theorem 1. [1] An arbitrary algebra of the family $R(M^{1,0}, 2)$ is isomorphic to the following algebra:

$$R(M^{1,0}, 2) : \begin{cases} [e_1, e_1] = e_n, & [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_1, x_1] = e_1, & [e_i, x_1] = (i-2)e_i, & 3 \leq i \leq n-1, \\ [e_n, x_1] = 2e_n, & [x_1, e_1] = -e_1, & \\ [e_i, x_2] = e_i, & & 2 \leq i \leq n-1. \end{cases}$$

In the following theorem we present table of multiplications a Leibniz 3-algebra constructed by solvable Leibniz algebras with a non-Lie quasi-filiform nilradical.

Theorem 2. Let L be a Leibniz 4-algebra constructed by the algebra $R(M^{1,0}, 2)$. The multiplication table of L has the following form:

$$L_1 : \begin{cases} [x_1, x_1, e_1, x_1] = e_1 \\ [x_1, x_1, x_1, e_1] = -e_1 \\ [e_i, x_1, x_1, e_1] = e_{i+1} & 2 \leq i \leq n-2 \\ [e_i, x_1, e_1, x_1] = -e_{i+1} \\ [e_1, x_1, x_1, e_1] = e_n \\ [e_1, x_1, e_1, x_1] = -e_n \end{cases}$$

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Local and 2-local $\frac{1}{2}$ -derivation on solvable Leibniz algebras with model nilradical

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Nowadays, local and 2-local operators have become popular for some non-associative algebras such as the Lie, Jordan, and Leibniz algebras. The notions of local derivations were introduced in 1990 by Kadison [3] and Larson, Sourour [4]. Later in 1997, Šemrl introduced the notions of 2-local derivations and 2-local automorphisms of algebras [5].

Investigation of local derivations on Lie algebras was initiated in [1] by Ayupov and Kudaybergenov. They proved that every local derivation on semisimple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations that are not derivations. In [2], local derivations of solvable Lie algebras are investigated, and it is shown that in the class of solvable Lie algebras there exist algebras that admit local derivations that are not derivations and also algebras for which every local derivation is a derivation.

Definition 1. Let $(\mathfrak{L}, [-, -])$ be an algebra with a multiplication $[-, -]$. A linear map φ is called a δ -derivation if it satisfies

$$\varphi[x, y] = \delta([\varphi(x), y] + [x, \varphi(y)]),$$

where δ from the ground field $\mathbb{F}$.

Since the notions of local operators can be defined for any type of operator, the notions of local and 2-local δ -derivations are defined as follows:

Definition 2. A linear map Δ is called a local δ -derivation, if for any $x \in \mathfrak{L}$, there exists a δ -derivation $\varphi_x : \mathfrak{L} \rightarrow \mathfrak{L}$ (depending on x) such that $\Delta(x) = \varphi_x(x)$. The set of all local δ -derivations on $\mathfrak{L}$ we denote by $\text{LocDer}_\delta(\mathfrak{L})$.

Definition 3. A map $\nabla : \mathfrak{L} \rightarrow \mathfrak{L}$ (not necessary linear) is called a 2-local δ -derivation, if for any $x, y \in \mathfrak{L}$, there exists a δ -derivation $\varphi_{x,y} \in \text{Der}_\delta(\mathfrak{L})$ such that

$$\nabla(x) = \varphi_{x,y}(x), \quad \nabla(y) = \varphi_{x,y}(y).$$

It should be noted that 2-local δ -derivation is not necessarily linear, but for any $x \in \mathfrak{L}$ and for any scalar λ , we have

$$\nabla(\lambda x) = \varphi_{x,\lambda x}(\lambda x) = \lambda \varphi_{x,\lambda x}(x) = \lambda \nabla(x).$$

Let $\mathcal{L}$ be a Leibniz algebra. For a Leibniz algebra $\mathcal{L}$ consider the following lower central and derived sequences:

$$\begin{aligned} \mathcal{L}^1 &= \mathcal{L}, & \mathcal{L}^{k+1} &= [\mathcal{L}^k, \mathcal{L}^1], & k &\geq 1, \\ \mathcal{L}^{[1]} &= \mathcal{L}, & \mathcal{L}^{[s+1]} &= [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}], & s &\geq 1. \end{aligned}$$

Definition 4. A Leibniz algebra $\mathcal{L}$ is called nilpotent (respectively, solvable), if there exists $k \in \mathbb{N}$ ($s \in \mathbb{N}$) such that $\mathcal{L}^k = 0$ (respectively, $\mathcal{L}^{[s]} = 0$). The minimal number k (respectively, s) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $\mathcal{L}$.

For a finite-dimensional nilpotent Leibniz algebra N and for the matrix of the linear operator ad_x denote by $C(x)$ the descending sequence of its Jordan blocks' dimensions. Consider the lexicographical order on the set $C(N) = \{C(x) \mid x \in N\}$.

Definition 5. The sequence

$$\left(\max_{x \in N \setminus N^2} C(x) \right)$$

is said to be the characteristic sequence of a nilpotent Leibniz algebra N .

Let N be a nilpotent Leibniz algebra with the characteristic sequence $(m_1, \dots, m_s)$, and with the table of multiplication

$$N_{m_1, \dots, m_s} : [e_i^t, e_1^1] = e_{i+1}^t, \quad 1 \leq t \leq s, \quad 1 \leq i \leq m_t - 1.$$

The algebra $N_{m_1, \dots, m_s}$ is usually said to be a model Leibniz algebra. For solvable Leibniz algebras with nilradical $N_{m_1, \dots, m_s}$ and the complement dimension space equal to s , we will use the notation $R(N_{m_1, \dots, m_s}, s)$.

Theorem 1.[6] A solvable Leibniz algebra $R(N_{m_1, \dots, m_s}, s)$ with nilradical $N_{m_1, \dots, m_s}$, such that $\dim R(N_{m_1, \dots, m_s}, s) - \dim N_{m_1, \dots, m_s} = s$, is isomorphic to the algebra:

$$R(N_{m_1, \dots, m_s}, s) : \begin{cases} [e_i^t, e_1^1] = e_{i+1}^t, & 1 \leq t \leq s, \quad 1 \leq i \leq m_t - 1, \\ [e_i^1, x_1] = ie_i^1, & 1 \leq i \leq m_1, \\ [e_i^t, x_1] = (i-1)e_i^t, & 2 \leq t \leq s, \quad 2 \leq i \leq m_t, \\ [e_i^t, x_t] = e_i^t, & 2 \leq t \leq s, \quad 1 \leq i \leq m_t, \\ [x_1, e_1^1] = -e_1^1, \end{cases}$$

where $\{x_1, \dots, x_s\}$ is a basis of the complementary vector space.

The following theorem is the main result of this note.

Theorem 2. Any local 2-local $\frac{1}{2}$ -derivation on the solvable Leibniz algebra $R(N_{m_1, \dots, m_s}, s)$ is a $\frac{1}{2}$ -derivation.

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RHIZAFORM ALGEBRAS

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Any anti-associative algebra gives rise to a Jacobi-Jordan algebra by $[x, y] = xy + yx$. Rhizaform algebras, which offer an approach to addressing anti-associativity, are defined by two operations whose sum is anti-associative, with the left and right multiplication operators forming bimodules of the sum of anti-associative algebras. This characterization parallels that of dendriform algebras, where the sum of operations preserves associativity.

Definition 1.[2] An algebra $(A, \diamond)$ over a field $\mathbb{F}$ is called a **Jacobi-Jordan algebra** if it satisfies the following two identities

$$\begin{aligned} x \diamond y - y \diamond x &= 0, \\ x \diamond (y \diamond z) + y \diamond (z \diamond x) + z \diamond (x \diamond y) &= 0, \end{aligned}$$

for all $x, y, z \in A$.

An algebra $(A, \circ)$ is called a **Jacobi-Jordan-admissible algebra** [1], where A is a vector space with a bilinear operation $\circ : A \otimes A \rightarrow A$, if the bilinear operation $[-, -] : A \otimes A \rightarrow A$ defined by

$$[x, y] = x \circ y + y \circ x, \quad \forall x, y \in A,$$

makes $(A, [-, -])$ a Jacobi-Jordan algebra, which is called the **sub-adjacent Jacobi-Jordan algebra** of $(A, \circ)$ and denoted by $(\mathfrak{J}(A), [-, -])$. Obviously, an anti-associative algebra is a Jacobi-Jordan-admissible algebra.

Definition 2.[1] A pre-Jacobi-Jordan algebra A is a vector space with a binary operation $(x, y) \mapsto xy$ satisfying

$$(xy)z + x(yz) + (yx)z + y(xz) = 0, \quad \forall x, y, z \in A.$$

Definition 3. Let A be a vector space with two bilinear operations $\succ$ and $\prec$. The triple $(A, \succ, \prec)$ is called a **rhizaform algebra** if the following equations hold:

$$\begin{aligned} (1) \quad & (x * y) \succ z = -x \succ (y \succ z), \\ (2) \quad & x \prec (y * z) = -(x \prec y) \prec z, \\ (3) \quad & x \succ (y \prec z) = -(x \succ y) \prec z, \quad \forall x, y, z \in A, \end{aligned}$$

where the bilinear operation $*$ is defined by $x * y = x \succ y + x \prec y$.

Theorem 1. Let $(A, \succ, \prec)$ be a rhizaform algebra. Then the followings hold.

(i). Define a bilinear operation $*$ by $x * y = x \succ y + x \prec y$, $\forall x, y \in A$. Then $(A, *)$ is an anti-associative algebra, called the **associated anti-associative algebra** of $(A, \succ, \prec)$. Furthermore, $(A, \succ, \prec)$ is called a **compatible rhizaform algebra structure** on $(A, *)$.

(ii). The bilinear operation $\circ : A \otimes A \rightarrow A$ given by

$$x \circ y = x \succ y + y \prec x, \quad \forall x, y \in A,$$

defines a pre-Jacobi-Jordan algebra, called the **associated pre-Jacobi-Jordan algebra** of $(A, \succ, \prec)$.

$$\begin{array}{ccc}
\mathbf{Rhiza} & \longrightarrow & \mathbf{Pre-J-J} \\
\downarrow & & \downarrow \\
\mathbf{AAs} & \longrightarrow & \mathbf{J-J}
\end{array}$$

(iii). Both $(A, *)$ and $(A, \circ)$ have the same sub-adjacent Jacobi-Jordan algebra $(\mathfrak{J}(A), [-, -])$ defined by

$$[x, y] = x \succ y + x \prec y + y \succ x + y \prec x, \quad \forall x, y \in A.$$

Moreover, the following diagram is commutative: where, **AAs** is the category of anti-associative algebras, **J-J** is the category of Jacobi-Jordan algebras, **Pre-J-J** is the category of pre-Jacobi-Jordan algebras, **Rhiza** is the category of Rhizaform algebras.

Example 1. Let $(A, \succ, \prec)$ be a 1-dimensional rhizaform algebra with a basis $\{e\}$. Assume that

$$e \succ e = \alpha e, \quad e \prec e = \beta e,$$

where $\alpha, \beta \in \mathbb{F}$. Then by Eqs.(1)–(3), we have

$$\begin{cases}
\alpha^2 e + \alpha \beta e = -\alpha^2 e, \\
\alpha \beta e + \beta^2 e = -\beta^2 e, \\
\alpha \beta = \alpha \beta.
\end{cases}$$

Then we have $\alpha(2\alpha + \beta) = 0$ and $\beta(2\beta + \alpha) = 0$. Hence $\alpha = \beta = 0$, that is, any 1-dimensional rhizaform algebra is trivial.

Proposition 1. Any two-dimensional rhizaform algebra can be included in one of the following classes of algebras:

$$\begin{aligned}
Rh_1 : & \quad e_1 \prec e_1 = e_2 \\
Rh_2[\lambda] : & \quad e_1 \succ e_1 = e_2, \quad e_1 \prec e_1 = \lambda e_2, \quad \lambda \in \mathbb{C}. \\
Rh_3 : & \quad \text{abelian}
\end{aligned}$$

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Compatible dendriform algebras

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Compatible algebraic structures are quite much interest in the sense that linear combinations of multiplications defined on the same two algebraic structures are still of the same kind of algebraic structures. Compatible algebraic structures have been widely studied in mathematics and mathematical physics. For example, compatible associative algebras were studied in connection with Cartan matrices of affine Dynkin diagrams, integrable matrix equations, infinitesimal bialgebras and quiver representations [2]–[4]. Compatible Lie algebras were studied in the contexts of the classical Yang-Baxter equation and principal chiral field, loop algebras over Lie algebras and elliptic theta functions [1],[5]. Compatible pre-Lie algebras have close relation with the classical Yang-Baxter equation in compatible Lie algebras [5].

Definition 1. Let $(A, \circ)$ and $(A, *)$ be two algebras of the same type. They are called compatible if for any $k_1, k_2 \in \mathbb{C}$, the following bilinear operation

$$x \star y = k_1 x \circ y + k_2 x * y, \quad \forall x, y \in A,$$

defines the same algebraic structure on A .

Definition 2. Let A be a vector space with a bilinear operation $*$. $(A, *)$ is called a pre-Lie algebra if it satisfies

$$(x * y) * z - x * (y * z) = (y * x) * z - y * (x * z), \quad \forall x, y, z \in A.$$

Proposition 1. Let $(A, \circ)$ and $(A, *)$ be two pre-Lie algebras. Then $(A, \circ, *)$ is a compatible pre-Lie algebra if and only if $\forall x, y, z \in A$,

$$\begin{aligned} (x \circ y) * z - x * (y \circ z) + (x * y) \circ z - x \circ (y * z) \\ = (y \circ x) * z - y * (x \circ z) + (y * x) \circ z - y \circ (x * z). \end{aligned}$$

Definition 3. A compatible associative algebra is a triple $(A, \cdot_1, \cdot_2)$ in which $(A, \cdot_1)$ and $(A, \cdot_2)$ are both associative algebras satisfying the following compatibility condition

$$(x \cdot_1 y) \cdot_2 z + (x \cdot_2 y) \cdot_1 z = x \cdot_1 (y \cdot_2 z) + x \cdot_2 (y \cdot_1 z),$$

for $x, y, z \in A$.

Proposition 2. Let $(\mathfrak{g}, [-, -]_{\mathfrak{g}1})$ and $(\mathfrak{g}, [-, -]_{\mathfrak{g}2})$ be two Lie algebras. Then $(\mathfrak{g}, [-, -]_{\mathfrak{g}1}, [-, -]_{\mathfrak{g}2})$ is a compatible Lie algebra if and only if for any $x, y, z \in \mathfrak{g}$, the following equation holds

$$[[x, y]_{\mathfrak{g}1}, z]_{\mathfrak{g}2} + [[y, z]_{\mathfrak{g}1}, x]_{\mathfrak{g}2} + [[z, x]_{\mathfrak{g}1}, y]_{\mathfrak{g}2} + [[x, y]_{\mathfrak{g}2}, z]_{\mathfrak{g}1} + [[y, z]_{\mathfrak{g}2}, x]_{\mathfrak{g}1} + [[z, x]_{\mathfrak{g}2}, y]_{\mathfrak{g}1} = 0.$$

Recall that a dendriform algebra $(A, \prec, \succ)$ is a vector space A with two binary operations denoted by $\prec$ and $\succ$ satisfying (for any $x, y, z \in A$)

$$(x \prec y) \prec z = x \prec (y \cdot z), \quad (x \succ y) \prec z = x \succ (y \prec z), \quad x \succ (y \succ z) = (x \cdot y) \succ z,$$

where $x \cdot y = x \prec y + x \succ y$.

Now, we introduce the notion of compatible dendriform algebras.

Definition 4. Let $(A, \succ_1, \prec_1)$ and $(A, \succ_2, \prec_2)$ be two dendriform algebras. They are called compatible if for any $k_1, k_2 \in \mathbb{C}$, the following bilinear operation

$$\begin{aligned} x \succ y &= k_1 x \succ_1 y + k_2 x \succ_2 y, \\ x \prec y &= k_1 x \prec_1 y + k_2 x \prec_2 y, \quad \forall x, y \in A, \end{aligned}$$

defines a dendriform algebra structure on A .

Proposition 3. Let $(A, \succ_1, \prec_1)$ and $(A, \succ_2, \prec_2)$ be two dendriform algebras. Then $(A, \succ, \prec)$ is a compatible dendriform algebra if and only if $\forall x, y, z \in A$,

$$\begin{aligned} x \succ_1 (y \succ_2 z) + x \succ_2 (y \succ_1 z) &= (x \cdot_1 y) \succ_2 z + (x \cdot_2 y) \succ_1 z, \\ (x \prec_1 y) \prec_2 z + (x \prec_2 y) \prec_1 z &= x \prec_1 (y \cdot_2 z) + x \prec_2 (y \cdot_1 z), \\ (x \succ_2 y) \prec_1 z + (x \succ_1 y) \prec_2 z &= x \succ_1 (y \prec_2 z) + x \succ_2 (y \prec_1 z), \end{aligned}$$

where $x \cdot_1 y = x \succ_1 y + x \prec_1 y$ and $x \cdot_2 y = x \succ_2 y + x \prec_2 y$.

Proposition 4. Let $(A, \succ_1, \prec_1, \succ_2, \prec_2)$ be a compatible dendriform algebra. Then $(A, \cdot_1, \cdot_2)$ is a compatible associative algebra.

Proposition 5. Let $(A, \succ_1, \prec_1, \succ_2, \prec_2)$ be a compatible dendriform algebra. Then $(A, \circ, *)$ is a compatible pre-Lie algebra, where

$$x \circ y = x \succ_1 y - y \prec_1 x, \quad x * y = x \succ_2 y - y \prec_2 x.$$

Theorem 1. Let $(A, \succ_1, \prec_1, \succ_2, \prec_2)$ be a compatible dendriform algebra. Then $(A, \cdot_1, \cdot_2)$ and $(A, \circ, *)$ have the same sub-adjacent compatible Lie algebra $(\mathfrak{g}(A), [\cdot]_1, [\cdot]_2)$ defined by

$$\begin{aligned} [x, y]_1 &= x \succ_1 y + x \prec_1 y - y \succ_1 x - y \prec_1 x, \\ [x, y]_2 &= x \succ_2 y + x \prec_2 y - y \succ_2 x - y \prec_2 x. \end{aligned}$$

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Local and 2-local $\frac{1}{2}$ -derivation on metabelian filiform Lie algebras

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Definition 3. A map $\nabla : \mathfrak{L} \rightarrow \mathfrak{L}$ (not necessary linear) is called a 2-local δ -derivation, if for any $x, y \in \mathfrak{L}$, there exists a δ -derivation $\varphi_{x,y} \in \text{Der}_\delta(\mathfrak{L})$ such that

$$\nabla(x) = \varphi_{x,y}(x), \quad \nabla(y) = \varphi_{x,y}(y).$$

It should be noted that 2-local δ -derivation is not necessarily linear, but for any $x \in \mathfrak{L}$ and for any scalar λ , we have

$$\nabla(\lambda x) = \varphi_{x,\lambda x}(\lambda x) = \lambda \varphi_{x,\lambda x}(x) = \lambda \nabla(x).$$

A non-abelian Lie algebra $\mathcal{L}$ is called a metabelian, if $[\mathcal{L}, \mathcal{L}]$ is abelian, that is, $\mathcal{L}^{(2)} = 0$.

We shall consider a metabelian filiform Lie algebra $\mathcal{L}$ of dimension $n \geq 7$ with a basis $\{e_1, \dots, e_n\}$ such that [6]

$$\begin{aligned} [e_1, e_i] &= e_{i+1}, & 2 \leq i \leq n-1, \\ [e_2, e_i] &= e_{i+2} + e_{i+3}, & 3 \leq i \leq n-3, \\ [e_2, e_{n-2}] &= e_n. \end{aligned}$$

Proposition 1. Any $\frac{1}{2}$ -derivations of the metabelian filiform Lie algebras $\mathcal{L}$ has the following matrix form:

$$\begin{aligned} D(e_1) &= \alpha_1 e_1 + \sum_{i=3}^n \alpha_i e_i, \\ D(e_2) &= \alpha_1 e_2 + \alpha_3 e_4 + \sum_{i=5}^{n-3} (\alpha_{i-2} + \alpha_{i-1}) e_i + \beta_1 e_{n-2} + \beta_2 e_{n-1} + \beta_3 e_n, \\ D(e_3) &= \alpha_1 e_3 + \frac{\beta_1 - \alpha_{n-3} - \alpha_{n-4}}{2} e_{n-1} + \frac{\beta_2 - \alpha_{n-2} - \alpha_{n-3}}{2} e_n, \\ D(e_4) &= \alpha_1 e_4 + \frac{\beta_1 - \alpha_{n-3} - \alpha_{n-4}}{4} e_n, \\ D(e_i) &= \alpha_1 e_i, \quad 5 \leq i \leq n. \end{aligned}$$

Theorem 1. The algebra $\mathcal{L}$ admits local and 2-local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations.

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Известно, что $n \times n$ - матрица над любой комплексной AW^* -алгеброй также является AW^* -алгеброй [1]. Однако в вещественном случае все гораздо сложнее так как существуют вещественные AW^* -алгебры с комплексификацией, которая не является AW^* -алгеброй [2]. В данной работе будет показано при каких условиях матрица на любой вещественной AW^* -алгебре также является вещественной AW^* -алгеброй.

Пусть A – вещественная или комплексная $*$ -алгебра и пусть S – непустое подмножество A . Положим $R(S) = \{x \in A : sx = 0, \forall s \in S\}$ и назовем $R(S)$ правым аннулятором S . Аналогично $L(S) = \{x \in A : xs = 0, \forall s \in S\}$ левый аннулятор S . $*$ -алгебра A называется бэровской $*$ -алгеброй, если для любого непустого $S \subset A$ имеем $R(S) = gA$, для некоторого проектора g . Поскольку $L(S) = R(S^*)^* = (hA)^* = Ah$, определение симметрично и может быть дано через левый аннулятор. Здесь $S^* = \{s^* | s \in S\}$. Вещественная (или комплексная) C^* -алгебра A , являющаяся $*$ -алгеброй Бэра, называется вещественной (соотв. комплексной) AW^* -алгеброй.

Предложение 1. [2] Абелева вещественная C^* -алгебра A является вещественной AW^* -алгеброй тогда и только тогда, когда A_s является вещественной AW^* -алгеброй.

В следующем примере дается абелева вещественная не W^* , AW^* -алгебра комплексификация которой не является AW^* -алгеброй

Пример 2. [2] Пусть B будет множеством всех последовательностей $x = \{\lambda_n\}$ комплексных чисел, таких, что $|\lambda_n|$ ограничено и $Im \lambda_n \rightarrow 0$ (т. е. ограниченных последовательностей, которые являются вещественными на бесконечности). С покомпонентной операцией и $x = \{\lambda_n\}$, B является вещественной $*$ -алгеброй. Полагая $\|x\| = \sup |\lambda_n|$, мы получаем, что B является вещественной C^* -алгеброй, поскольку $\|x^*x\| = \|x\|^2$ и $1 + x^*x$ обратимо для любого $x \in B$, где $1 = \{1\}$ является единицей в B . Проекторы в B являются последовательностями нулей и единиц. Эрмитова часть B_s алгебры B совпадает с m_r алгеброй всех ограниченных вещественных последовательностей - это абелева вещественная W^* -алгебра, в частности, это вещественная AW^* -алгебра. По предложению 1 B является вещественной AW^* -алгеброй. Но B не является вещественной W^* -алгеброй (так как она не является поточечно полной, а ее поточечное пополнение есть m алгебра всех ограниченных комплексных последовательностей)

Теорема 3. [1] Если A – любая комплексная AW^* -алгебра, то $M_n(A)$ – $n \times n$ матрица над A является AW^* -алгеброй.

Результат известен только в комплексном случае, в следующей теореме мы покажем, что для вещественных AW^* -алгебр данное свойство выполняется при определенных условиях.

Теорема 4. [2] Пусть R – вещественная C^* -алгебра и пусть $A = R + iR$ – ее комплексификация. Предположим, что A – AW^* -алгебра. Тогда R – вещественная AW^* -алгебра.

Возникает вполне закономерный вопрос, насколько существенно условие, что $A = R + iR$ AW^* - алгебра? Данное условие играет существенную роль, при переходе на

матричные алгебры для вещественных AW^* -алгебр, что очевидно из следующей теоремы.

Теорема 5 Пусть R – вещественная AW^* -алгебра и $A = R + iR$ ее комплексификация, не являющаяся AW^* -алгеброй. Тогда $M_2(R)$ не является AW^* -алгеброй.

Доказательство Если R – вещественная AW^* -алгебра удовлетворяющая условию теоремы 4 то по теореме 3 очевидно, что $M_2(R)$ – AW^* -алгебра.

Напомним, что семейство проекторов порождает аннулятор тогда и только тогда, когда имеет супремум. Рассмотрим случай если опущено условие, что $A = R + iR$ AW^* -алгебра и применим пример 2. Очевидно, мы сразу теряем возможность применить теорему 3 и 4. Остается только проверить, будет ли $M_2(R)$ AW^* -алгеброй. Известно, что матрица над алгеброй из примера 2 не является AW^* -алгеброй. Таким образом $M_2(R)$ не является AW^* -алгеброй. Теорема доказана.

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GALILEY FAZOSIDA NOL EGRILIKKA EGA BURALGAN SIRTDLAR

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Ushbu maqoladada Galiley fazosida buralgan sirtlar o'rganilgan. Bunday sirtlar ikkita almashtirish (Evklid va Galiley) natijasida hosil bo'lib, ularning uch xil turi aniqlangan. Har bir tur uchun Gauss va o'rtacha egriliklar topilgan hamda nol egrilikka ega sirtlar tasniflangan. Natijada, I va III-tur sirtlar uchun yassi sirtlar mavjud emasligi, II-tur sirtlar uchun esa yassi va minimal sirt tushunchalari ekvivalentligi isbotlangan.

Galiley fazosi G_3 klassik Keli-Kleyin fazolaridan biri bo'lib, majrux metrikali fazodir. Bu fazoda izotrop va noizotrop yo'nalishlar mavjudligi sababli sirtlarning differensial-geometrik xossalari Evklid fazosidan tubdan farq qiladi. Mazkur ishda Galiley fazosida buralgan sirtlar o'rganilib, ularning uchta turi bo'yicha nol egrilikka ega sirtlar tasniflanadi.

Ta'rif. Galiley fazosida egri chiziqning ikki xil (Evklid va Galiley) almashtirish natijasida hosil bo'lgan sirt buralgan sirt deyiladi.

Galiley 3-fazosida burilgan sirtlar uch turga bo'linadi:

I-tur sirt: Profil egri chiziq yz -tekislikda joylashgan bo'lib, sirt quyidagicha aniqlanadi:

$$\varphi(s, t) = (cs, f(t) \cos(bs) + g(t) \sin(bs) + \frac{c}{2}s^2, -f(t) \sin(bs) + g(t) \cos(bs)),$$

bu yerda $b, c \in \mathbb{R}$ doimiylardir.

II-tur sirt: Profil chiziq xy -tekislikda joylashgan bo'lib, sirt quyidagicha aniqlanadi:

$$\varphi(s, t) = (f(t) + (bc + d)s, bsf(t) + g(t) + \frac{b^2c}{2}s^2, (a + f(t))s + \frac{(bc+d)}{2}s^2).$$

III-tur sirt: Bu tur ikki Yevklid almashtirish natijasida hosil bo'ladi:

$$\varphi(s, t) = (f(t) + bcs, \cos s(a + bsf(t) + g(t) + \frac{b^2c}{2}s^2), -\sin s(a + bsf(t) + g(t) + \frac{b^2c}{2}s^2)).$$

1-teorema. I-tur buralgan sirtlar orasida (aylanma sirtlardan tashqari) nol Gauss egrilikka ega sirtlar mavjud emas.

2-teorema. I-tup buralgan sirt minimal bo'ladi, agar u quyidagi ko'rinishdagi chiziqli sirt bo'lsa:

$$\varphi(s, t) = (cs, q \cos(bs) + \frac{c}{2}s^2, -q \sin(bs)) + t(0, p \cos(bs) + \sin(bs), \cos(bs) - p \sin(bs)).$$

3-teorema. II-tup buralgan sirt tekis ($K = 0$) yoki minimal ($H = 0$) bo'ladi, agar u quyidagilardan biri bo'lsa:

tekislik;

chiziqli sirt (parabolik silindr);

siklik sirt.

4-teorema. III-tup buralgan sirtlarda nol Gauss egrilikka ega sirtlar mavjud emas.

Galiley fazosida burilgan sirtlarning uch turi aniqlanib, ularning nol egrilikka ega holatlari tekshirildi. I va III-tur sirtlarda yassi sirtlar mavjud emas, II-tur sirtlarda esa yassi va minimal sirt tushunchalari ekvivalentdir. Bu natijalar Galiley va pseudo-Galiley fazolarida maxsus sirtlarning tasnifini chuqurlashtirish uchun asos bo'qladi.

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GALILEY FAZOSIDA ASIMPTOTIK CHIZIQQA EGA SIRTLAR OILASI

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Ushbu maqolada Galiley fazosida berilgan asimptotik egri chiziq orqali o'tuvchi sirtlarning oilasi o'rganilgan. Frene formulasi yordamida bunday sirtlarning parametrik ifodasi topilgan hamda berilgan egri chiziqning parametrik sirtga nisbatan izoasimptotik bo'lishi uchun zarur va yetarli shartlar aniqlangan. Natijada, umumiy asimptotik chiziqqa ega bo'lgan sirtlarning umumiy formulalar keltirildi.

Asimptotik chiziqlar differensial geometriyada muhim tadqiqot yo'nalishlaridan biridir. Asimptotik chiziq – bu sirt ustida normal egriligi nolga teng bo'lgan yo'nalishda yotuvchi egri chiziqdir. Bu faqat sirtning Gauss egriligi manfiy yoki nol bo'lgan joylarda mumkin bo'lib qoladi. Ushbu ishda Galiley fazosida berilgan asimptotik chiziq orqali o'tuvchi sirtlarning oilasi uchun shartlar keltirilgan. Galiley fazosida sirt tenglamasi quyidagi ko'rinishda beriladi:

$$\varphi(u, v) = (u, \varphi_2(u, v), \varphi_3(u, v)),$$

bu yerda $\varphi_i(u, v)$ ($i = 1, 2, 3$) – C^3 sinfga mansub silliq funksiyalar. Egri chiziq tenglamasi esa

$$r(u) = (u, f(u), g(u))$$

quyidagi ko'rinishda bo'ladi. Bunday chiziq uchun urinma, normal va binormal vektorlar:

$$t = (1, f'(u), g'(u)), \quad n = \frac{1}{\kappa(u)}(0, f''(u), g''(u)), \quad b = \frac{1}{\kappa(u)}(0, -g''(u), f''(u)),$$

bo'lib, bu yerda $\kappa(u)$ egri chiziq egriligi.

Sirt quyidagicha aniqlansin:

$$\varphi(u, v) = r(u) + [x(u, v)t(u) + y(u, v)n(u) + z(u, v)b(u)].$$

Berilgan $r(u)$ egri chiziq bu sirt uchun asimptotik bo'lishi uchun quyidagi shartlar bajarilishi zarur:

$$x(u, v_0) = y(u, v_0) = z(u, v_0) = 0,$$

$$-(1 + x_u)z_v + z_u x_v = 0,$$

$$(1 + x_u)y_v - y_u x_v \neq 0.$$

Bu holatda $r(u)$ chiziq $\varphi(u, v)$ sirtida asimptotik chiziq deyiladi.

1-teorema Agar sirtning parametrik ko'rinishidagi funksiyalar

$$x(u, v) = a(u)X(v), \quad y(u, v) = b(u)Y(v), \quad z(u, v) = c(u)Z(v),$$

bo'lsa, u holda sirt $r(u)$ ni asimptotik chiziq sifatida o'z ichiga oladi, agar

$$X(v_0) = Y(v_0) = Z(v_0) = 0, \quad b(u) \neq 0, \quad \frac{dY(v_0)}{dv} \neq 0, \quad c(u) = 0, \quad \frac{dZ(v_0)}{dv} = 0.$$

2-teorema Agar $x(u, v), y(u, v), z(u, v)$ quyidagicha aniqlansa:

$$x(u, v) = f(a(u)X(v)), \quad y(u, v) = g(b(u)Y(v)), \quad z(u, v) = h(c(u)Z(v)),$$

u holda asimptotiklik sharti quyidagicha ifodalanadi:

$$f(0) = g(0) = h(0) = 0, \quad X(v_0) = Y(v_0) = Z(v_0) = 0, \quad g'(0) \neq 0, \quad h'(0) = 0.$$

Galiley fazosida asimptotik chiziqqa ega sirlarning umumiy oilasi aniqlandi. Berilgan egri chiziqni o'z ichiga oluvchi bunday sirlarni yaratish uchun zarur bo'lgan parametrik shartlar keltirildi. Ushbu natijalar Galiley fazosidagi sirlarning lokal geometriyasini o'rganishda va kompyuter yordamida geometriya sohalarida qo'llanishi mumkin.

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PERFEKT LI ALGEBRALARI DIFFERENSIALLASHLAR ALGEBRASINING TO'LALIGI

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Li algebralarining differentsiallashlari algebraik tuzilmalarning ichki va tashqi simmetriyalarini tushinishda muhim rol o'ynaydi. Differentsiallashlarni o'rganish algebraning birinchi kogomologik fazosi $H^1(\mathfrak{g}, \mathfrak{g})$ bilan chambarchas bog'liq bo'lib, bu fazo tashqi differentsiallashlar fazosini tasniflaydi.

Bundan tashqari, differentsiallashlar Li algebralarining klassifikatsiyasi, yarim to'g'ri ko'paytmalar qurilishi va Li algebraning ba'zi kengaytmalarini o'rganishda muhim o'rin tutadi. Shu sababli, differentsiallashlar algebrasini $\text{Der}(\mathfrak{g})$ tahlil qilish berilgan Li algebraning strukturaviy xossalarini va simmetriyalarini chuqur tushunish uchun zarur bosqich hisoblanadi. Quyida Li algebralarining differentsiallashlariga oid asosiy tushuncha va ta'riflarni keltiramiz.

Ta'rif 1. Aytaylik, $\mathfrak{g}$ biror $\mathbb{K}$ maydon ustida Li algebra bo'lsin. Agar $D : \mathfrak{g} \rightarrow \mathfrak{g}$ chiziqli akslantirish quyidagi Leibniz qoidasini qanoatlantirsa:

$$D([x, y]) = [D(x), y] + [x, D(y)], \quad \forall x, y \in \mathfrak{g}.$$

U holda D *differentsiallash* deyiladi.

Li algebraning barcha differentsiallashlari fazosini $\text{Der}(\mathfrak{g})$ bilan belgilaymiz. Bu fazo quyidagicha aniqlangan amalga nisbatan Li algebra hosil qiladi:

$$[D_1, D_2] = D_1 D_2 - D_2 D_1 \in \text{Der}(\mathfrak{g}), \quad D_1, D_2 \in \text{Der}(\mathfrak{g}).$$

Ta'rif 2. Har qanday $x \in \mathfrak{g}$ element uchun $D_x(y) = [x, y]$ kabi differentsiallash *ichki differentsiallash* deyiladi va $D_x = \text{ad}_x$ kabi belgilanadi. Barcha ichki differentsiallashlar to'plami $\text{ad}(\mathfrak{g})$ orqali belgilanadi:

$$\text{ad}(\mathfrak{g}) = \{ \text{ad}_x \mid x \in \mathfrak{g} \}.$$

Ta'rif 3. Agar $\mathfrak{g}$ Li algebraning barcha differentsiallashlari ichki va markazi faqat nol elementdan iborat bo'lsa, u holda $\mathfrak{g}$ to'la Li algebra deyiladi.

Misol 1. $\mathfrak{sl}_2(\mathbb{C})$ Li algebra qaralaylik. Ma'lumki, bu algebra sodda va shu sababli perfekt algebra hisoblanadi, ya'ni $\mathfrak{sl}_2(\mathbb{C}) = [\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})]$. Ma'lumki har qanday sodda Li algebrasining markazi faqat nol elementdan iborat hamda har qanday differentsiallashi ichki bo'ladi ([1] adabiyotga qarang):

$$\text{Der}(\mathfrak{sl}_2(\mathbb{C})) = \text{ad}(\mathfrak{sl}_2(\mathbb{C})).$$

Shuning uchun $\mathfrak{sl}_2(\mathbb{C})$ algebra to'la algebradir.

Endi perfekt Li algebralarining differentsiallashlar algebrasi haqida ma'lum bo'lgan natijalarni keltiramiz. Quyidagi [2] adabiyotda, agar chekli o'lchamli perfekt li algebrasining markazi trivial bo'lsa, u holda uning differentsiallashlar algebrasi to'la bo'lishi isbotlangan. Lekin, markazi trivial bo'lmagan bir qancha Li algebralar mavjudki, ularning differentsiallashlar algebrasi ham to'la bo'ladi.

Aytaylik, S_3 biror 13-o'lchamli fazo va $\{e, h, f, s_{12}, s_{13}, s_{23}, x_1, x_2, x_3, y_1, y_2, y_3, z\}$ uning bazisi bo'lsin. U holda S_3 quyidagicha aniqlangan ko'paytirish jadvaliga nisbatan Li algebrasi tashkil qiladi:

$$\begin{aligned} [h, x_i] &= x_i & [h, y_i] &= -y_i, & [e, y_i] &= x_i, & [f, x_i] &= y_i, \\ [s_{jk}, x_i] &= \delta_{ki}x_j - \delta_{ji}x_k, & [s_{jk}, y_i] &= \delta_{ki}y_j - \delta_{ji}y_k, & [x_i, y_i] &= z, \end{aligned}$$

bu yerda, $i, j, k \in \{1, 2, 3\}$ va δ_{ij} Kroneker simvoli. Qayd etish joizki, S_3 algebraning mazkazi trivial emas.

Teorema 1. S_3 algebraning differensiallashlar algebrasi to'la Li algebra bo'ladi.

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NOMERLANGAN GRUPPALAR TASNIFI

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N.X.Qosimov [1] ishlarida hisoblanuvchi ravishda ajralib turadigan algebralarning eng umumiy effektiv, strukturaviy va topologik xususiyatlari o'rganilib, bunday algebralarning eng muhim turlari bayon qilingan. Negativ nomerlash va pozitiv algebralar A.I.Maltsev, Yu.L.Yershov, A.S.Morozov, S.P.Odintsov va V.L.Selivanov tomonidan har xil nuqtai nazardan o'ganilgan.

Ta'rif. Bizga berilgan “*” binar amal aniqlangan $(G, *)$ grupp va $\nu : \omega \rightarrow G$ akslantirish uchun shunday $K : G \rightarrow 2^\omega$ hisoblanuvchi funksiya topilib

$$\forall n, m \in \omega \quad K(\nu(n) * \nu(m)) = \phi_n(K(\nu(m)))$$

tenglik o'rinli bo'lsa, (G, ν) juftlikka *nomerlangan grupp* deyiladi.

Ta'rif. [1] Agar ν nomeratsiya negativ (pozitiv, hisoblanuvchi) bo'lsa u holda (G, ν) nomerlangan grupp *negativ (pozitiv, hisoblanuvchi)* deyiladi.

Ta'rif. [1] ν nomeratsiya *yechilgan* deyiladi, agar $\{(n, m) \mid \nu(n) = \nu(m)\}$ munosabat hisoblanuvchi bo'lsa.

Nomerlangan ekivalentlik $\eta_\nu = \{(n, m) \mid \nu(n) = \nu(m)\}$.

Ta'rif. [2] Agar nomerlangan ekivalentlik hisoblanuvchi ajraluvchi bo'lsa, nomerlangan grupp *hisoblanuvchi ajraluvchi* deyiladi.

Quyidagi teorema shuni ko'rsatadiki, negativ gruppalar hisoblanuvchi ajraluvchi gruppalarining sodda bo'lmagan misollari hisoblanadi.

Teorema. Har qanday (G, ν) negativ grupp *hisoblanuvchi ajratiluvchi nomerlangan grupp* bo'ladi.

Isboti. η_ν nomerlangan ekivalentlikni $\mathcal{L}$ yopilma operatorini, Γ orqali belgilaymiz. Ya'ni $\Gamma(E)$ operator E - yopiq to'plamni o'z ichiga oladigan eng kichik to'plam va u ω ni o'z ichiga oladi. Shuni aytishimiz mumkinki, W to'plam z raqamni rad etsa, agar $z \notin W$ bo'lsa, ya'ni $\bar{W} = \omega \setminus W$ va ixtiyoriy $A, B \subseteq \omega$ bizga berilgan bo'lsin, $\Gamma(A)$ - yopiq, hisoblanuvchi W to'plamni, ya'ni W ni ω dan ajratuvchini quyidagicha hosil qilamiz:

0-qadam: $D_0 = \{0\}$, $E_0 = \emptyset$

1-qadam: $D_1 = D_0 \cup \{1\}$, $E_1 = E_0 \cup \{0\}$

$\vdots$

e+1-qadam: Biz D_e ga tegishli bo'lmagan birinchi $x \in \omega$ ni tanlaymiz va tekshirishni boshlaymiz “ushbu to'plamlarning har biri W_e tomonidan rad etilganligini”. Agar W_e tomonidan $D_e \cup \{x\}$ rad etilsa, u holda bilamizki

$$D_{e+1} = D_e, \quad E_{e+1} = E_e \cup \{x\}.$$

Agar W_e tomonidan $E_e \cup \{x\}$ rad etilsa, agar unday bo'lmasa, u holda W_e tomonidan D_e va $E_e \cup \{x\}$ rad etiladi. Ya'ni ν ni negativligidan “ W_e tomonidan $D_e \cup \{x\}$ (yoki $E_e \cup \{x\}$) rad etilishi” hisoblanuvchi sanaluvchi munosabat.

$$D = \bigcup_{e \in \omega} D_e, \quad E = \bigcup_{e \in \omega} E_e$$

deb hisoblaylik.

Agar hisoblanuvchi ravishda sanaluvchi to'plamning to'ldiruvchisi ham hisoblanuvchi ravishda sanaluvchi to'plam bo'lsa, u holda to'plam hisoblanuvchidir. Shunday qilib bunda $D \subseteq \omega$ hisoblanuvchi sanaluvchi va $E \subseteq \omega$ ham hisoblanuvchi sanaluvchi. U holda bu to'plam hisoblanuvchi deyiladi.

Endi tekshiramiz, ya'ni D ni Γ - yopiqligini. X va Y lar berilgan, agar $X \subseteq D$ bo'lsa, u holda qandaydir e uchun $X \subseteq D_e$, $Y \subseteq E_e$ bo'ladi. U holda $D_e \cup \{x\}$ to'plamni keyingi bironta qadamda W_e rad qilmaydi. Bu esa shuni bildiradiki qandaydir $i \geq e$ qadamda $D_e \cup \{x\}$ to'plam W_i tomonidan rad etiladi. Ya'ni $x \in E_{i+1}$ agar $i + 1 > e$ bo'lsa u holda $x \in E$ bo'ladi. Teskari holda $D_e \cup \{x\}$ tomonidan rad etilmay qoladi, bundan $x \in D$. Bu ziddiyat. Demak D ham Γ - yopiq hisoblanuvchi to'plam hisoblanadi. Teorema isbotlandi.

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ЛОКАЛЬНО ПОЗИТИВНЫЕ И НЕГАТИВНЫЕ ГРУППЫ

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Базовые понятия, используемые в работе, содержатся в [1–4]. Напомним, что всюду определенная функция из множества натуральных чисел $\mathbb{N}$ в $\mathbb{N}$ называется *вычислимой*, если существует вычисляющий ее алгоритм [1]. Подмножество $A \subseteq \mathbb{N}$ называется *вычислимым* (*перечислимым*, соответственно *коперечислимым*), если вычислима его характеристическая функция (χ_A (соответственно A – область значений некоторой вычислимой функции)). Эти определения естественным образом переносятся на многоместные функции и отношения.

Следуя Ю. Л. Ершову и С. С. Гончарову [2–4], приведем ряд основных определений. Если $\mathfrak{A}$ счетная универсальная алгебра эффективной сигнатуры Σ , то ее *нумерацией* будем называть всякое отображение ν множества натуральных чисел $\mathbb{N}$ на основное множество алгебры $|\mathfrak{A}|$, для которого существует эффективное семейство $\{f_i\}$ вычисляемых функций, представляющих Σ -операции алгебры $\mathfrak{A}$ в нумерации ν , т. е. всякая операция $\sigma \in \Sigma$ представляется соответствующей ей такой вычислимой функцией f_σ , что $\nu(f_\sigma(x_1, \dots, x_n)) = \sigma(\nu(x_1), \dots, \nu(x_n))$. Если η эквивалентность на $\mathbb{N}$, то алгебру $\mathfrak{B}$ будем называть *представимой* над η , если существует нумерация этой алгебры с нумерационной эквивалентностью, равной η , т. е. найдется такое эффективное семейство $\{f_i\}$ вычисляемых функций, согласованных с эквивалентностью η (из η -эквивалентности аргументов следует η -эквивалентность значений функций на этих аргументах), что $\mathfrak{B}$ изоморфна фактор-алгебре вычислимой алгебры $\mathfrak{A}_f$ по конгруэнции η^f .

Фундаментальные алгебраические понятия группы являются классическим объектом математической логики, которая, в частности занимается описанием алгоритмических свойств групп, заданных теми или иными представлениями. Важнейшими среди алгоритмических представлений этих объектов являются вычисляемые, т.е. такие, которые изоморфны группам, носителями которых являются вычисляемые подмножества множества натуральных чисел, бинарная операция группы представлена подходящей вычислимой операцией.

Определение 1. Нумерованной группой называется пара $(\mathcal{G}, \nu)$, где $\mathcal{G}$ группа с операцией “ $\cdot$ ”, $\nu : \mathbb{N} \rightarrow \mathcal{G}$ – отображение со следующим свойством:

Для “ $\cdot$ ” бинарной операции существует такая рекурсивная функция f , что $\nu(f(x, y)) = \nu(x) \cdot \nu(y)$.

В [5] введены следующие понятия:

Определение 2. Нумерация алгебры называется локально позитивной (негативной), если хотя бы один смежный класс ядра этой нумерации является перечислимым (коперечислимым).

Отметим, что, в то время как позитивность и негативность являются “глобальными” понятиями, подразумевающими наличие единообразной эффективной процедуры порождения нумерационной эквивалентности или соответственно ее дополнения, “локальность”, вообще говоря, означает отсутствие такой процедуры. При этом,

к примеру, число локально позитивных (негативных) эквивалентностей даже с не более чем двухэлементными смежными классами есть континуум, тогда как число позитивных (негативных) эквивалентностей счетно. Очевидно, что всякая эквивалентность с конечными смежными классами является как локально позитивной, так и локально негативной, однако ее алгоритмическая сложность может быть какой угодно. Тем не менее свойства симметрии в группах и полях позволяют равномерно эффективно переходить от локальной позитивности (негативности) их представлений к глобальной, т. е. алгоритмы позитивности (негативности) оказываются изначально присутствующими в структурах самих этих систем, что проявляется и в свойствах их алгоритмических представлений. В [5] доказан следующий результат.

Предложение 1. Для всякой локально позитивной (локально негативной) группы любой смежный класс ее ядра перечислим (коперечислим).

Теорема 1. Если нумерация группы $\mathcal{G}$ является локально позитивной и локально негативной, то она вычислима.

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ХАРАКТЕРИЗАЦИЯ ВЫЧИСЛИМО ОТДЕЛИМЫХ ГРУПП

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В работе [3] развит теория вычислимо отделимых нумерованных алгебр. Модели являются естественным обобщением алгебр. В [4] дана структурная характеристика вычислимо отделимых моделей. Очень важно изучать эффективные свойства нумерованных алгебраических классических систем, таких как группы и кольца. В данной работе рассматривается равномерно вычислимо отделимые группы и получены аналогичные результаты для конкретных алгебр, а именно для нумерованных групп.

Все неопределяемые понятия можно найти в [1–4].

Если в определении вычислимой отделимости модели потребовать наличие эффективного процесса построения для отделяющего множества по элементу a и по номеру предикатного символа P , то получим понятие равномерно вычислимо отделимой модели.

Для групп формально дадим следующее определение.

Определение 1. Нумерованной группой называется пара $(\mathcal{G}, \nu)$, где $\mathcal{G}$ группа с операцией “ $\cdot$ ”, $\nu : \mathbb{N} \rightarrow \mathcal{G}$ – отображение со следующим свойством:

Для “ $\cdot$ ” бинарной операции существует такая вычислимая функция f на множестве натуральных чисел $\mathbb{N}$, что $\nu(f(x, y)) = \nu(x) \cdot \nu(y)$.

Если нумерация ν вычислимом (положительном, отрицательном), то нумерованная группа $(\mathcal{G}, \nu)$ называется вычислимом (положительном, отрицательном). Нумерация ν назовем положительной (отрицательной), если $\ker \nu (\mathbb{N}^2 \setminus \ker \nu)$ вычислимо перечислимо. Нумерация ν назовем вычислимой, если отношение $\ker \nu$ вычислимо. Нумерационная эквивалентность $\equiv_\nu$.

Определение 2. Нумерованная группа называется вычислимо отделимой, если таковой является еще нумерационная эквивалентность.

Следующая теорема показывает, что отрицательные группы являются важными неочевидными примерами вычислимо отделимых группой.

Теорема 1. Всякая отрицательная группа $(\mathcal{G}, \nu)$ является вычислимо отделимой нумерованной группой.

Следующая теорема характеризует вычислимо отделимые группы.

Теорема 2. Нумерованная группа вычислимо отделима тогда и только тогда, когда она аппроксимируется отрицательными группами.

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2-SHO‘BA:MATEMATIKANALIZ.

On the absence of odd periodic points of a quadratic stochastic operator with variable coefficients**Abdurakhimova Sh.B.¹**Department of Mathematics, Namangan State University, Namangan, Uzbekistan;
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In this work, we present some results on the dynamics of a Volterra quadratic stochastic operator whose coefficient varies according to the rule of a piecewise-linear function.

Let S^{m-1} be the m -dimensional simplex:

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\}. \quad (1)$$

We consider a Volterra operator on S^1 . This dynamics is generated by the operator $W : (x, y, a) \in \mathcal{S} \rightarrow \mathcal{S}$, (where $\mathcal{S} = S^1 \times [-1, 1]$) having the following form

$$W : \begin{cases} x' = x(1 - ay), \\ y' = y(1 + ax), \\ a' = f(a) \end{cases} \quad (2)$$

where $a \in [-1, 1]$, $f : [-1, 1] \rightarrow [-1, 1]$ is piecewise-linear function defined as

$$f(a) = \begin{cases} 2a + 2, & \text{if } -1 \leq a < -\frac{1}{2}, \\ -2a, & \text{if } -\frac{1}{2} \leq a \leq \frac{1}{2}, \\ 2a - 2, & \text{if } \frac{1}{2} < a \leq 1. \end{cases} \quad (3)$$

The operator W maps a point (x, y, a) in the space $\mathcal{S}$ to another point (x', y', a') in the same space. The first two equations describe how x and y evolve. The third equation $a' = f(a)$ determines the evolution of the parameter a , this is a one-dimensional discrete dynamical system independent of x and y .

A periodic point of period k for W is a point (x, y, a) such that $W^k(x, y, a) = (x, y, a)$, where W^k denotes k -th iteration the operator W (see [1], [2], [3]). Since dynamics of a is independent, any periodic point of W must have an a component that is periodic for the map f . Therefore, to have periodic points for W , we need f to have periodic points.

Remark. If the operator W has a periodic point of period k , then the function f also has a periodic point of period k . But the fact that the function f has a periodic point of period k doesn't mean that the operator W also has a periodic point of period k .

The function f , that is defined above has no odd-periodic points for periods $k \geq 3$. Consequently, we have

Theorem. The operator W has no odd-periodic points for periods $k \geq 3$.

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Gibbs measures for the asymmetric clock model on a Cayley tree order two

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The asymmetric p -state clock model, also known as the chiral model, was introduced by Ostlund and Huse in 1981 [2]. Let $\Gamma^k = (V, L)$ be the Cayley tree, where each vertex has $k+1$ neighbors with V being the set of vertices and L the set of edges (see [1]). For an arbitrary vertex $x^0 \in V$, we put $W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \{x \in V \mid d(x, x^0) \leq n\}$, where $d(x, y)$ is the distance between x and y in the Cayley tree, i.e., the number of edges of the path between x and y . Let $S(x) = \{y \in W_{n+1} : d(y, x) = 1\}$ be the set of direct successors of x . We assume that the spin variables take values in the set $\Phi = \{1, 2, \dots, p\}$. A configuration σ on a subset $A \subseteq V$ is defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$, while the set of all configurations on V is given by $\Omega = \Omega_V$ with $\sigma = \sigma_V$.

The p -state asymmetric clock model is defined by the Hamiltonian

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \cos \left(\frac{2\pi}{p} (\sigma(x) - \sigma(y) - \Delta) \right), \quad (1)$$

where $J \in \mathbb{R}$ is the coupling constant, and $\Delta \in \mathbb{R}$ is an additional parameter of the model.

Let $h : x \mapsto h_x = (h_{0,x}, h_{1,x}, \dots, h_{m,x}) \in \mathbb{R}^{m+1}$ be a vector function of $x \in V \setminus \{x^0\}$. Consider a probability distribution $\mu^{(n)}$ on Ω_{V_n} :

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}), \quad (2)$$

where $\sigma_n \in \Omega_{V_n}$, $\beta = \frac{1}{T}$, $T > 0$ is temperature and

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp(-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}(x), x}).$$

A probability distribution $\mu^{(n)}$ is said to be compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu^{(n-1)}(\sigma_{n-1}), \quad (3)$$

where $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$. In this case, there is a unique measure μ on Ω such that $\mu(\{\sigma \mid_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$, for all n and $\sigma_n \in \Omega_{V_n}$. Such a measure is called a splitting Gibbs measure (SGM) corresponding to the Hamiltonian H and to the function $x \mapsto h_x, x \neq x_0$ [1].

The following statement gives a condition on h_x guaranteeing that the distribution of $\mu^{(n)}(\sigma_n)$ is compatible.

Theorem 1. *The probability distributions $\mu^{(n)}(\sigma_n), n = 1, 2, \dots$, determined by the formula (2), are compatible if and only if, for any $x \in V \setminus \{x^0\}$, the following equation holds:*

$$h_x^* = \sum_{y \in S(x)} F(h_y^*, p, \theta). \quad (4)$$

Here, $\theta = \exp(J\beta \cos(\frac{2\pi}{p}\Delta))$, where h_x^* denotes the vector

$$h_x^* = (h_{1,x} - h_{p,x}, h_{2,x} - h_{p,x}, \dots, h_{p-1,x} - h_{p,x})$$

and the vector function $F(\cdot, p, \theta) : \mathbb{R}^{p-1} \rightarrow \mathbb{R}^{p-1}$ is given by

$$F(h, p, \theta) = (F_1(h, p, \theta), \dots, F_{p-1}(h, p, \theta)), \quad h = (h_1, h_2, \dots, h_{p-1}),$$

with components

$$F_i = \ln \frac{\sum_{j=1}^{p-1} \exp(J\beta \cos(\frac{2\pi}{p}(i-j-\Delta)) + h_{j,y}) + \exp(J\beta \cos(\frac{2\pi}{p}(i-p-\Delta)))}{\sum_{j=1}^{p-1} \exp(J\beta \cos(\frac{2\pi}{p}(p-j-\Delta)) + h_{j,y}) + \theta},$$

$i = 1, \dots, p-1$.

Namely, for any boundary condition satisfying the functional equation (4) there exists a unique SGM.

Proof: *Necessity.* Suppose that (3) holds; we aim to prove (4). Substituting (2) into (3), we obtain, for any configuration $\sigma_{n-1} : x \in V \mapsto \sigma_{n-1}(x) \in \Phi$:

$$\begin{aligned} \sum_{\omega_n \in \Omega_{W_n}} Z_n^{-1} \exp \left(-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} (J\beta \cos(\frac{2\pi}{p}(\sigma_{n-1}(x) - \omega_n(y) - \Delta)) \right. \\ \left. + h_{\omega_n(y),y}) \right) = Z_{n-1}^{-1} \exp \left(-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x} \right), \end{aligned}$$

where $\omega_n : x \in W_n \mapsto \omega_n(x)$.

From this, we derive:

$$\begin{aligned} \frac{Z_{n-1}}{Z_n} \sum_{\omega_n \in \Omega_{W_n}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \exp(J\beta \cos(\frac{2\pi}{p}(\sigma_{n-1}(x) - \omega_n(y) - \Delta)) + h_{\omega_n(y),y}) = \\ \prod_{x \in W_{n-1}} \exp(h_{\sigma_{n-1}(x),x}). \end{aligned} \quad (5)$$

Consequently, for all $i \in \Phi$,

$$\exp(h_{i,x} - h_{p,x}) = \prod_{y \in S(x)} \frac{\sum_{j \in \Phi} \exp(J\beta \cos(\frac{2\pi}{p}(i-j-\Delta)) + h_{j,y})}{\sum_{j \in \Phi} \exp(J\beta \cos(\frac{2\pi}{p}(p-j-\Delta)) + h_{j,y})}. \quad (6)$$

Introducing $\theta = \exp(J\beta \cos(\frac{2\pi}{p}\Delta))$ as in (5) and denoting $h_{i,x}^* = h_{i,x} - h_{p,x}$, we obtain (4) from (6).

Sufficiency. From (4), we recover (6) and (5), leading to (2). This completes the proof.

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BILATERAL NONLINEAR RECURRENCE EQUATION OVER $\mathbb{Q}_p$

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Let $\mathbb{Q}_p$ be a field of p -adic numbers. For each $a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$ we denote

$$B_n(a) = \{x \in \mathbb{Q}_p : |x - a|_p < p^n\}.$$

We notice that $B_n(a)$ is a clopen subset of $\mathbb{Q}_p$. Recall that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ and $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ are the set of all p -adic integers and p -adic units, respectively (for further details on p -adic analysis, the reader is referred to [1,2]).

The range of the p -adic exponential function is denoted by $\mathcal{E}_p$, and one has

$$\mathcal{E}_p = \begin{cases} B_{-1}(1), & \text{if } p = 2; \\ B_0(1), & \text{if } p \neq 2. \end{cases}$$

By $\ell_p^\infty(\mathbb{Z})$ we denote the set of all bounded bilateral sequences over $\mathbb{Q}_p$, i.e. $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in \ell_p^\infty(\mathbb{Z})$ if and only if $\max\{|x_i|_p : i \in \mathbb{Z}\} < \infty$. Here we have used "max" instead of "sup" because of the discreteness of the p -adic norm.

We notice that $\ell_p^\infty(\mathbb{Z})$ is a non-Archimedean normed space over $\mathbb{Q}_p$ where the norm is defined by

$$\|\mathbf{x}\|_p = \max\{|x_i|_p : i \in \mathbb{Z}\}, \quad \mathbf{x} \in \ell_p^\infty(\mathbb{Z}).$$

The set $c_p^0(\mathbb{Z}) := \left\{ \mathbf{x} \in \ell_p^\infty(\mathbb{Z}) : \lim_{n \rightarrow \infty} |x_{\pm n}|_p = 0 \right\}$ is a closed subset of $\ell_p^\infty(\mathbb{Z})$. Consequently, $(c_p^0(\mathbb{Z}), \|\cdot\|_p)$ is a non-Archimedean Banach space.

Let $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ be a bilateral sequence on $\mathbb{Q}_p$. Then the following are equivalent:

- $\mathbf{x} \in c_p^0(\mathbb{Z})$;
- $\sum_{i=-\infty}^{\infty} x_{\pi(i)} < \infty$ for any permutation π of integers.

We consider the following nonlinear bilateral recurrence equation

$$x_{n+1} = ((x_1 + x_{-1})\lambda_n + (\theta + \theta^{-1})(1 - \lambda_n))x_n - x_{n-1}, \quad \forall n \in \mathbb{Z}, \quad (1)$$

where $\theta \in \mathcal{E}_p \setminus \{1\}$ and $\lambda = (\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots) \in c_p^0(\mathbb{Z})$.

Theorem. Let $\lambda = (\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots) \in c_p^0(\mathbb{Z})$ be a bilateral vector satisfying $\lambda_0 = 1$, $\lambda_{\pm n} \in p\mathbb{Z}_p$ for all $n \in \mathbb{N}$. Then, for every $\theta \in \mathcal{E}_p \setminus \{1\}$, the set of solutions to equation (1) in $\mathcal{E}_p^\mathbb{Z}$ has the cardinality of the continuum.

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On antiferromagnetic coupled Ising-Potts model

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We present a result concerning Gibbs measures for a Hamiltonian defined on a Cayley tree $\Gamma^k = (V, L)$ (see [1] for the basic definitions). Following [2], we study a coupled Ising-Potts model on Cayley trees of order $k \geq 2$. The spin variables are vectors (s, σ) , where $s \in I = \{-1, 1\}$ and $\sigma \in \Phi = \{1, \dots, q\}$, with $q \geq 2$.

A coupled configuration (s, σ) on V is a function assigning to each vertex $x \in V$ a pair $(s(x), \sigma(x)) \in I \times \Phi$.

The Hamiltonian of the coupled Ising-Potts model is given by

$$H(s, \sigma) = -J \sum_{\langle x, y \rangle \in L} s(x)s(y)\delta_{\sigma(x)\sigma(y)}, \quad (1)$$

where $J \in \mathbb{R}$ is a coupling constant, $\langle x, y \rangle$ denotes nearest-neighbor vertices, and δ_{ij} is the Kronecker symbol.

In [2], it was shown that each translation-invariant Gibbs measure for the Hamiltonian (1) (in the case $q = k = 2$) corresponds to a positive solution of the following system:

$$u_1 = \left(\frac{\theta^{-1}v_1 + \theta u_1 + v_2 + 1}{\theta + \theta^{-1}v_2 + u_1 + v_1} \right)^2 \quad (2)$$

$$v_1 = \left(\frac{\theta^{-1}u_1 + \theta v_1 + v_2 + 1}{\theta + \theta^{-1}v_2 + u_1 + v_1} \right)^2 \quad (3)$$

$$v_2 = \left(\frac{\theta^{-1} + \theta v_2 + u_1 + v_1}{\theta + \theta^{-1}v_2 + u_1 + v_1} \right)^2 \quad (4)$$

Denote by $n_{ti}(H)$ the set of translation-invariant Gibbs measures of the coupled Ising-Ising model.

By an analysis of system (2)-(4) we have proved the following:

Theorem 1. The following holds:

$$n_{ti}(H) \geq \begin{cases} 1, & \text{if } \theta \in (\frac{1}{2}(5 - \sqrt{21}), \frac{1}{2}(5 + \sqrt{21})) \\ 2, & \text{if } \theta = \frac{1}{2}(5 - \sqrt{21}), \frac{1}{2}(5 + \sqrt{21}) \\ 3, & \text{if } \theta \in (0, \frac{1}{2}(5 - \sqrt{21})) \cup (\frac{1}{2}(5 + \sqrt{21}), +\infty) \end{cases}$$

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p -Adic generalized $G_3^{(2)}$ -periodic Gibbs measure for the Ising model with $G_3^{(2)}$ -periodic external field on the Cayley tree

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Let $\mathbb{Q}_p$ be a field of p -adic numbers (see [1]). Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots)$$

where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 \neq 0$, $x_j \in \{0, 1, \dots, p-1\}$, $j \in \mathbb{N}$ (see [1]). In this case $|x|_p = p^{-\gamma(x)}$.

Let $\Gamma^k = (V, L)$ be a Cayley tree and $\Phi = \{-1, 1\}$ (see [2]). We consider p -adic Ising model with $G_3^{(2)}$ -periodic external field on the Cayley tree.

For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\varphi^{(n)} \in \Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1} \vee \varphi^{(n)})(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \varphi^{(n)}(x), & \text{if } x \in W_n. \end{cases}$$

A formal p -adic Hamiltonian $H : \Omega \rightarrow \mathbb{Q}_p$ of the p -adic Ising model is defined by

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y) + \sum_{x \in V} \alpha_x \sigma(x), \quad (1)$$

where $0 < |J|_p < p^{-1/(p-1)}$ and $0 < |\alpha_x|_p < p^{-1/(p-1)}$.

We define a function $h : x \rightarrow h_x, \forall x \in V \setminus \{x_0\}$, $h_x \in \mathbb{Q}_p$ and consider p -adic probability distribution $\mu_h^{(n)}$ on Ω_{V_n} defined by

$$\mu_h^{(n)}(\sigma_n) = \frac{1}{Z_n^{(h)}} \exp_p\{H_n(\sigma_n)\} \prod_{x \in W_n} h_x^{\varphi(x)} \quad n = 1, 2, \dots, \quad (2)$$

where $Z_n^{(h)}$ is the normalizing constant.

A p -adic probability distribution $\mu_h^{(n)}$ is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \varphi) = \mu_h^{(n-1)}(\sigma_{n-1}). \quad (3)$$

In this case, by the p -adic analogue of the Kolmogorov theorem there exists a unique measure μ_h on the set Ω such that $\mu_h(\{\sigma|_{V_n} \equiv \sigma_n\}) = \mu_h^{(n)}(\sigma_n)$ for all n and $\sigma_n \in \Omega_{V_n}$. (see [3]) A limit p -adic measure generated by (2) is called a p -adic generalized Gibbs measure.

From [3] it is known that the sequence of p -adic probability distributions $\{\mu_h^{(n)}\}_{n \geq 1}$, determined by formula (2), is consistent if and only if for any $x \in V \setminus \{x_0\}$, the following equality holds:

$$\hat{h}_x = \prod_{y \in S(x)} \frac{\theta \eta_y \hat{h}_y + 1}{\theta + \eta_y \hat{h}_y},$$

where $\theta = \exp_p\{2J\}$, $\eta_y = \exp_p(2\alpha_y)$, $\widehat{h}_x = \eta_x \exp_p\{2h_x\}$.

Let $\theta = 1 + \theta_1 p + o[p]$, $\eta = 1 + \eta_{10} p^{10} + o[p^{10}]$ and

$$\eta_x = \begin{cases} \eta, & \text{if } |x| - \text{even}, \\ 1, & \text{if } |x| - \text{odd}. \end{cases}$$

Theorem 1. *Let $p > 3$. Then for the Ising model with $G_3^{(2)}$ -periodic external field on the Cayley tree there exist two $G_3^{(2)}$ -periodic p -adic generalized Gibbs measures.*

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Ground states for the mixed type spin-1 and spin-2 Ising model on the Cayley tree of order two

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In the work, we consider the mixed type spin-1 and spin-2 Ising model on the Cayley tree of order two. Let $\Gamma^2 = (V, L)$ be the Cayley tree of order two, here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y . The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is the number of edges in the shortest path from x to y . Let x^0 be a any vertex and we call it the root of tree. Denote

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\},$$

$$L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\},$$

$$\Gamma_+^2 = \{x \in V : d(x^0, x) - \text{even}\}, \quad \Gamma_-^2 = \{x \in V : d(x^0, x) - \text{odd}\}.$$

To define the model properly, let us consider spin state spaces as follows: $\Phi = \{-1; 0; 1\}$ and $\Psi = \{-2; -1; 0; 1; 2\}$. The corresponding configuration spaces are defined by $\Omega_+ = \Phi^{\Gamma_+^2}$ and $\Omega_- = \Psi^{\Gamma_-^2}$.

For the configuration $\xi \in \Xi = \Omega_+ \times \Omega_-$, the associated sites belong to successive generations of the tree. At odd-numbered levels of the tree, spins with Ψ elements are placed in the vertices at even-numbered levels, i.e.

$$\xi(x) = \begin{cases} \sigma(x) & \text{if } x \in \Gamma_+^2 \\ s(x) & \text{if } x \in \Gamma_-^2, \end{cases}$$

where $\sigma \in \Phi = \{-1; 0; 1\}$ and $s \in \Psi = \{-2; -1; 0; 1; 2\}$.

Let $S_1(x) = \{y \in G_2 : \langle x, y \rangle\}$, $x \in G_2$. Consider the quotient group $G_2/G_2^* = \{H_1, \dots, H_r\}$, where G_2^* is a normal subgroup of index r with $r \geq 1$.

Definition 1. A configuration $\xi(x)$ is called to be G_2^* -periodic if $\xi(x) = \xi_i$ for all $x \in G_2$ with $x \in H_i$. A G_2 -periodic configuration is called to be *translation-invariant*.

The (formal) Hamiltonian of the mixed type spin-1 va spin-2 Ising model is defined by

$$H(\xi) = -J \sum_{\langle x, y \rangle} \xi(x)\xi(y),$$

where $J \in \mathbb{R}$ are coupling constants and $\xi \in \Xi$.

For a pair of configurations ξ and φ that coincide almost everywhere, i.e., everywhere except for a finite number of positions, we consider a relative Hamiltonian $H(\xi, \varphi)$, the difference between the energies of the configurations ξ and φ has the form

$$H(\xi, \varphi) = -J \sum_{\langle x, y \rangle} (\xi(x)\xi(y) - \varphi(x)\varphi(y)).$$

Let M be the set of unit balls with vertices in V . We call the restriction of a configuration ξ to the ball $b \in M$ a *bounded configuration* ξ_b .

Define the energy of a ball b for configuration ξ by

$$U(\xi_b) \equiv U(\xi_b, J) = -\frac{1}{2}J \sum_{\substack{\langle x,y \rangle \in L: \\ x,y \in b}} \xi(x)\xi(y).$$

Let $G_2^{(2)} = \{x \in G_2 : |x| \text{ even}\}$, $G_2 \setminus G_2^{(2)} = \{x \in G_2 : |x| \text{ odd}\}$.

In the considered setting, one easily finds that $U(\xi_b) \in \{U_1, \dots, U_{13}\}$ for every ξ_b , where

$$U_1 = 3J, U_2 = \frac{5}{2}J, U_3 = 2J, U_4 = \frac{3}{2}J, U_5 = J, U_6 = \frac{1}{2}J, U_7 = 0, U_8 = -\frac{1}{2}J, U_9 = -J, \\ U_{10} = -\frac{3}{2}J, U_{11} = -2J, U_{12} = -\frac{5}{2}J, U_{13} = -3J.$$

Definition 2. A configuration φ is called a *ground state* of the relative Hamiltonian H if

$$U(\varphi_b) = \min\{U_1, \dots, U_{13}\} \text{ for any } b \in M.$$

For any $i = 1, \dots, 13$ we put

$$A_i = \{J \in \mathbb{R} : U_i = \min\{U_1, \dots, U_{13}\}\}. \quad (1)$$

It is easy to see

$$A_1 = \{J : J \leq 0\}, A_{13} = \{J : J \geq 0\}, A_2 = A_3 = A_4 = \dots = A_{12} = 0.$$

Let us consider the quotient group $G_2/G_2^* = \{H_1, \dots, H_r\}$, where G_2^* is a normal subgroup of index r with $r \geq 1$.

A configuration $\xi(x)$ is said to be $G_k^{(2)}$ -periodic if

$$\xi(x) = \begin{cases} \sigma(x) & \text{if } x \in \Gamma_+^2 \\ s(x) & \text{if } x \in \Gamma_-^2 \end{cases}$$

$\forall x \in G_2, x \in H_i \ i = \overline{1, r}$.

Theorem 1. Let $k = 2$.

1. If $J \in A_1$, then for the mixed spin-1 and spin-2 Ising model the configurations $\xi_1(x) = (-1; -2)$, $\xi_2(x) = (1; 2)$ are $G_2^{(2)}$ -periodic ground states.
2. If $J \in A_{13}$, then for the mixed spins Ising model the configurations $\xi_3(x) = (-1; 2)$, $\xi_4(x) = (1; -2)$ are $G_2^{(2)}$ -periodic ground states.

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On dynamics of a separable quadratic stochastic operators

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Let $E = \{1, \dots, m\}$ be a finite set and the set of all probability distribution on E

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\} \quad (1)$$

be the $(m-1)$ -dimensional simplex. A QSO is a mapping defined as $V : S^{m-1} \rightarrow S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$x'_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j, \quad k \in E, \quad (2)$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^m P_{ij,k} = 1 \quad \text{for all } i, j, k \in E. \quad (3)$$

The trajectory (orbit) $\{\mathbf{x}^{(n)}\}_{n \geq 0}$, of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{(n+1)}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots$$

Separable QSOs were introduced as follows: let $A = (a_{ik})_{i,k=1}^m$, $B = (b_{jk})_{j,k=1}^m$ be the matrices with the real entries. Consider the QSO (2), (3) with additional condition

$$p_{ij,k} = a_{ik} b_{jk}, \quad \forall i, j, k \in E, \quad (4)$$

where $a_{ik}, b_{jk} \in \mathbb{R}$ entries of matrices A and B such that the conditions (3) are satisfied for the coefficients (4). Then the QSO V corresponding to the coefficients (4) has the form

$$x'_k = (V(\mathbf{x}))_k = (A(\mathbf{x}))_k \cdot (B(\mathbf{x}))_k, \quad \forall k \in E, \quad (5)$$

where $(A(\mathbf{x}))_k = \sum_{i=1}^m a_{ik} x_i$, $(B(\mathbf{x}))_k = \sum_{j=1}^m b_{jk} x_j$.

Definition The operator (5) is called a separable QSO.

Let us consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1+a & 1 \\ 1-a & 1 & 1+c \\ 1 & 1-c & 1 \end{pmatrix}, \quad (6)$$

where $a, c \in [-1, 1]$.

Then corresponding SQSO $W : S^2 \rightarrow S^2$ is:

$$W : \begin{cases} x'_1 = x_1 (1 - ax_2), \\ x'_2 = x_2 (1 + ax_1 - cx_3), \\ x'_3 = x_3 (1 + cx_2). \end{cases} \quad (7)$$

Let a face of the simplex S^2 be the set $\Gamma_\alpha = \{\mathbf{x} \in S^2 : x_i = 0, i \notin \alpha \subset \{1, 2, 3\}\}$. Let the set $\text{int } S^2 = \{\mathbf{x} \in S^2 : x_1 x_2 x_3 > 0\}$ and let the set $\partial S^2 = S^2 \setminus \text{int } S^2$ be the interior and the boundary of the simplex S^2 , respectively. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the vertexes of the two-dimensional simplex.

Theorem 1. *For the SCSO W (7), the following assertions true:*

- (i) *The faces $\Gamma_{\{1,2\}}$, $\Gamma_{\{1,3\}}$, $\Gamma_{\{2,3\}}$ of the simplex S^2 are invariant sets;*
- (ii) $\text{Fix}(W) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$;

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DEGENERATE CASES OF LOTKA-VOLTERRA MAPPINGS ACTING IN THE SIMPLEX S^4

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Lotka–Volterra systems are important models in mathematics, biology, and economics. Their trajectories and limit sets, especially in simplices, help to understand long-term behavior. This work studies degenerate cases of Lotka–Volterra mappings in the simplex S^4 , showing how skew-symmetric matrices and tournament graphs describe the convergence of trajectories.

For any $x^0 \in S^4$, let

$$\omega(x^0) = \{x^0, x^{(1)}, x^{(2)}, \dots\}$$

be the set of limit points of the positive trajectory, and

$$\alpha(x^0) = \{x^0, x^{(-1)}, x^{(-2)}, \dots\}$$

be the set of limit points of the negative trajectory.

Definition 1. A graph with vertices $1, 2, \dots, m$ is called a partially oriented graph, in which any two vertices are connected by a directed edge; such a graph is called a tournament. It is known [2] that matrices in general position form an open and dense subset everywhere in the set of all skew-symmetric matrices, and moreover, the corresponding partially oriented graph is a tournament.

Definition 2. A skew-symmetric matrix is called a matrix in general position if all its principal minors of even order are positive. The Lotka–Volterra operator consists in studying the trajectories of points and defining the set of initial points P and the set of terminal points Q of the trajectories. It is clear that with each skew-symmetric matrix one can associate the corresponding partially oriented graph.

Theorem 1. For any interior point x^0 of the simplex S^4 , both the positive and the negative trajectories converge, and, moreover $\alpha(x^0) \subset P$ and $\omega(x^0) \subset Q$.

Theorem 2. Every trajectory converges and, moreover, $\alpha(x^0) \subset P$, $\omega(x^0) \subset Q$.

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THE FOUR-STATE CHUI-WEEKS MODEL ON A CAYLEY TREE OF ORDER TWO: TRANSLATION-INVARIANT GROUND STATES

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The Chui-Weeks model on the Cayley tree (see [2]) is defined by the following Hamiltonian

$$H(\sigma) = J \sum_{\langle x,y \rangle \in L} |\sigma(x) - \sigma(y)| + \alpha \sum_{x \in V} \delta_{\sigma(x),0}, \quad (1)$$

where $J, \alpha \in \mathbb{R}$, α is an external field, and $\sigma \in \Phi^V$, $\Phi = \{0, 1, 2, 3\}$, as well as $\delta_{i,j}$ denotes the Kronecker delta, that is, $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

Recall that model (1) coincides with the SOS model under the condition $\alpha = 0$ (see, e.g., [1]).

Let M be the set of all unit balls with vertices in V and let $S_1(x)$ be the set of all nearest-neighbor vertices of $x \in V$. We denote the restriction of a configuration σ to the ball $b \in M$ by σ_b . The energy of configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{J}{2} \sum_{x \in S_1(c_b)} |\sigma(x) - \sigma(c_b)| + \frac{\alpha}{k+2} \sum_{x \in b} \delta_{\sigma(x),0}.$$

where $J = (J, \alpha) \in \mathbb{R}^2$ and c_b is the center of the unit ball b .

The Hamiltonian (1) can be written as

$$H(\sigma) = \sum_{b \in M} U(\sigma_b).$$

We proved the following lemma.

Lemma. *Let $k = 2$. Then for each configuration φ_b , we have the following*

$$U(\varphi_b) \in \{U_i : i = 1, 2, \dots, 29\},$$

where

$$\begin{aligned} U_1 &= 0; U_2 = \alpha; U_3 = \frac{J}{2}; U_4 = \frac{J}{2} + \frac{\alpha}{4}; U_5 = \frac{J}{2} + \frac{3\alpha}{4}; U_6 = J; U_7 = J + \frac{\alpha}{4}; \\ U_8 &= J + \frac{\alpha}{2}; U_9 = J + \frac{3\alpha}{4}; U_{10} = \frac{3J}{2}; U_{11} = \frac{3J}{2} + \frac{\alpha}{4}; U_{12} = \frac{3J}{2} + \frac{\alpha}{2}; U_{13} = \frac{3J}{2} + \frac{3\alpha}{4}; \\ U_{14} &= 2J; U_{15} = 2J + \frac{\alpha}{4}; U_{16} = 2J + \frac{\alpha}{2}; U_{17} = \frac{5J}{2}; U_{18} = \frac{5J}{2} + \frac{\alpha}{4}; U_{19} = \frac{5J}{2} + \frac{\alpha}{2}; \\ U_{20} &= 3J; U_{21} = 3J + \frac{\alpha}{4}; U_{22} = 3J + \frac{\alpha}{2}; U_{23} = 3J + \frac{3\alpha}{4}; U_{24} = \frac{7J}{2} + \frac{\alpha}{4}; \\ U_{25} &= \frac{7J}{2} + \frac{\alpha}{2}; U_{26} = 4J + \frac{\alpha}{4}; U_{27} = 4J + \frac{\alpha}{2}; U_{28} = \frac{9J}{2} + \frac{\alpha}{4}; U_{29} = \frac{9J}{2} + \frac{3\alpha}{4}. \end{aligned}$$

Definition. A configuration φ is called a ground state for the Hamiltonian (1), if

$$U(\varphi_b) = \min\{U_i : i = 1, 2, \dots, 29\}$$

for any $b \in M$.

We denote $A_\xi = \{(J, \alpha) \in \mathbb{R}^2 : U_\xi = \min\{U_i : i = 1, 2, \dots, 29\}\}$.

Calculations show that:

$$A_1 = \{(J, \alpha) \in \mathbb{R}^2 : J \geq 0, \alpha \geq 0\}; A_2 = \{(J, \alpha) \in \mathbb{R}^2 : 18J \geq \alpha, \alpha \leq 0\};$$

$$A_3 = A_6 = A_{10} = A_{14} = A_{17} = \{(J, \alpha) \in \mathbb{R}^2 : J = 0, \alpha \geq 0\}; A_4 = A_5 = A_7 = A_8 = A_9$$

$$= A_{11} = A_{12} = A_{13} = A_{15} = A_{16} = A_{18} = A_{19} = A_{21} = A_{22} = \dots$$

$$= A_{27} = \{(J, \alpha) \in \mathbb{R}^2 : J = 0, \alpha = 0\}; A_{20} = \{(J, \alpha) \in \mathbb{R}^2 : -\alpha \leq 6J \leq 0\};$$

$$A_{28} = \{(J, \alpha) \in \mathbb{R}^2 : 0 \leq \alpha \leq -6J\}; A_{29} = \{(J, \alpha) \in \mathbb{R}^2 : 18J \leq \alpha \leq 0\}$$

and $\bigcup_{i=1}^{29} A_i = \mathbb{R}^2$.

We study all translation-invariant ground states for the Chui-Weeks model on the Cayley tree of order two. The following theorem describes all translation-invariant ground states for the four-state Chui-Weeks model.

Theorem. *Let $k = 2$. Then, for the four-state Chui-Weeks model, the following assertions hold:*

- (i) *The configurations $\sigma(x) = i, i \in \{1, 2, 3\}$, for all $x \in V$ are translation-invariant ground states iff $(J, \alpha) \in A_1$;*
- (ii) *The configuration $\sigma(x) = 0$ for all $x \in V$ is the translation-invariant ground state iff $(J, \alpha) \in A_2$;*
- (iii) *If $(J, \alpha) \in \mathbb{R}^2 \setminus (A_1 \cup A_2)$, then there is no translation-invariant ground state.*

Proof. Let $k = 2$.

(i) We consider the configuration $\sigma(x) = i, i \in \{1, 2, 3\}$, for all $x \in V$. For any $b \in M$ by Lemma we have $U(\sigma_b) = U_1 = 0$. Thus the configuration $\sigma(x) = i, i \in \{1, 2, 3\}$, for all $x \in V$ is ground state iff $(J, \alpha) \in A_1$;

(ii) We consider the configuration $\sigma(x) = 0$ for all $x \in V$. For any $b \in M$ by Lemma we have $U(\sigma_b) = U_2 = \alpha$. Thus the configuration $\sigma(x) = 0$ for all $x \in V$ is ground state if $(J, \alpha) \in A_2$;

(iii) It is obvious. Theorem is proved.

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FIXED POINTS OF THE 1-VOLTERRA QUADRATIC STOCHASTIC OPERATOR ON S^2

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Let us denote by $\mathbb{R}$ and $\mathbb{R}_+$ the set of all real and non-negative real numbers, respectively. Then for a given $m \geq 2$ we denote by S^{m-1} the set of all probability vectors (or distributions), i.e.

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}_+^m : \sum_{k=1}^m x_k = 1 \right\}.$$

One can see that S^{m-1} is a norm closed subset of $\mathbb{R}^m$. Moreover, S^{m-1} is a simplex, i.e. every $\mathbf{x} \in S^{m-1}$ is written as follows $\mathbf{x} = \sum_{k=1}^m x_k \mathbf{e}_k$, where $\{\mathbf{e}_k\}_{k=1}^m$ is the standard basis of $\mathbb{R}^m$.

The quadratic stochastic operator (QSO) is a mapping of the simplex into itself, i.e.

$$V : x'_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j, \quad k = 1, \dots, m. \quad (1)$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k} = P_{ij,k} \geq 0, \quad \forall i, j, k \in \{1, \dots, m\}, \quad (2)$$

$$\sum_{k=1}^m P_{ij,k} = 1, \quad \forall i, j \in \{1, \dots, m\}. \quad (3)$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbf{P} = (P_{ij,k})_{i,j,k=1}^m$ with conditions (2), (3).

ℓ -Volterra QSO. Fix $\ell \in \{1, \dots, m\}$ and assume that elements $P_{ij,k}$ of the matrix $\mathbf{P}$ satisfy

$$P_{ij,k} = 0, \quad \forall k \in \{1, \dots, \ell\}, \quad i, j \in \{1, \dots, m\}, \quad (4)$$

$$P_{ij,k} > 0 \text{ for at least one pair } (i, j), i \neq k, j \neq k, \text{ for any } \forall k > \ell. \quad (5)$$

Definition. [4] For any fixed $\ell \in \{1, \dots, m\}$, the QSO defined by (1) – (5) is called ℓ -Volterra QSO.

The following type of 1-Volterra operator on S^2 was studied in [2,3]

$$V_{a,b,c} : \begin{cases} x' = x(x + 2ay + 2bz), \\ y' = y^2 + 2(1-b)xz + 2cyz, \\ z' = z^2 + 2(1-a)xy + 2(1-c)yz, \end{cases} \quad (6)$$

where $a, b, c \in (0; 1)$.

By $Fix(V_{a,b,c})$ we denote a set of all fixed points of the operator $V_{a,b,c}$, i.e. $\mathbf{x} \in Fix(V_{a,b,c})$ if and only if $V(\mathbf{x}) = \mathbf{x}$.

It should be noted that in [2,3] the authors investigated the operator V defined by (6) in the special case $a = b$.

Theorem. Let $V_{a,b,c}$ be a QSO given by (6) and $a \neq b$. If $\text{sgn}(1 - 2a) \neq \text{sgn}(2b - 1)$ then

$$\text{Fix}(V_{a,b,c}) = \begin{cases} \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, & \text{if } c \neq \frac{1}{2}, \\ \{\mathbf{e}_1\} \cup \{(0, y, 1 - y) : y \in [0; 1]\}, & \text{if } c = \frac{1}{2}. \end{cases}$$

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THE DYNAMIC OF A HOMEOMORPHISM OF 2D SIMPLEX

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Let $E = \{1, \dots, m\}$ be a finite set and set of all probability distributions on E

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}_+^m : \sum_{i=1}^m x_i = 1\},$$

the $(m-1)$ - dimensional simplex, where $\mathbb{R}_+^m = \{\mathbf{x} \in \mathbb{R}^m : x_i \geq 0, i = 1, \dots, m\}$.

A mapping $V : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ is called *stochastic operator* (SO) if $V(S^{m-1}) \subset S^{m-1}$. The simplest non-linear SOs are quadratic stochastic operators.

A map $V : S^{m-1} \rightarrow S^{m-1}$ is called a quadratic stochastic operator (QSO) if

$$(V\mathbf{x})_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \dots, m$, where

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \text{for all } i, j; \quad \sum_{k=1}^m p_{ij,k} = 1.$$

A QSO is said to be a *Volterra QSO* if the coefficients of heredity satisfy the following additional assumption $p_{ij,k} = 0$, for $k \notin \{i, j\}$, $i, j, k = 1, \dots, m$.

The orbit (trajectory) $\{\mathbf{x}^{(n)}\}_{n=0}^\infty$ of an operator V for an initial point $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{n+1}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots$$

Denote by $\omega_V(\mathbf{x}^{(0)})$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^\infty$. One of the main problems in mathematical biology consists to study of asymptotical behaviour of the trajectories for a given QSO. In other words, the main task is the description of the set $\omega_V(\mathbf{x}^{(0)})$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO.

A point $\mathbf{x} \in S^{m-1}$ is called a fixed point of V if $V(\mathbf{x}) = \mathbf{x}$. Denote the set of all fixed points by $\text{Fix}(V)$. We let a face of the simplex S^{m-1} be the set $\Gamma_\alpha = \{\mathbf{x} \in S^{m-1} : x_i = 0, i \notin \alpha \subset E\}$ and let $\mathbf{e}_i = (\delta_{1i}, \delta_{2i}, \dots, \delta_{mi}) \in S^{m-1}$, $i = 1, \dots, m$, the vertices of the simplex S^{m-1} , where δ_{ij} is the Kronecker delta.

Let $DV(\mathbf{x}^*) = (\partial V_i / \partial x_j)(\mathbf{x}^*)$ be the Jacobian of V at the point $\mathbf{x}^*$. A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DV(\mathbf{x}^*)$ has no eigenvalues on the unit circle. A hyperbolic fixed point $\mathbf{x}^*$ is called: (i) *attracting* if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are in the unite ball; (ii) *repelling* if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are outside the unite ball; (iii) a *saddle* otherwise [1].

Let π be a permutation of the set E . Define a map $T_\pi : S^{m-1} \rightarrow S^{m-1}$ by the formula $T_\pi(\mathbf{x}) = (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(m)})$.

In [2] it has been proven that any quadratic homeomorphism of the simplex is a superposition of the permutation T_π and the Volterra operator V_0 , that is $T_\pi \circ V_0$.

We shall study the dynamics of the following homeomorphism of the simplex S^2

$$W : \begin{cases} x'_1 = x_1 (1 + ax_2 + bx_3) (1 + bx_2 + ax_3 - abx_1x_2 - abx_1x_3 + (bc - ac)x_2x_3), \\ x'_2 = x_3 (1 - bx_1 - cx_2) (1 - ax_1 + cx_2 - a(a + c)x_1x_2 - abx_1x_3 + c^2x_2x_3), \\ x'_3 = x_2 (1 - ax_1 + cx_3) (1 - bx_1 - cx_3 - abx_1x_2 - b(b - c)x_1x_3 + c^2x_2x_3), \end{cases}$$

where $a, b, c \in (0, 1]$.

It is easy to see that W is a superposition of the Volterra QSO and its permutation. Denote $\mathbf{x}^* = (0, x^*, 1 - x^*)$, where $x^* \in [0, 1]$ is a real solution of the quartic equation

$$(1 - x)(1 - cx) (1 + c(1 + c)x - c^2x^2) - x = 0.$$

Theorem. For the SO W the following assertions are true:

- i) the face $\Gamma_{\{2,3\}}$ is an invariant set;
- ii) $\text{Fix}(W) = \{\mathbf{e}_3, \mathbf{x}^*\}$, here $\mathbf{e}_1$ is attracting and $\mathbf{x}^*$ is repelling points;
- iii) for any $\mathbf{x}^{(0)} \in S^2$ we have

$$\omega_W(\mathbf{x}^{(0)}) = \begin{cases} \{\mathbf{e}_1\}, & \text{if } x_1^{(0)} > 0, \\ \{\mathbf{x}^*\}, & \text{if } x_2^{(0)} = x^*, x_1^{(0)} = 0, \\ \{\mathbf{e}_2, \mathbf{e}_3\}, & \text{if } x_2^{(0)} \neq x^*, x_1^{(0)} = 0. \end{cases}$$

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ON DYNAMICS OF A SEPARABLE CUBIC STOCHASTIC OPERATOR

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Let $E = \{1, \dots, m\}$ be a finite set and the set of all probability distributions on E

$$S^{m-1} = \{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \text{ for any } i \text{ and } \sum_{i=1}^m x_i = 1\},$$

the $(m-1)$ -dimensional simplex.

A cubic stochastic operator (CSO) is a mapping $W : S^{m-1} \rightarrow S^{m-1}$ of the form

$$W : x'_\ell = \sum_{i,j,k \in E} P_{ijk,\ell} x_i x_j x_k, \quad \ell \in E, \quad (1)$$

where $P_{ijk,\ell}$ are the coefficients of heredity such that

$$P_{ijk,\ell} \geq 0, \quad \forall i, j, k, \ell \in E, \quad \sum_{\ell \in E} P_{ijk,\ell} = 1, \quad \forall i, j, k \in E. \quad (2)$$

Let

$$A = (a_{i\ell})_{i,\ell=1}^m, \quad B = (b_{j\ell})_{j,\ell=1}^m, \quad C = (c_{k\ell})_{k,\ell=1}^m$$

be the matrices with the real entries.

Consider a CSO (1), (2) with additional condition

$$P_{ijk,\ell} = a_{i\ell} b_{j\ell} c_{k\ell}, \quad \text{for all } i, j, k, \ell \in E, \quad (3)$$

where $a_{i\ell}, b_{j\ell}, c_{k\ell} \in \mathbb{R}$ entries of matrices A, B and C such that the conditions (1) are satisfied for the coefficients (3). Then the CSO W corresponding to the coefficients (3) has the form

$$x'_\ell = (W(\mathbf{x}))_\ell = (A(\mathbf{x}))_\ell \cdot (B(\mathbf{x}))_\ell \cdot (C(\mathbf{x}))_\ell, \quad \ell \in E, \quad (4)$$

where $(A(\mathbf{x}))_\ell = \sum_{i=1}^m a_{i\ell} x_i$, $(B(\mathbf{x}))_\ell = \sum_{j=1}^m b_{j\ell} x_j$ and $(C(\mathbf{x}))_\ell = \sum_{k=1}^m c_{k\ell} x_k$.

Definition The CSO (4) is called separable cubic stochastic operator (SCSO).

Let us consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1+a & 1 \\ 1-a & 1 & 1 \\ 1+b & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1-b \\ 1 & 1 & 1+c \\ 1 & 1-c & 1 \end{pmatrix},$$

where $ab + bc = ac$ and $a, b, c \in [-1, 1]$.

Then corresponding SCSO $W : S^2 \rightarrow S^2$ is:

$$W : \begin{cases} x'_1 = x_1 (x_1 + (1+a)x_2 + x_3) (x_1 + x_2 + (1-b)x_3), \\ x'_2 = x_2 ((1-a)x_1 + x_2 + x_3) (x_1 + x_2 + (1+c)x_3), \\ x'_3 = x_3 ((1+b)x_1 + x_2 + x_3) (x_1 + (1-c)x_2 + x_3), \end{cases} \quad (5)$$

where $ab + bc = ac$ and $a, b, c \in [-1, 1]$.

Using the equation $x_1 + x_2 + x_3 = 1$ we rewrite the operator (5) as follows

$$W : \begin{cases} x'_1 = x_1(1 + ax_2)(1 - bx_3), \\ x'_2 = x_2(1 - ax_1)(1 + cx_3), \\ x'_3 = x_3(1 + bx_1)(1 - cx_2), \end{cases} \quad (6)$$

where $ab + bc = ac$ and $a, b, c \in [-1, 1]$.

Let a face of the simplex S^2 be the set $\Gamma_\alpha = \{\mathbf{x} \in S^2 : x_i = 0, i \notin \alpha \subset \{1, 2, 3\}\}$. Let the set $\text{int } S^2 = \{\mathbf{x} \in S^2 : x_1x_2x_3 > 0\}$ and let the set $\partial S^2 = S^2 \setminus \text{int } S^2$ be the interior and the boundary of the simplex S^2 , respectively. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the vertices of the two-dimensional simplex.

Theorem 1. *For the SCSO W (6), the following assertions true:*

- (i) *The faces $\Gamma_{\{1,2\}}$, $\Gamma_{\{1,3\}}$, $\Gamma_{\{2,3\}}$ of the simplex S^2 are invariant sets;*
- (ii) $\text{Fix}(W) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$;
- (iii) *for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}}$ we have*

$$\omega_W(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_1, & \text{if } 0 < a \leq 1, \\ \mathbf{e}_2, & \text{if } -1 \leq a < 0; \end{cases}$$

- (iv) *for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,3\}}$ we have*

$$\omega_W(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_3, & \text{if } 0 < b \leq 1, \\ \mathbf{e}_1, & \text{if } -1 \leq b < 0; \end{cases}$$

- (v) *for any $\mathbf{x}^{(0)} \in \Gamma_{\{2,3\}}$ we have*

$$\omega_W(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_2, & \text{if } 0 < c \leq 1, \\ \mathbf{e}_3, & \text{if } -1 \leq c < 0. \end{cases}$$

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Regularity of a quadratic operator

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Consider a map $V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3) \rightarrow \mathbf{x}' = V(\mathbf{x}) = (x'_1, x'_2, x'_3)$ defined by

$$V : \begin{cases} x'_1 = x_1 - \beta x_1 x_2 - \alpha x_1, \\ x'_2 = x_2 + \beta x_1 x_2 - \gamma x_2, \\ x'_3 = x_3 + \gamma x_2 + \alpha x_1. \end{cases} \quad (1)$$

Such operators can be considered as discrete analogues of the models which arise in the problems of epidemiology (see e.g. [1,2,3]).

The orbit (trajectory) can be written as $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^n(\mathbf{x}^{(0)})$, $n \geq 0$. We are interested in studying the limit $\lim_{n \rightarrow \infty} \mathbf{x}^{(n)}$, for any initial condition $\mathbf{x}^{(0)} \in \mathbb{R}^3$. Let $\omega_V(\mathbf{x}^{(0)})$ be the set of limit points of the orbit $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$.

Let $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}$ be the two-dimensional simplex. Denote by $\mathbf{e}_i = (\delta_{1i}, \delta_{2i}, \delta_{3i}) \in S^2$, $i = 1, 2, 3$, the vertices of S^2 , where δ_{ij} is the Kronecker delta symbol. We set $\Gamma_\tau := \{\mathbf{x} \in S^2 : x_i = 0, i \notin \tau \subset \{1, 2, 3\}\}$ be a face of S^2 .

A point $\mathbf{a} \in S^2$ is called a *fixed* point of V if $V(\mathbf{a}) = \mathbf{a}$ and we denote by $\text{Fix}(V)$ the set of all fixed points.

Proposition-1. (1) maps S^2 to itself if and only if

$$0 \leq \alpha \leq 1, \quad 0 \leq \gamma \leq 1, \quad \gamma - 1 \leq \beta \leq 1 - \alpha. \quad (2)$$

Theorem-1. For the operator (1) the set of fixed points has the form:

$$\text{Fix}(V) = \begin{cases} S^2, & \text{if } \alpha = \beta = \gamma = 0, \\ \Gamma_{\{1,3\}}, & \text{if } \alpha = \beta = 0, \gamma \neq 0, \\ \Gamma_{\{1,3\}} \cup \Gamma_{\{2,3\}}, & \text{if } \alpha = \gamma = 0, \beta \neq 0, \\ \Gamma_{\{2,3\}}, & \text{if } \beta = \gamma = 0, \alpha \neq 0, \\ \Gamma_{\{1,3\}}, & \text{if } \alpha = 0, \beta\gamma \neq 0, \\ \{\mathbf{e}_3\}, & \text{if } \beta = 0, \alpha\gamma \neq 0, \\ \Gamma_{\{2,3\}}, & \text{if } \gamma = 0, \alpha\beta \neq 0, \\ \{\mathbf{e}_3\}, & \text{if } \alpha\beta\gamma \neq 0; \end{cases}$$

Here, the point $\mathbf{e}_3$ is an attracting fixed point, and all other fixed points are of non-hyperbolic type.

Theorem-2. For the quadratic operator (1) with parameters given in (2) for an arbitrary initial point $\mathbf{x}^{(0)} \in S^2 \setminus \text{Fix}(V)$, the set of limit points has the form:

$$\omega_V(\mathbf{x}^{(0)}) = \begin{cases} \left\{ \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \right) \right\}, & \text{if } \alpha = \beta = \gamma = 0, \\ \left\{ \left(x_1^{(0)}, 0, 1 - x_1^{(0)} \right) \right\}, & \text{if } \alpha = \beta = 0, \gamma \neq 0, \\ \left\{ \left(0, 1 - x_3^{(0)}, x_3^{(0)} \right) \right\}, & \text{if } \alpha = \gamma = 0, \beta > 0, \\ \left\{ \left(1 - x_3^{(0)}, 0, x_3^{(0)} \right) \right\}, & \text{if } \alpha = \gamma = 0, \beta < 0, \\ \left\{ \left(0, x_2^{(0)}, 1 - x_2^{(0)} \right) \right\}, & \text{if } \alpha \neq 0, \beta = \gamma = 0, \\ \{ (1 - x_3^*, 0, x_3^*) \}, & \text{if } \alpha = 0, \beta\gamma \neq 0, \\ \{ (0, 0, 1) \}, & \text{if } \beta = 0, \alpha\gamma \neq 0, \\ \{ (0, 1 - x_3^*, x_3^*) \}, & \text{if } \gamma = 0, \alpha\beta \neq 0, \\ \{ (0, 0, 1) \}, & \text{if } \alpha\beta\gamma \neq 0. \end{cases}$$

Corollary. For any parameters (2) the quadratic operator (1) is a regular transformation.

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Convergence of orbits for a quadratic operator

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Consider the space $\mathbb{R}^m$ with the norm $\|\mathbf{x}\| = \sum_{i=1}^m |x_i| = 1$. Let $S^{m-1} = \{\mathbf{x} \in \mathbb{R}_+^m : \|\mathbf{x}\| = 1\}$ be the $(m-1)$ -dimensional simplex, where $\mathbb{R}_+^m = \{\mathbf{x} \in \mathbb{R}^m : x_i \geq 0, i = 1, \dots, m\}$.

A map V of S^{m-1} into itself is called a quadratic stochastic operator (QSO) if

$$(V\mathbf{x})_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j,$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \dots, m$, where

$$p_{ij,k} = p_{ji,k} \geq 0 \text{ for all } i, j, k = 1, \dots, m \text{ and } \sum_{k=1}^m p_{ij,k} = 1, i, j = 1, \dots, m.$$

Assume that $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots} \subset S^{m-1}$ is the orbit (trajectory) of the initial point $\mathbf{x}^{(0)} \in S^{m-1}$, where $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$ for all $n = 0, 1, 2, \dots$. Let $\omega_V(\mathbf{x}^{(0)})$ be the set of limit points of the orbit $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$. Since S^{m-1} is a compact set, and $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots} \subset S^{m-1}$ it follows that $\omega_V(\mathbf{x}^{(0)}) \neq \emptyset$.

A point $\mathbf{x} \in S^{m-1}$ is called a fixed point of V if $V(\mathbf{x}) = \mathbf{x}$. Denote the set of all fixed points by $\text{Fix}(V)$. We let a face of S^{m-1} be the set $\Gamma_\alpha = \{\mathbf{x} \in S^{m-1} : x_k = 0, k \notin \alpha\}$, where $\alpha \subset \{1, \dots, m\}$; let $\mathbf{e}_i = (\delta_{1,i}, \delta_{2,i}, \dots, \delta_{m,i}) \in S^{m-1}$, $i \in \{1, \dots, m\}$, denote the vertices of the simplex S^{m-1} , where $\delta_{i,j}$ is the Kronecker delta.

The QSO V is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$. The asymptotical behavior of orbits for Volterra QSOs was deeply studied [2].

Let $DV(\mathbf{x}^*) = (\partial V_i / \partial x_j)(\mathbf{x}^*)$ be the Jacobian of V at the point $\mathbf{x}^*$. A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DV(\mathbf{x}^*)$ has no eigenvalues on the unit circle. A hyperbolic fixed point $\mathbf{x}^*$ is called: (i) *attracting* if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are in the unite ball; (ii) *repelling* if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are outside the unite ball; (iii) a *saddle* otherwise [1].

In [3], quadratic operators with nonstochastic matrices were studied, and necessary and sufficient conditions on the operator coefficients were established for the quadratic operator to preserve the simplex.

Consider a non-Volterra quadratic operator defined on the two-dimensional simplex which has the form

$$V_\alpha : \begin{cases} x'_1 = x_1^2 + (x_2 + x_3)^2 - \alpha x_1 x_2 - \alpha x_1 x_3, \\ x'_2 = (2 + \alpha) x_1 x_2, \\ x'_3 = (2 + \alpha) x_1 x_3, \end{cases} \quad (1)$$

where $\alpha \in [-2, 1]$.

For $\alpha \in (-1, 1]$ denote $S_\alpha = \{\mathbf{x} \in S^2 : x_1 = 1/(2 + \alpha)\}$.

In the following theorem we give description fixed points and their types.

Proposition 1. For the operator V_α the following statements are hold:

- i) the faces $\Gamma_{\{1,2\}}$ and $\Gamma_{\{1,3\}}$ are invariant sets with respect to operator V_α ;
- ii)

$$Fix(V_\alpha) = \begin{cases} \{\mathbf{e}_1\}, & \text{if } \alpha \in [-2, -1], \\ \{\mathbf{e}_1\} \cup S_\alpha, & \text{if } \alpha \in (-1, 1]; \end{cases}$$

- iii) the point $\mathbf{x} \in S_\alpha$ is a non-hyperbolic and the vertex

$$\mathbf{e}_1 \text{ has the type a (an)} \begin{cases} \text{attracting,} & \text{if } \alpha \in [-2, -1], \\ \text{non-hyperbolic,} & \text{if } \alpha = -1, \\ \text{repelling,} & \text{if } \alpha \in (-1, 1]. \end{cases}$$

The following theorem describes the orbit of any point $\mathbf{x}^{(0)} \in S^2$.

Theorem 1. For the operator V_α the following statements are hold:

- i) if $\mathbf{x}^{(0)} \in \Gamma_{\{2,3\}}$ then for any $\alpha \in [-2, 1]$ we have $V_\alpha(\mathbf{x}^{(0)}) = \mathbf{e}_1$;
- ii) if $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_1, \mathbf{e}_2\}$ then we have

$$\lim_{n \rightarrow \infty} V_\alpha^n(\mathbf{x}^{(0)}) = \begin{cases} (1, 0, 0), & \text{if } \alpha \in [-2, -1], \\ \left(\frac{1}{2+\alpha}, \frac{1+\alpha}{2+\alpha}, 0\right), & \text{if } \alpha \in (-1, 1]; \end{cases}$$

- iii) if $\mathbf{x}^{(0)} \in \Gamma_{\{1,3\}} \setminus \{\mathbf{e}_1, \mathbf{e}_3\}$ then we have

$$\lim_{n \rightarrow \infty} V_\alpha^n(\mathbf{x}^{(0)}) = \begin{cases} (1, 0, 0), & \text{if } \alpha \in [-2, -1], \\ \left(\frac{1}{2+\alpha}, 0, \frac{1+\alpha}{2+\alpha}\right), & \text{if } \alpha \in (-1, 1]; \end{cases}$$

- iv) if $\mathbf{x}^{(0)} \in \text{int } S^2 \setminus Fix(V_\alpha)$, then we have

$$\lim_{n \rightarrow \infty} V_\alpha^n(\mathbf{x}^{(0)}) = \begin{cases} (1, 0, 0), & \text{if } \alpha \in [-2, -1], \\ \left(\frac{1}{2+\alpha}, \frac{(1+\alpha)x_2^{(0)}}{(2+\alpha)(x_2^{(0)} + x_3^{(0)})}, \frac{(1+\alpha)x_3^{(0)}}{(2+\alpha)(x_2^{(0)} + x_3^{(0)})}\right), & \text{if } \alpha \in (-1, 1]. \end{cases}$$

Remark 1. The operator given by (1) for the case $\alpha \in [0, 2]$ was investigated in [3].

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On dynamics of a SEIR epidemic model

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In recent years, the study of infectious disease spread within populations has become increasingly relevant. Mathematical modeling of infectious diseases aids in studying the spread of diseases. In the classical SEIR model, the population is divided into four parts: (i) S – part of the population that is susceptible; (ii) E – the part of the population that is exposed; (iii) I – the part of the population that is infectious; (iv) R – the part of the population that has recovered from the disease.

Consider the following epidemic model

$$\begin{cases} S_{n+1} = S_n - \beta S_n I_n + \mu R_n, \\ E_{n+1} = (1 - \gamma_1) E_n + \beta S_n I_n, \\ I_{n+1} = (1 - \gamma_2) I_n + \gamma_1 E_n, \\ R_{n+1} = (1 - \mu) R_n + \gamma_2 I_n, \end{cases} \quad (1)$$

where n is a non-negative integer number. where S, E, I and R are defined as above and $S + E + I + R = 1$. The parameters β – is the infection rate, γ_1 – denotes the rate at which an exposed person is converted to an infector, γ_2 – denotes the rate at which infectors are recovered, μ – be the rate at which removals lose the immunization and become susceptible.

The nonlinear system (1) is analyzed using the quadratic stochastic operators approach.

Let $S^{m-1} = \{\mathbf{x} \in \mathbb{R}_+^m : \sum_{i=1}^m x_i = 1\}$ be the $(m-1)$ -dimensional simplex. A map V of S^{m-1} into itself is called a *quadratic stochastic operator* (QSO) if

$$V(\mathbf{x})_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad k = 1, 2, \dots, m, \quad \forall \mathbf{x} \in S^{m-1},$$

where $p_{ij,k} = p_{ji,k} \geq 0$ for all $i, j, k = 1, \dots, m$ and $\sum_{k=1}^m p_{ij,k} = 1$, for all $i, j = 1, \dots, m$.

Assume $\{\mathbf{x}^{(n)} \in S^{m-1} : n = 0, 1, 2, \dots\}$ is the orbit (trajectory) of the initial point $\mathbf{x}^{(0)} \in S^{m-1}$, where $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$ for all $n = 0, 1, 2, \dots$. We denote by $\omega_V(\mathbf{x}^{(0)})$ the set of ω -limiting points of the orbit $\{\mathbf{x}^{(n)}\}_{n \geq 0}$.

We let a face of the simplex S^{m-1} be the set $\Gamma_\alpha = \{\mathbf{x} \in S^{m-1} : x_k = 0, k \notin \alpha\}$, where $\alpha \subset \{1, \dots, m\}$; let $\mathbf{e}_i = (\delta_{1,i}, \delta_{2,i}, \dots, \delta_{m,i}) \in S^{m-1}$, $i \in \{1, \dots, m\}$, denote the vertices of S^{m-1} , where $\delta_{i,j}$ is the Kronecker delta.

The QSO V is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$. The asymptotical behavior of orbits for Volterra QSOs was deeply studied [1].

Consider following operator $V : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $\mathbf{x} \mapsto \mathbf{x}' = V(\mathbf{x})$, defined by

$$V : \begin{cases} x'_1 = x_1 + \mu x_4 - \beta x_1 x_3, \\ x'_2 = (1 - \gamma_1)x_2 + \beta x_1 x_3, \\ x'_3 = (1 - \gamma_2)x_3 + \gamma_1 x_2, \\ x'_4 = (1 - \mu)x_4 + \gamma_2 x_3, \end{cases} \quad (2)$$

where $x'_1 = S_1$, $x'_2 = E_1$, $x'_3 = I_1$, $x'_4 = R_1$, $x_1 = S_0$, $x_2 = E_0$, $x_3 = I_0$, $x_4 = R_0$.

In [2], the dynamical properties of a discrete-time SIRD reinfection model were investigated. Using the approach in [2], we analyze the discrete-time nonlinear operator defined in (2).

Proposition 1. The operator (2) maps S^3 to itself if and only if

$$\beta, \gamma_1, \gamma_2, \mu \in [0, 1]. \quad (3)$$

In the next theorem, we describe the set of limit points of the orbits of any initial points under the operator V . We proved that the operator (2) has the regularity property.

Theorem 1. For any $\mathbf{x}^{(0)} \in S^3 \setminus \text{Fix}(V)$ and the QSO (2) with parameters (3), the set of limit points has the form

$$\omega_V(\mathbf{x}^{(0)}) = \begin{cases} \{(1 - x_2^{(0)} - x_3^{(0)}, x_2^{(0)}, x_3^{(0)}, 0)\}, & \text{if } \beta = \gamma_1 = \gamma_2 = 0, \mu > 0, \\ \{(x_1^{(0)}, 0, 1 - x_1^{(0)} - x_4^{(0)}, 0, x_4^{(0)})\}, & \text{if } \beta = \gamma_2 = \mu = 0, \gamma_1 > 0, \\ \{(x_1^{(0)}, x_2^{(0)}, 0, 1 - x_1^{(0)} - x_2^{(0)})\}, & \text{if } \beta = \gamma_1 = \mu = 0, \gamma_2 > 0, \\ \{(0, 1 - x_3^{(0)} - x_4^{(0)}, x_3^{(0)}, x_4^{(0)})\}, & \text{if } \gamma_1 = \gamma_2 = \mu = 0, \beta > 0, \\ \{(x_1^*, 0, 1 - x_1^*, 0)\}, & \text{if } \beta = \gamma_2 = 0, \gamma_1 \mu > 0, \\ \{(1 - x_2^{(0)}, x_2^{(0)}, 0, 0)\}, & \text{if } \beta = \gamma_1 = 0, \gamma_2 \mu > 0, \\ \{(1 - x_2^* - x_3^{(0)}, x_2^*, x_3^{(0)}, 0)\}, & \text{if } \gamma_1 = \gamma_2 = 0, \beta \mu > 0, \\ \{(x_1^{(0)}, 0, 0, 1 - x_1^{(0)})\}, & \text{if } \beta = \mu = 0, \gamma_1 \gamma_2 > 0, \\ \{(0, 0, 1 - x_4^{(0)}, x_4^{(0)})\}, & \text{if } \gamma_2 = \mu = 0, \beta \gamma_1 > 0, \\ \{(0, x_2^*, 0, 1 - x_2^*)\}, & \text{if } \gamma_1 = \mu = 0, \beta \gamma_2 > 0, \\ \{\mathbf{e}_1\}, & \text{if } \beta = 0, \gamma_1 \gamma_2 \mu > 0, \\ \{(x_1^*, 0, 0, 1 - x_1^*)\}, & \text{if } \mu = 0, \beta \gamma_1 \gamma_2 > 0, \\ \{\mathbf{e}_3\}, & \text{if } \gamma_2 = 0, \beta \gamma_1 \mu > 0, \\ \{(1 - x_2^*, x_2^*, 0, 0)\}, & \text{if } \gamma_1 = 0, \beta \gamma_2 \mu > 0, \\ \{\mathbf{e}_1\}, & \text{if } \beta \gamma_1 \gamma_2 \mu > 0, \beta \leq \gamma_2. \end{cases}$$

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Dynamics Under Volterra Operator on the Four-Dimensional Simplex of Idempotent Measures in the Special Case

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In [1] all linear operators which maps n -dimensional simplex of idempotent measures to itself are described. These linear operators on the set of idempotent measures are "idempotent" analogies of Markov chains, where a state of the Markov chain is an idempotent measure, but linearity of the evolution operator is defined by the usual operations $+$ and $\cdot$. Moreover, in [1], the dynamical systems generated by the linear maps of the set of idempotent measures are studied. In this paper we analyze quadratic operators mapping $\mathcal{I}_4$ to itself, focusing on Volterra-type operators. We give limits of trajectories for all initial points in the case $a_{ij} = -1$.

Consider a cubic matrix $P = (p_{ij,k})_{i,j,k=1}^n$ with $p_{ij,k} \in \mathbf{R}_{\max}$.

Definition 1. The quadratic map

$$Q : x = (x_1, \dots, x_n) \in \mathbf{R}_{\max}^n \rightarrow x' = Q(x) = (x'_1, \dots, x'_n) \in \mathbf{R}_{\max}^n$$

is defined as

$$x'_k = \sum_{i,j=1}^n p_{ij,k} x_i x_j, \quad k = 1, 2, \dots, n. \quad (1)$$

where $p_{ij,k} \in \mathbf{R}_{\max}$.

Definition 2. Quadratic operator (1), corresponding to matrix $P = (p_{ij,k})_{i,j,k=1}^n$, with $p_{ij,k} \leq 0$ for all i, j, k , is called quadratic Volterra operator if it satisfies condition:

$$p_{ij,k} = 0, \quad k \notin \{i, j\}. \quad (2)$$

By this definition, it is easy to see that a Volterra operator (denoted by V) has the following form:

$$V : x'_k = x_k \sum_{i=1}^n (p_{ik,k} + p_{ki,k}) x_i, \quad k = 1, 2, \dots, n. \quad (3)$$

Definition 3. [see 2] A fixed point $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ of an operator V is called hyperbolic if its Jacobian $J_V(x_0)$ has no eigenvalues on the unit circle.

Definition 4. [see 2] A hyperbolic fixed point $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ called:

- 1) **attracting** if all eigenvalues of the Jacobi matrix $J_V(x_0)$ are less than 1 in absolute value;
- 2) **repelling** if all eigenvalues of the Jacobi matrix $J_V(x_0)$ are greater than 1 in absolute value;
- 3) **saddle** otherwise.

Definition 5. [see 3] Quadratic Volterra operator (4) with condition (2) is called a *quadratic absolute stochastic* Volterra operator (QASVO).

From condition (2) it follows that for QASVO we have $p_{ii,i} = -1$ for any $i = 1, \dots, n$. Let's denote by $a_{ij} = p_{ij,i} + p_{ji,i}$, if $i \neq j$, else if $i = j$ then $a_{ii} = p_{ii,i}$.

We study trajectories of non-fixed points for QASVO in the case $n = 4$. We have to study following cases for $i, j, k \in \{1, 2, 3, 4\}$: $a_{ij} = a_{ik} = a_{jk} = -1$.

In this case we have fixed point as $(0, 0, 0, 0)$ and all points of the peace of plane $x_i + x_j + x_k = -1$, where $x_i \leq 0$, $x_j \leq 0$, $x_k \leq 0$ for $i, j, k \in \{1, 2, 3, 4\}$. Now, let us find trajectories of non-fixed point under QASVO $V : \mathcal{I}_4 \rightarrow \mathcal{I}_4$. Under condition (4) the system $x^{(m+1)} = V(x^{(m)})$ gets following system:

$$\begin{cases} x_1^{(m+1)} = -x_1^{(m)} \left(x_1^{(m)} + x_2^{(m)} + x_3^{(m)} + x_4^{(m)} \right), \\ x_2^{(m+1)} = -x_2^{(m)} \left(x_1^{(m)} + x_2^{(m)} + x_3^{(m)} + x_4^{(m)} \right), \\ x_3^{(m+1)} = -x_3^{(m)} \left(x_1^{(m)} + x_2^{(m)} + x_3^{(m)} + x_4^{(m)} \right), \\ x_4^{(m+1)} = -x_4^{(m)} \left(x_1^{(m)} + x_2^{(m)} + x_3^{(m)} + x_4^{(m)} \right). \end{cases} \quad (4)$$

Proposition 1. Suppose $a_{ij} = -1$, $i, j \in \{1, 2, 3, 4\}$. For the QASVO $V : \mathcal{I}_4 \rightarrow \mathcal{I}_4$:

- a) the fixed point $(0, 0, 0, 0)$ is an attracting fixed point;
- b) all fixed points $(x_1, x_2, x_3, x_4) \in \mathcal{I}_4$, where, $x_i + x_j + x_k = -1$, $x_i \leq 0$, $x_j \leq 0$, $x_k \leq 0$, and $x_l = 0$ for $i, j, k, l \in \{1, 2, 3, 4\}$ are repelling fixed points.

Let denote by

$$D_{ijk} = \{x \in \mathcal{I}_4 : -1 < x_i + x_j + x_k \leq 0, x_l = 0\}, \quad i, j, k, l \in \{1, 2, 3, 4\}.$$

Theorem 1. Let $n = 4$ and $a_{ij} = -1$ for $i, j \in \{1, 2, 3\}$. For arbitrary $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, 0)$ the trajectory has the following limit

$$\lim_{m \rightarrow \infty} V^m(x^{(0)}) = \begin{cases} \left\{ \begin{array}{l} (0, 0, 0, 0), \quad \text{if } x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, 0) \in D_{123}; \\ x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, 0), \quad \text{if } x_1^{(0)} + x_2^{(0)} + x_3^{(0)} = -1; \end{array} \right. \\ \left\{ \begin{array}{l} \text{if } x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, 0) \in \mathcal{I}_4 \setminus \bar{D}_{123}, \text{ then} \\ (-\infty, 0, 0, 0), \quad \text{if } x_1^{(0)} \neq 0, \quad x_2^{(0)} = 0, \quad x_3^{(0)} = 0, \\ (0, -\infty, 0, 0), \quad \text{if } x_1^{(0)} = 0, \quad x_2^{(0)} \neq 0, \quad x_3^{(0)} = 0, \\ (0, 0, -\infty, 0), \quad \text{if } x_1^{(0)} = 0, \quad x_2^{(0)} = 0, \quad x_3^{(0)} \neq 0, \\ (-\infty, -\infty, 0, 0), \quad \text{if } x_1^{(0)} \neq 0, \quad x_2^{(0)} \neq 0, \quad x_3^{(0)} = 0, \\ (-\infty, 0, -\infty, 0), \quad \text{if } x_1^{(0)} \neq 0, \quad x_2^{(0)} = 0, \quad x_3^{(0)} \neq 0, \\ (0, -\infty, -\infty, 0), \quad \text{if } x_1^{(0)} = 0, \quad x_2^{(0)} \neq 0, \quad x_3^{(0)} \neq 0, \\ (-\infty, -\infty, -\infty, 0), \quad \text{otherwise.} \end{array} \right. \end{cases}$$

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Fixed points of Quadratic Operators on the 4D-Simplex of Idempotent Measures

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Idempotent mathematics employs associative operations $\oplus$ (idempotent addition, $x \oplus x = x$) and $\odot$, forming semifields/semirings. A key example is the tropical semiring $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$, where $\oplus$ is $\max$, $\odot$ is $+$, $\mathbf{0} = -\infty$, and $\mathbf{1} = 0$ (see [1],[2]). Tropical geometry studies polyhedral complexes (tropical varieties) with applications in algebraic geometry, optimization, and biology [3].

Idempotent measures, used in control theory and nonlinear PDEs (e.g., Hamilton-Jacobi equations), define the simplex

$$\mathcal{I}_n = \{(x_1, \dots, x_n) \in \mathbf{R}_{\max}^n : \max_i x_i = 0\}.$$

Note that $\max_i x_i = 0$ is equivalent to $x_1 \oplus \dots \oplus x_n = \mathbf{1}$ (see[1]).

Consider a cubic matrix $P = (p_{ij,k})_{i,j,k=1}^n$ with $p_{ij,k} \in \mathbf{R}_{\max}$.

Definition 1. The quadratic map

$$Q : x = (x_1, \dots, x_n) \in \mathbf{R}_{\max}^n \rightarrow x' = Q(x) = (x'_1, \dots, x'_n) \in \mathbf{R}_{\max}^n$$

is defined as

$$x'_k = \sum_{i,j=1}^n p_{ij,k} x_i x_j, \quad k = 1, 2, \dots, n, \quad (1)$$

where $p_{ij,k} \in \mathbf{R}_{\max}$.

Denote

$$M_n = \{x = (x_1, \dots, x_n) \in \mathbf{R}_{\max}^n : x_i \leq 0, \sum_{k=1}^n x_k = -1\}.$$

We note that any quadratic map (1) with $p_{ij,k} \leq 0$ and

$$\sum_{k=1}^n p_{ij,k} = -1 \quad (2)$$

maps M_n to itself.

Definition 2. Quadratic operator (1), corresponding to matrix $P = (p_{ij,k})_{i,j,k=1}^n$, with $p_{ij,k} \leq 0$ for all i, j, k , is called quadratic Volterra operator if it satisfies condition:

$$p_{ij,k} = 0, \quad k \notin \{i, j\}. \quad (3)$$

By this definition, it is easy to see that a Volterra operator (denoted by V) has the following form:

$$V : \quad x'_k = x_k \sum_{i=1}^n (p_{ik,k} + p_{ki,k}) x_i, \quad k = 1, 2, \dots, n. \quad (4)$$

Theorem 1. A quadratic operator V given by cubic matrix $P = (p_{ij,k})_{i,j,k=1}^n$ with $p_{ij,k} \leq 0$ maps $\mathcal{I}_n$ to itself, if and only if it satisfies one of following conditions:

- i) Each k -th matrix $P_k = (p_{ij,k})_{i,j=1}^n$ of the cubic matrix P contains exactly one non-zero row and exactly one non-zero column.
- ii) Cubic matrix P has at least one zero matrix, i.e., $\exists m \leq n$ such that all elements of m -th matrix P_m consists of only zeroes.

Condition (3) implies that Volterra operator satisfies the first condition of Theorem 1.

Theorem 2. A quadratic Volterra operator, maps $\mathcal{I}_n$ to itself.

Definition 3. Quadratic Volterra operator (4) with condition (2) is called a *quadratic absolute stochastic* Volterra operator (QASVO).

From condition (2) it follows that for QASVO we have $p_{ii,i} = -1$ for any $i = 1, \dots, n$.

Definition 4. A solution to $V(x) = x$ is called fixed point. The set of all fixed points denoted by $\text{Fix}(V)$.

Proposition 1. If $x = (x_1, \dots, x_n)$ is a fixed point of a quadratic map (1) with conditions $p_{ij,k} \leq 0$ and (2) then

$$\sum_{i=1}^n x_i \in \{-1, 0\}.$$

All fixed points of Volterra operator $V : \mathcal{I}_n \rightarrow \mathcal{I}_n$ are defined and described their characteristics in the case $p_{ik,k} + p_{ki,k} = -1$ for general n . And also, fixed points of Volterra operator $V : \mathcal{I}_n \rightarrow \mathcal{I}_n$ in the case $p_{ik,k} + p_{ki,k} \neq -1$ for cases $n = 2$ and $n = 3$ are found and studied their dynamics. Finding all fixed points in the case $p_{ik,k} + p_{ki,k} \neq -1$ is a difficult problem for general n . We consider the case $n = 4$.

Let's denote $a_{ij} = p_{ij,i} + p_{ji,i}$, if $i \neq j$, and if $i = j$ then $a_{ii} = p_{ii,i}$.

The set of all fixed points of QASVO $V : \mathcal{I}_4 \rightarrow \mathcal{I}_4$ given in following Proposition.

Proposition 2. Assume $n = 4$, the set of all fixed points for QASVO $V : \mathcal{I}_4 \rightarrow \mathcal{I}_4$ is:

$$\text{Fix}(V) = \{(0, 0, 0, 0), (-1, 0, 0, 0), (0, -1, 0, 0), (0, 0, -1, 0), (0, 0, 0, -1)\}$$

$$\bigcup \{(x_1, x_2, x_3, x_4) \in \mathcal{I}_4, \text{ where, } x_i + x_j + x_k = -1, \ x_i \leq 0, \ x_j \leq 0, \ x_k \leq 0, \ x_l = 0, \\ \text{if } \Delta_{ijk} = 0 \text{ and } a_{ij} = a_{ik} = a_{jk} = -1\}$$

$$\bigcup \{(x_1, x_2, x_3, x_4) \in \mathcal{I}_4, \text{ where, } x_i = \alpha_{ijk}, \ x_j = \beta_{ijk}, \ x_k = \gamma_{ijk}, \ x_l = 0, \\ \text{if } (\Delta_{ijk} > 0, \ a_{ij} \geq -1, \ a_{ik} \leq -1, \ a_{jk} \geq -1) \text{ or} \\ (\Delta_{ijk} < 0, \ a_{ij} \leq -1, \ a_{ik} \geq -1, \ a_{jk} \leq -1) \text{ for } i < j < k.\}$$

where $i, j, k, l \in \{1, 2, 3, 4\}$.

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IDEMPOTENT O'LCHOVLAR SIMPLEKSINI O'ZINI O'ZIGA AKSLANTIRUVCHI OPERATORLAR

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Maks-plus $R_{\max}$ va min-plus $R_{\min}$ algebralar tipik misollar bo'ladi. R – haqiqiy sonlar maydoni bo'lsin. Unda $R_{\max} = R \cup \{-\infty\}$ dagi amallar:

$$x \oplus y = \max \{x, y\} \text{ va } x \otimes y = x + y.$$

Shu kabi $R_{\min} = R \cup \{+\infty\}$ dagi amallar: $\oplus = \min$, $\otimes = +$.

Yangi qo'shish amali idempotent bo'ladi, ya'ni barcha x lar uchun $x \oplus x = x$.

Yuqorida berilgan idempotent qo'shish va ko'paytirish amalli $R_{\max} = R \cup \{-\infty\}$ ni qaraymiz. $(n-1)$ –o'lchovli idempotent o'lchovlar simpleksini quyidagi tarzda aniqlaymiz:

$$\begin{aligned} I_n &= \left\{ (x_1, \dots, x_n) \in R_{\max}^n : \max_{1 \leq i \leq n} x_i = 0 \right\} = \\ &= \left\{ (x_1, \dots, x_n) \in R_{\max}^n : x_1 \oplus \dots \oplus x_n = 1 \right\}. \end{aligned}$$

Faraz qilaylik $(P)_{ijk\dots t,l}^n$ operatorida $\{i, j, k, \dots, t\} = M$ va $[M] = m$ bo'lsin.

Teorema: $(P)_{ijk\dots t,l}^n$ operator I_n ni o'zini o'zi akslantirishi uchun quyidagi ikki shartni birini bajarishi zarur va yetarli:

1. l ning kamida bitta qiymatida $(P)_{ijk\dots t,l} = 0$ va

$$\begin{cases} P_{ijk\dots t,l} \leq 0, & \text{agar } m = 2h \text{ bo'lsa,} \\ P_{ijk\dots t,l} \geq 0, & \text{agar } m = 2h - 1 \text{ bo'lsa,} \end{cases}$$

2. Agar $m = 2h$ bo'lsa $\begin{cases} P_{ijk\dots t,l} \leq 0 & \text{agar } \ell \in \{i, j, k, \dots, t\} \\ P_{ijk\dots t,l} = 0 & \text{agar } \ell \notin \{i, j, k, \dots, t\} \end{cases}$ agar $m = 2h - 1$ bo'lsa $\begin{cases} P_{ijk\dots t,l} \geq 0 & \text{agar } \ell \in \{i, j, k, \dots, t\} \\ P_{ijk\dots t,l} = 0 & \text{agar } \ell \notin \{i, j, k, \dots, t\} \end{cases}$
Yuqoridagi operatorlarda

$$\pi_{ijk\dots t} = \begin{pmatrix} P_{ijk\dots t,i} & P_{ijk\dots t,j} & P_{ijk\dots t,k} & \dots & P_{ijk\dots t,t} \\ P_{ijk\dots t,i} & P_{ijk\dots t,j} & P_{ijk\dots t,k} & \dots & P_{ijk\dots t,t} \end{pmatrix}$$

almashtirish yordamida hosil qilingan barcha operatorlar.

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ALTERNATIVE GIBBS MEASURES FOR THE HC-MODEL ON THE CAYLEY TREE IN THE "HINGE" CASE

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Let $\mathfrak{S}^k = (V, L)$ be the Cayley tree of order $k \geq 2$. Let $\Phi = \{0, 1, 2\}$ and $\sigma \in \Omega = \Phi^V$ be a configuration. We denote by Ω the set of all configurations on V .

We consider the set Φ as the set of vertices of the graph G . A configuration σ is called a *G-admissible configuration* on the Cayley tree, if $\{\sigma(x), \sigma(y)\}$ is an edge of the graph G for any pair of nearest neighbors x, y in V . We denote by Ω_G^a the set of admissible configurations.

The set of activities [1] for the graph G is a function $\lambda : G \rightarrow R_+$. For given G and λ , the Hamiltonian of the G -HC model is defined as

$$H_G^\lambda(\sigma) = - \sum_{x \in V} \ln \lambda_{\sigma(x)}, \text{ if } \sigma \in \Omega^a.$$

Let $L(G)$ be the set of edges of the graph G , and let $A \equiv A^G = (a_{ij})_{i,j \in Z}$ denote the adjacency matrix of G .

We consider the specific graph $G = \text{hinge}: \{0, 0\}\{0, 1\}\{0, 2\}\{1, 1\}\{2, 2\}$ (see [3]).

The notion of a Gibbs measure is introduced in the standard way (see, e.g., [1]–[3]). It is known [4] that every Gibbs measure for the HC-model on the Cayley tree can be associated with a set of quantities $z = \{z_x, x \in G_k\}$, satisfying

$$z_{i,x} = \lambda \prod_{y \in S(x)} \frac{a_{i0} + a_{i1}z_{1,y} + a_{i2}z_{2,y}}{a_{00} + a_{01}z_{1,y} + a_{02}z_{2,y}}, i = 1, 2. \quad (1)$$

Translation-invariant and periodic Gibbs measures have been studied in [3]–[7]. We construct new solutions to (1), i.e., we take m and r to be non-negative integers such that $0 \leq m \leq k$ and $0 \leq r \leq k$. We consider the following matrix

$$M = \begin{pmatrix} m & k - m \\ r & k - r \end{pmatrix}.$$

This matrix determines how many times the values z and t occur in the set $S(x)$ for each $z_x \in \{z, t\}$. More precisely, the boundary condition $z = \{z_x, x \in G_k\}$ with values $z = (z_1, z_2) \in R_+^2$ and $t = (t_1, t_2) \in R_+^2$ is defined by the following steps:

- if at vertex x we have $z_x = z$, then m vertices from the set $S(x)$ take the value z , and the remaining $k - m$ vertices take the value t ;
- if at vertex x we have $z_x = t$, then r vertices from $S(x)$ take the value t (Note: r is used here, not the value r as in the original text, which would be inconsistent), and the remaining $k - r$ vertices take the value z .

A measure corresponding to z_x , constructed in the manner described above and which is not translation-invariant (TI), is called an alternative Gibbs measure (AGM) and is denoted as $\mu_{m,r}$.

For the AGM, from system (1) we obtain the following system of equations

$$\begin{cases} z_1 = \lambda \cdot \left(\frac{1+z_1}{1+z_1+z_2} \right)^m \cdot \left(\frac{1+t_1}{1+t_1+t_2} \right)^{k-m}, & t_1 = \lambda \cdot \left(\frac{1+t_1}{1+t_1+t_2} \right)^r \cdot \left(\frac{1+z_1}{1+z_1+z_2} \right)^{k-r}, \\ z_2 = \lambda \cdot \left(\frac{1+z_2}{1+z_1+z_2} \right)^m \cdot \left(\frac{1+t_2}{1+t_1+t_2} \right)^{k-m}, & t_2 = \lambda \cdot \left(\frac{1+t_2}{1+t_1+t_2} \right)^r \cdot \left(\frac{1+z_2}{1+z_1+z_2} \right)^{k-r}. \end{cases} \quad (2)$$

We consider the invariant set $I_4 = \{(z_1, z_2, t_1, t_2) \in R^4 : z_1 = z_2, t_1 = t_2\}$.

The following theorem has been proven.

Theorem. Let $k \geq 2$ and $\lambda > 0$. For the HC model in the case $G = \text{hinge}$ on I_4 , AGMs (not translation-invariant) exist for $\lambda > 0$ if and only if $m + r \leq k - 6$.

Remark. In the paper [8], alternative Gibbs measures for the HC-model in the case of the graph "wand" were studied.

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Gibbs Measures in the Hard-Core Model on the Cayley Tree in the case hinge

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Let $\mathfrak{S}^k = (V, L)$ is the Cayley tree. For fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \bigcup_{s=0}^n W_s$, where $d(x, y)$ is the distance between vertices x and y on the Cayley tree. Let $\Phi = \{0, 1, 2\}$ and $\sigma \in \Omega = \Phi^V$ be the configuration, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. The set of all configurations on V (V_n) is denoted by Ω (Ω_{V_n}).

Consider the set Φ as the set of vertices of some graph G . The σ configuration is called *G-admissible configuration* on the Cayley tree (in V_n) if $\{\sigma(x), \sigma(y)\}$ is edge of the graph G for any the nearest pair of neighbors x, y from V (from V_n). Let us denote the set of *G-admissible configurations* via $\Omega^{a,G}$ ($\Omega_{V_n}^{a,G}$) (see [2]).

The activity set [1] for the graph G is the function $\lambda: \{i, j\} \in K \rightarrow \lambda_{ij} \in \mathbb{R}_+$. Value λ_{ij} of function λ on edge $\{i, j\} \in K$ is called its “activity”.

In the hinge-graph G , configuration are admissible only if, for any pair of nearest-neighbor vertices $\{x, y\}$, we have that $\sigma(x)\sigma(y) \neq 2$.

For given G and λ , it is known if $\sigma \in \Omega^G$, then the Hamiltonian of the *HC* model as

$$H(\sigma) = \begin{cases} -J \sum_{\langle x, y \rangle} \ln \lambda_{\sigma(x), \sigma(y)}, & \text{if } \sigma \in \Omega^G, \\ +\infty, & \text{if } \sigma \notin \Omega^G. \end{cases}$$

Let $z: x \mapsto z_x = (z_{0,x}, z_{1,x}, z_{2,x}) \in \mathbb{R}_+^3$ vector-valued function on V . For $n \geq 1$ and $\lambda > 0$ consider the probability measure $\mu^{(n)}$ on $\Omega_{V_n}^{a,G}$, defined as

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \prod_{\langle x, y \rangle \in V_n} \lambda_{\sigma_n(x), \sigma_n(y)} \prod_{x \in W_n} z_{\sigma(x), x}. \quad (1)$$

The sequence of probability measures $\mu^{(n)}$ is said to be consistent if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}^{a,G}$ the equality holds

$$\sum_{\omega_n \in \Omega_{V_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \mathbf{1}(\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}^{a,G}) = \mu^{(n-1)}(\sigma_{n-1}). \quad (2)$$

In this case, by the theorem Kolmogorov (see [3]) there is a unique measure μ on $(\Omega^{a,G}, B)$ such that $\mu(\sigma : \sigma|_{V_n} = \sigma_n) = \mu^{(n)}(\sigma_n)$ for all n and any $\sigma_n \in \Omega_{V_n}^{a,G}$.

We call the resulting hinge-graph model the *HC model*. Consequently, for the *HC* model we have the next system of the equations

$$\begin{cases} z'_{1,x} = \lambda_{1,1} \prod_{y \in S(x)} \frac{\lambda_{10} a_{10} + \lambda_{11} a_{11} z'_{1,y} + \lambda_{12} a_{12} z'_{2,y}}{\lambda_{00} a_{00} + \lambda_{01} a_{01} z'_{1,y} + \lambda_{02} a_{02} z'_{2,y}}, \\ z'_{2,x} = \lambda_{2,2} \prod_{y \in S(x)} \frac{\lambda_{20} a_{20} + \lambda_{21} a_{21} z'_{1,y} + \lambda_{22} a_{22} z'_{2,y}}{\lambda_{00} a_{00} + \lambda_{01} a_{01} z'_{1,y} + \lambda_{02} a_{02} z'_{2,y}}, \end{cases} \quad (3)$$

and, for any $z = \{z_x : x \in V\}$ satisfying (3), there exists a unique GM μ for the HC . We therefore restrict our attention to a smaller class of measures, namely the translation-invariant GMS, i.e., $z_{i,x} \equiv z_i$.

More precisely, we denote $\theta = \exp(J)$ and consider the activity $\lambda = (\lambda_{ij})_{\{i,j\} \in K}$ defined as

$$\lambda_{ij} = \begin{cases} 0, & \text{if } i = 1, j = 2, \\ 1, & \text{if } i = j, \\ \theta, & \text{otherwise.} \end{cases} \quad (4)$$

Let $x = \sqrt[k]{z_1}$, $y = \sqrt[k]{z_2}$. Then searching only for translation-invariant measures, the functional equation (3) reduces to

$$x = \frac{\theta + x^k}{1 + \theta x^k + \theta y^k}, \quad y = \frac{\theta + y^k}{1 + \theta x^k + \theta y^k}. \quad (5)$$

Let the invariant set $I_1 = \{(x, y, t) \in R^3 : x = 1, y = t\}$. Then on the set I_1 , the system (5) is equivalent to

$$\theta = \frac{x^k - x}{2x^{k+1} - 1} =: f(x, k). \quad (6)$$

By analyzing the function in equation (6) for $k \geq 2$, we obtained the following theorem.

Theorem 2. Let $k \geq 2$. Then for the three state HC model in the case $G = \text{hinge}$, there exist critical values $\theta_{cr}(k) = f(x_0, k)$ such that the number of TISGMs on I_1 is 1 if $\theta > \theta_{cr}(k)$, 2 if $\theta = \theta_{cr}(k)$, 3 if $0 < \theta < \theta_{cr}(k)$, where x_0 is point of maximum of the function $f(x, k)$, defined by (6).

Theorem 3. Let $k = 2$. Then, for the three state HC model in the case $G = \text{hinge}$, there exist critical values $\theta_{cr} \approx 0.13507$ and $\theta_c \approx 1.23607$ such that the number of TISGMs: 1 if $\theta \geq \theta_c$; 5 if $\theta \geq \theta_c$; 6 if $\theta \geq \theta_c$; 7 if $\theta \geq \theta_c$.

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On dynamical systems generated by the evolution operator for the case of CSO

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Let m be a natural number and let $E_m = \{1, 2, \dots, m\}$ be a finite set. Then the set of all probability distributions on the E_m is

$$S^{m-1} = \left\{ x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \forall i, \sum_{i=1}^m x_i = 1 \right\},$$

which is called $m - 1$ -dimensional simplex.

In general, a cubic operator $V : x \in \mathbb{R}^m \rightarrow x' = V(x) \in \mathbb{R}^m$ corresponding to a matrix $\mathbb{P} = \{P_{ijk,l}\}_{i,j,k,l=1}^m$ is defined by:

$$V : x'_l = \sum_{i,j,k=1}^m P_{ijk,l} x_i x_j x_k, \quad l = 1, \dots, m. \quad (1)$$

Let W be an operator defined on simplex S^{m-1} . A point $\mathbf{x} \in S^{m-1}$ is called a *periodic* point of W if there exists an n so that $W^n(\mathbf{x}) = \mathbf{x}$. The smallest positive integer n satisfying the above is called the prime period or least period of the point $\mathbf{x}$. A period-one point is called a *fixed* point of W . Denote by $\text{Fix}(W)$ the set of all fixed points of the operator W , i.e.

$$\text{Fix}(W) = \{\mathbf{x} \in S^2 : W(\mathbf{x}) = \mathbf{x}\}.$$

Let $DW(\mathbf{x}^*) = (\partial W_i / \partial x_j)(\mathbf{x}^*)$ be a Jacobian of W at the point $\mathbf{x}^*$.

Definition 1.[3] A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DW(\mathbf{x}^*)$ has no eigenvalues on the unit circle in $\mathbb{C}$.

Definition 2.[3] A hyperbolic fixed point $\mathbf{x}^*$ is called:

- i) attracting if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are in the unit disk;
- ii) repelling if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are outside the closed unit disk;
- iii) a saddle otherwise.

We consider the following CSO on the two-dimensional simplex :

$$W = \begin{cases} x' = cz^3 + 3x^2z + 3xz^2 + 3xyz, \\ y' = dz^3 + 3y^2z + 3yz^2 + 3xyz, \\ z' = (x + y)^3, \end{cases} \quad (2)$$

where $c + d = 1$, $c > 0$, $d > 0$.

Theorem. The dynamical system generated by operator (2) has the following assertions hold

1. The set of fixed points is $Fix(W) = \{(c(1 - z^*), d(1 - z^*), z^*)\}$,
where $z^* = 1 - \frac{1}{6}(108 + 12\sqrt{93})^{\frac{1}{3}} + 2(108 + 12\sqrt{93})^{-\frac{1}{3}}$;
2. The fixed point is a saddle;
3. For any initial point $\lambda \in S^2 \setminus Fix(W)$ the trajectory will converge to 2-periodic orbit, namely to the periodic points $(0, 0, 1)$ and $(c, d, 0)$.

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ON THE EXTREMALITY OF TRANSLATION-INVARIANT GIBBS MEASURES FOR THE BLUME-CAPEL MODEL ON THE CAYLEY TREE

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Let $\Gamma^k = (V, L)$ is the Cayley tree. For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V | d(x^0, x) = n\}, V_n = \{x \in V | d(x^0, x) \leq n\},$$

where $d(x, y)$ is the distance between vertices x and y on the Cayley tree.

We consider a model in which spin variables take values from the set $\Phi = \{-1, 0, 1\}$. We then define a configuration σ on V as a function $x \in V \rightarrow \sigma(x) \in \Phi$. The set of all configurations coincides with $\Omega = \Phi^V$. Let $A \subset V$. We denote the space of configurations defined on a set A by Ω_A .

The Hamiltonian of the Blume-Capel model is given by the formula

$$H(\sigma) = J \sum_{\langle x, y \rangle} (\sigma(x) - \sigma(y))^2, \quad (1)$$

where $J > 0$.

The set of the direct successors of x is denoted by $S(x)$, i.e.,

$$S(x) = \{y \in W_{n+1} | d(x, y) = 1\}, x \in W_n.$$

Let $h : x \mapsto h_x = (h_{-1,x}, h_{0,x}, h_{1,x})$ be a vector-valued function on $x \in V \setminus \{x^0\}$. We consider the probability measure $\mu^{(n)}$ on Ω_A ,

$$\mu^n(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}\}. \quad (2)$$

Here $\sigma \in \Omega_{V_n}$, and Z_n is a normalization factor, $Z_n = \sum_{\bar{\sigma}_n \in \Omega_{V_n}} \exp\left\{-\beta H(\bar{\sigma}_n) + \sum_{x \in W_n} h_{\bar{\sigma}(x), x}\right\}$, where $h_{\bar{\sigma}, x} \in R$. The probability measure $\mu^{(n)}$ is said to be consistent if for all $n \geq 1$ and any $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}). \quad (3)$$

In this case, there is a unique measure μ on Ω_V such that

$$\mu(\{\sigma | V_n = \sigma_n\}) = \mu^{(n)}(\sigma_n)$$

for all $n \geq 1$ and any $\sigma_n \in \Omega_{V_n}$.

A condition for $h_{i,x}$ ensuring the consistency of the measures $\mu^{(n)}$ is formulated in the next theorem.

Teorema 1. *Let $k \geq 2$. The sequence of probabilistic measures $\mu^{(n)}$, $n = 1, 2, \dots$, defined (3) is consisted if and only if the equalities*

$$\begin{cases} z_{1,x} = \prod_{y \in S(x)} \frac{\theta^4 z_{-1,y} + \theta + z_{1,y}}{\theta z_{-1,y} + 1 + \theta z_{1,y}}, \\ z_{-1,x} = \prod_{y \in S(x)} \frac{z_{-1,y} + \theta + \theta^4 z_{1,y}}{\theta z_{-1,y} + 1 + \theta z_{1,y}}, \end{cases} \quad (4)$$

where $\theta = \exp\{J\beta\}$, $\beta = 1/T$, $z_{i,x} = \exp(h_{i,x} - h_{0,x})$, $i = -1, 1$ hold for any $x \in V$.

Translation-invariant Gibbs measures (TIGM) correspond to solutions (4) with $z_{i,x} = z_i > 0$ for all $x \in V$ and $i = -1, 1$. We introduce the notation $z_1 = z_1, z_{-1} = z_2$. Then (4) has the form

$$\begin{cases} z_1 = \left(\frac{\theta^4 z_2 + \theta + z_1}{\theta z_1 + \theta z_2 + 1} \right)^k, \\ z_2 = \left(\frac{\theta^4 z_1 + \theta + z_2}{\theta z_1 + \theta z_2 + 1} \right)^k. \end{cases} \quad (5)$$

From the solution $z_1 = z_2 = z$ of system (5), we obtain the following Theorem 2 and Theorem 3. The case $z_1 \neq z_2$ will not be considered here.

Theorem 2. *Let $k = 2$. Then for the Blume-Capel model there is critical value $\theta_{cr} \approx 0.1351717506$, such that:*

- a) for $0 < \theta < \theta_{cr}$ there are 3 translation-invariant Gibbs measures: μ_1, μ_2, μ_3 ;
- b) for $\theta = \theta_{cr}$ there are 2 translation-invariant Gibbs measures: μ_1, μ_2 ;
- c) for $\theta_{cr} < \theta$ there exists only one translation-invariant Gibbs measure: μ_1 .

Using the methods employed in [2]-[4], we analyze the extremality of the measures μ_1, μ_2 and μ_3 . Consequently, the following theorem holds.

Theorem 3. *Let $k = 2$. For the Blume-Capel model, the following statements hold:*

- a) the measure μ_1 is extremal if $\theta \in (\theta_1, \theta_2)$;
- b) the measure μ_2 is extremal if $\theta \in (\theta_3, \theta_4)$;
- c) the measure μ_3 is extremal if $\theta \in (\theta_5, \theta_6)$,

where $\theta_{cr} \approx 0.1351717506$, $\theta_1 \approx 0.144632$, $\theta_2 \approx 1.59963$, $\theta_3 \approx 0.09442$, $\theta_4 \approx 0.1126974$, $\theta_5 \approx 0.0771265$, and $\theta_6 \approx 0.085221$.

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Biological interpretation of Gibbs measures for the DNA-Ising molecule model on the Cayley tree

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The Cayley tree [1]. A Cayley tree $\mathfrak{T}^k$ of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles in which each vertex is connected to exactly $k + 1$ edges. We denote the tree by $\mathfrak{T}^k = (V, L, i)$, where V is the set of vertices, L is the set of edges, and i is the incidence function that assigns to each edge $l \in L$ its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then the vertices x and y are called *nearest neighbors*, and the edge connecting them is denoted by $l = \langle x, y \rangle$.

For a fixed vertex $x^0 \in V$, we define the set $W_n = \{x \in V : d(x^0, x) = n\}$,

$$V_n = \{x \in V : d(x^0, x) \leq n\}, L_n = \{l = \langle x, y \rangle \in L : x, y \in V_n\}. \quad (1)$$

Here, $d(x, y)$ denotes the distance between vertices x and y on the Cayley tree, which is defined as the minimal number of edges in any path connecting x and y , i.e., $d(x, y) = \min\{d : \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle\}$.

Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. As shown in [3], all vertices of the Cayley tree can be partitioned into equivalence classes indexed by integers. Moreover, each vertex in the equivalence class labeled by m lies on a unique path, referred to as a $\mathbb{Z}$ -path.

Model. A configuration $\sigma = \{\sigma(x), x \in V\}$ on the vertex set V of a Cayley tree is defined as a function assigning to each vertex $x \in V$ a value $\sigma(x)$ from the set $\{-1, 1\}$. Here, -1 corresponds to the DNA base pairs A and T , and 1 corresponds to the base pairs C and G . The set of all configurations defined on the vertex set V is denoted by Ω . Similarly, configurations on the subset V_n are defined, and the collection of all such configurations is denoted by Ω_n . The restriction of a configuration to a $\mathbb{Z}$ -path is referred to as DNA.

The Ising model is employed to describe the configuration energies σ of the DNA molecule set, given by the Hamiltonian

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y), \quad (2)$$

where $J > 0$ is the coupling constant between neighboring DNA vertices, $\sigma(x) \in \{-1, 1\}$.

Markov chains. The transition matrix of a Markov chain (with a given Gibbs measure) is defined as (see [4])

$$\mathbb{P}^{\langle x, y \rangle} = (P_{k,j}^{\langle x, y \rangle})_{k,j=1,2} = \begin{cases} \begin{pmatrix} \frac{\theta^2 u}{\theta^2 u + 1} & \frac{1}{\theta^2 u + 1} \\ \frac{u}{u + \theta^2} & \frac{\theta^2}{u + \theta^2} \end{pmatrix}, & \langle x, y \rangle \in \mathbb{Z}\text{-path}, \\ \begin{pmatrix} \frac{\theta^2 v}{\theta^2 v + 1} & \frac{1}{\theta^2 v + 1} \\ \frac{v}{v + \theta^2} & \frac{\theta^2}{v + \theta^2} \end{pmatrix}, & \langle x, y \rangle \notin \mathbb{Z}\text{-path}, \end{cases}$$

where (u, v) is a solution of for the system of the functional equations given in the work[] Note that each matrix $\mathbb{P}^{\langle x, y \rangle}$ does not depend on $\langle x, y \rangle$ itself, but depends on the $\mathbb{Z}$ -path to which it belongs.

Stationary distributions are easy to find:

$$\pi^{\langle x, y \rangle} = \begin{cases} \left(\frac{\theta^2 u^2 + u}{\theta^2 u^2 + 2u + \theta^2}, \frac{u + \theta^2}{\theta^2 u^2 + 2u + \theta^2} \right), & \langle x, y \rangle \in \mathbb{Z}\text{-path}, \\ \left(\frac{\theta^2 v^2 + v}{\theta^2 v^2 + 2v + \theta^2}, \frac{v + \theta^2}{\theta^2 v^2 + 2v + \theta^2} \right), & \langle x, y \rangle \notin \mathbb{Z}\text{-path}. \end{cases}$$

Theorem 1. *Regardless of the edge $\langle x, y \rangle$ for $k = 2, 3, 4, 5$ we have the following limit relations*

$$\lim_{T \rightarrow 0} \pi^{\langle x, y \rangle, 1} = \left(\frac{1}{2}, \frac{1}{2} \right), \quad \lim_{T \rightarrow 0} \pi^{\langle x, y \rangle, 2} = \left(\frac{1}{2}, \frac{1}{2} \right), \quad \lim_{T \rightarrow 0} \pi^{\langle x, y \rangle, 3} = (1, 0)$$

in case of low temperatures and

$$\lim_{T \rightarrow +\infty} \pi^{\langle x, y \rangle, 1} = \lim_{T \rightarrow T_c} \pi^{\langle x, y \rangle, i} = \left(\frac{1}{2}, \frac{1}{2} \right), \quad i = 1, 2, 3$$

in case of high temperatures.

Corollary 1. Utilizing this proposition, the structure of DNA is analyzed for $k = 6, 7, 8, 9$ under both low and high temperature regimes.

In the case of $T \rightarrow 0$, the DNA set has the following stationary states (configurations).

For measures of $\mu_1^{(k)}$, the base pairs $-1 = A + T$ and $1 = C + G$ at each point of the DNA molecule are found with equal probability $1/2$ for states -1 and 1.

For measures of $\mu_2^{(k)}$, all DNAs are rigid and interact, and $\sigma(x) = -1$ for all $x \in \mathbb{Z}$ -path.

For measures of $\mu_3^{(k)}$, all DNAs are rigid and interact, and $\sigma(x) = 1$ for all $x \in \mathbb{Z}$ -path.

In the case $T \rightarrow +\infty$ the sequence of states -1 and 1 in the DNA molecule on the $\mathbb{Z}$ -path is free, the states are independent and identically distributed. There is a state -1 with probability $1/2$ and a state 1 with probability $1/2$.

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Some Gibbs measures for the DNA-Ising molecule model on the Cayley tree

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Every DNA molecule is composed of a double helix structure formed by two complementary nucleotide strands. These strands are held together by hydrogen bonds between base pairs: cytosine pairs with guanine (C-G), and adenine pairs with thymine (A-T). [1].

The Cayley tree [2]. A Cayley tree $\mathfrak{S}^k$ of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles in which each vertex is connected to exactly $k + 1$ edges. We denote the tree by $\mathfrak{S}^k = (V, L, i)$, where V is the set of vertices, L is the set of edges, and i is the incidence function that assigns to each edge $l \in L$ its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then the vertices x and y are called *nearest neighbors*, and the edge connecting them is denoted by $l = \langle x, y \rangle$.

For a fixed vertex $x^0 \in V$, we define the set $W_n = \{x \in V : d(x^0, x) = n\}$,

$$V_n = \{x \in V : d(x^0, x) \leq n\}, L_n = \{l = \langle x, y \rangle \in L : x, y \in V_n\}. \quad (1)$$

Here, $d(x, y)$ denotes the distance between vertices x and y on the Cayley tree, which is defined as the minimal number of edges in any path connecting x and y , i.e., $d(x, y) = \min\{d : \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle\}$.

Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. As shown in [3], all vertices of the Cayley tree can be partitioned into equivalence classes indexed by integers. Moreover, each vertex in the equivalence class labeled by m lies on a unique path, referred to as a $\mathbb{Z}$ -path.

Model. A configuration $\sigma = \{\sigma(x), x \in V\}$ on the vertex set V of a Cayley tree is defined as a function assigning to each vertex $x \in V$ a value $\sigma(x)$ from the set $\{-1, 1\}$. Here, -1 corresponds to the DNA base pairs A and T , and 1 corresponds to the base pairs C and G . The set of all configurations defined on the vertex set V is denoted by Ω . Similarly, configurations on the subset V_n are defined, and the collection of all such configurations is denoted by Ω_n . The restriction of a configuration to a $\mathbb{Z}$ -path is referred to as DNA.

The Ising model is employed to describe the configuration energies σ of the DNA molecule set, given by the Hamiltonian

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y), \quad (2)$$

where $J > 0$ is the coupling constant between neighboring DNA vertices, $\sigma(x) \in \{-1, 1\}$.

We find a finite-dimensional distribution of the probability measure μ on the set Ω_n of all possible configurations on V_n

$$\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ \beta H_n(\sigma_n) + \sum_{y \in W_n} h_{\sigma(y), y} \right\}, \quad (3)$$

where Z_n^{-1} is the normalizing coefficient, $\beta = \frac{1}{T}$, $T > 0$ is the temperature,

$$H_n(\sigma_n) = \sum_{\langle x, y \rangle \in L_n} \sigma_n(x)\sigma_n(y),$$

and $\{h_{i,x} \in \mathbb{R}, i = -1, 1, x \in V_n\}$.

Theorem 1. *The probability distributions μ_n in (3) are consistent if and only if the equations*

$$\begin{cases} z_x = \frac{\theta^2 \hat{z}_{x_0} + 1}{\hat{z}_{x_0} + \theta^2} \cdot \frac{\theta^2 \hat{z}_{x_1} + 1}{\hat{z}_{x_1} + \theta^2} \prod_{t \in S_0(x)} \frac{\theta^2 \hat{z}_t + 1}{\hat{z}_t + \theta^2}, & \langle x_{\downarrow}, x \rangle \notin \mathbb{Z}\text{-path}, \\ \hat{z}_x = \frac{\theta^2 \hat{z}_{x_1} + 1}{\hat{z}_{x_1} + \theta^2} \prod_{t \in S_1(x)} \frac{\theta^2 \hat{z}_t + 1}{\hat{z}_t + \theta^2}, & \langle x_{\downarrow}, x \rangle \in \mathbb{Z}\text{-path}, \end{cases} \quad (4)$$

holds for any $x \in V \setminus \{x^0\}$. Here, $\theta = e^{-J\beta}$,

$$z_x = e^{h_{1,x} - h_{-1,x}} = u, \quad \langle x_{\downarrow}, x \rangle \notin \mathbb{Z}\text{-path},$$

$$\hat{z}_x = e^{h_{1,x} - h_{-1,x}} = v, \quad \langle x_{\downarrow}, x \rangle \in \mathbb{Z}\text{-path}.$$

The Gibbs measures corresponding to such solutions are called *translation-invariant*. From (4), we get

$$\begin{aligned} u &= \left(\frac{\theta^2 v + 1}{v + \theta^2} \right)^2 \left(\frac{\theta^2 u + 1}{u + \theta^2} \right)^{k-2}, \\ v &= \left(\frac{\theta^2 v + 1}{v + \theta^2} \right) \left(\frac{\theta^2 u + 1}{u + \theta^2} \right)^{k-1}, \end{aligned} \quad (5)$$

where $u, v > 0$. Clearly, $u = v = 1$ satisfies system (5) for any $k \geq 2$ and $\theta > 0$.

Let us denote by $\mu_i^{(k)}$ the Gibbs measures corresponding to the solutions $z_i^{(k)}, i = 1, 2, 3$ given in the work [2]. Let us define the critical temperature

$$T_c := T_c(k) = \frac{J}{\ln \sqrt{\frac{k-1}{k+1}}}.$$

Theorem 2. *For the DNA-Ising molecule model on the Cayley tree of order $k \geq 2$ the following statements are true:*

- 1) if $T \geq T_c$, then there exists a unique translation-invariant Gibbs measure μ_1 ;
- 2) if $T < T_c$, then there exist three translation-invariant Gibbs measures μ_1, μ_2, μ_3 , i.e. a phase transition occurs.

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TRANSLATION-INVARIANT GIBBS MEASURES FOR HC-BLUME-CAPEL MODEL IN THE "TRIANGLE" CASE ON THE CAYLEY TREE

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The definition of the Gibbs measure and other concepts related to the theory of Gibbs measures can be found, for example, in [2]. In this paper, we study translation-invariant Gibbs measures for the HC-Blume-Capel model in the case of a "triangle" on a Cayley tree. A Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles such that each vertex has precisely $k + 1$ edges, where V is the set of vertices of the graph Γ^k , L is the set of its edges. Let i be an incidence function associating each edge $l \in L$ to its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called the nearest neighbors of a vertex and can be written as $\langle x, y \rangle$. The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined as

$$d(x, y) = \min\{d | \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle\}.$$

We consider a model in which spin variables take values from the set $\Phi = \{-1, 0, +1\}$, which has the following form (see [1])

$$\text{triangle: } \{-1, 1\}, \{1, 0\}, \{0, -1\}, \{0, 0\}.$$

We then define a configuration σ on V as a function $x \in V \rightarrow \sigma(x) \in \Phi$. The set of all configurations coincides with $\Omega = \Phi^V$. Let $A \subset V$. We denote the space of configurations defined on a set A by Ω_A .

The hamiltonian of the Blume-Capel model is given by the formula

$$H(\sigma) = J \sum_{\langle x, y \rangle} (\sigma(x) - \sigma(y))^2, \quad (1)$$

where $J > 0$.

For a fixed $x^0 \in V$, we write $x \prec y$ if a path from x^0 to y runs through x .

A vertex y called a "direct descendant" of a vertex x if $x \prec y$ and $d(x, y) = 1$.

We let $S(x)$ denote the set of "direct descendants" of a vertex $x \in V$.

Let $h : x \mapsto h_x = (h_{-1,x}, h_{0,x}, h_{1,x})$ be a vector-valued function on $x \in V \setminus \{x^0\}$. We consider the probability measure $\mu^{(n)}$ on Ω_A ,

$$\mu^n(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}\}. \quad (2)$$

Here $\sigma \in \Omega_{V_n}$, and Z_n is a normalization factor,

$$Z_n = \sum_{\bar{\sigma}_n \in \Omega_{V_n}} \exp\left\{-\beta H(\bar{\sigma}_n) + \sum_{x \in W_n} h_{\bar{\sigma}(x), x}\right\},$$

where $h_{\bar{\sigma}, x} \in R$.

The probability measure $\mu^{(n)}$ is said to be consistent if for all $n \geq 1$ and any $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}). \quad (3)$$

In this case, there is a unique measure μ on Ω_V such that

$$\mu(\{\sigma | V_n = \sigma_n\}) = \mu^{(n)}(\sigma_n)$$

for all $n \geq 1$ and any $\sigma_n \in \Omega_{V_n}$.

The following theorem gives necessary and sufficient conditions on $h_{i,x}$ under which (3) holds.

Theorem 1. *Let $k \geq 2$. Probability distribution $\mu^{(n)}(\sigma_n), n = 1, 2, \dots$ in (2) consistent if and only if for any $x \in V$ the following hold:*

$$\begin{cases} z_{1,x} = \prod_{y \in S(x)} \frac{a_{1,-1}\theta^4 z_{-1,y} + a_{1,0}\theta + a_{1,1}z_{1,y}}{a_{0,-1}\theta z_{-1,y} + a_{0,0} + a_{0,1}\theta z_{1,y}}, \\ z_{-1,x} = \prod_{y \in S(x)} \frac{a_{-1,-1}z_{-1,y} + a_{-1,0}\theta + a_{-1,1}\theta^4 z_{1,y}}{a_{0,-1}\theta z_{-1,y} + a_{0,0} + a_{0,1}\theta z_{1,y}}, \end{cases} \quad (4)$$

where $\theta = \exp\{-J\beta\}, \beta = 1/T, z_{i,x} = \exp(h_{i,x} - h_{0,x}), i = -1, 1$.

Translation-invariant Gibbs measures correspond to solutions of (4) with $z_{i,x} = z_i$ for all $x \in V$ and $i = -1, 1$. For convenience, let's denote $z_1 = z_1, z_{-1} = z_2$. Then the system of equations (4) in the case of "triangle" will look like:

$$\begin{cases} z_1 = \left(\frac{\theta + \theta^4 z_2}{1 + \theta z_1 + \theta z_2} \right)^k, \\ z_2 = \left(\frac{\theta + \theta^4 z_1}{1 + \theta z_1 + \theta z_2} \right)^k. \end{cases} \quad (5)$$

Let $z_1 = z_2 = z$. Then from (5) we get

$$z = \left(\frac{\theta + \theta^4 z}{1 + 2\theta z} \right)^k. \quad (6)$$

The following theorem is true.

Theorem 2. *Let $k \geq 2$. Then for any $\theta > 0$ the equation (6) has a unique solution.*

Then from this theorem we obtain the following

Theorem 3. *Then for the HC-Blume-Capel model in the "triangle" case for $\theta > 0$ there is only one translation-invariant Gibbs measure μ_0 .*

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PHASE TRANSITIONS IN THE THREE-STATE HARD-CORE MODEL WITH COMPETING INTERACTIONS ON THE CAYLEY TREE

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The Hard-Core (HC) model is significant in statistical physics as a fundamental representation of a gas consisting of particles with finite size. This model can also be obtained as a limiting case of the antiferromagnetic Ising model (see, e.g., [1]). Moreover, the HC model has found applications in the study of communication networks (see, e.g., [2]).

In this paper, we derive a functional equation for the HC model with one-level next-nearest-neighbour interactions, exploiting the self-similarity of the Cayley tree. The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree. We denote by V the set of the vertices of the tree and by L the set of edges of the tree. Two vertices x and y , where $x, y \in V$, are called nearest-neighbour if there exists an edge $l \in L$ connecting them, which is denoted by $l = \langle x, y \rangle$. The distance on this tree, denoted by $d(x, y)$, is defined as the number of nearest-neighbour pairs of the minimal path between the vertices x and y .

For a fixed $x^0 \in V$, called the root, we set $W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \bigcup_{m=0}^n W_m$. For the sake of simplicity, we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called second nearest-neighbour if $d(x, y) = 2$. The second nearest-neighbour vertices x and y are called prolonged second nearest-neighbour if $|x| \neq |y|$ and is denoted by $> \overline{x, y} <$. The second nearest-neighbour vertices $x, y \in V$ that are not prolonged are called one-level next-nearest-neighbour since $|x| = |y|$ and are denoted by $> \overline{x, y} <$.

Let $\Phi = \{0, 1, 2\}$ and $\sigma \in \Omega = \Phi^V$ be the configuration, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. In other words, in this model, each vertex x is assigned to one of the values $\sigma(x) \in \Phi$. The set of all configurations on V (V_n) is denoted by Ω (Ω_{V_n}). Obviously, $\Omega \subset \Phi^V$.

Consider the set Φ as the set of vertices of some graph G . Using the graph G we define a G -admissible configuration as follows. The σ configuration is called G -admissible configuration on the Cayley tree (in V_n) if there is an edge $\{\sigma(x), \sigma(y)\}$ in G for any nearest-neighbour pair of x, y from V (from V_n). Let us denote the set of G -admissible configurations by Ω^G ($\Omega_{V_n}^G$).

For given G , we define the Hamiltonian of the G -HC-model with one-level next-nearest-neighbour interactions as

$$H(\sigma) = -J \sum_{x \in V} \mathbf{1}(\sigma(x) \geq 1) - J_1 \sum_{> \overline{x, y} <} \delta_{\sigma(x), \sigma(y)},$$

where the sum in the first term ranges all vertices, the second sum runs over all one-level next-nearest-neighbours and $J, J_1 \in \mathbb{R}$ are the coupling constants.

Let Λ be a finite subset of V . We will denote by $\sigma(\Lambda)$ the restriction of σ to Λ . Let $\bar{\sigma}(V \setminus \Lambda)$ be a fixed boundary configuration. The total energy of $\sigma(\Lambda)$ under condition $\bar{\sigma}(V \setminus \Lambda)$ is defined as

$$H(\sigma(\Lambda) \mid \bar{\sigma}(V \setminus \Lambda)) = -J \sum_{x \in \Lambda} \mathbf{1}(\sigma(x) \geq 1) - J_1 \sum_{> \overline{x, y} <: x, y \in \Lambda} \delta_{\sigma(x), \sigma(y)} - J \sum_{\langle x, y \rangle: x \in \Lambda, y \notin \Lambda} \mathbf{1}(\sigma(y) \geq 1).$$

Then the conditional Gibbs measure μ_Λ of a configuration $\sigma(\Lambda)$ is defined as

$$\mu_\Lambda(\sigma(\Lambda) \mid \bar{\sigma}(V \setminus \Lambda)) = \frac{\exp(-\beta H(\sigma(\Lambda) \mid \bar{\sigma}(V \setminus \Lambda)))}{Z_\Lambda(\bar{\sigma}(V \setminus \Lambda))},$$

where β is the inverse temperature and $Z_\Lambda(\bar{\sigma}(V \setminus \Lambda))$ is partition function in volume Λ .

Let $L(G)$ be the set of edges of a graph G . We let $A \equiv A^G = (a_{ij})_{i,j}$ denote the adjacency matrix of the graph G , i.e.,

$$a_{ij} = a_{ij}^G = \begin{cases} 1, & \text{if } \{i, j\} \in L(G), \\ 0, & \text{if } \{i, j\} \notin L(G). \end{cases}$$

In [3], the fertile three-state HC -models are identified that correspond to graphs of the hinge, pipe, wand, and wrench types.

Let $k = 2$. Then taking an advantage of structure of the Cayley tree, under some conditions we obtain the following system of equations

$$\begin{cases} u = \lambda \cdot \frac{a_{10}\theta + a_{11}\theta u^2 + a_{12}\theta v^2 + 2a_{10}a_{11}u + 2a_{10}a_{12}v + 2a_{11}a_{12}uv}{a_{00}\theta + a_{01}\theta u^2 + a_{02}\theta v^2 + 2a_{00}a_{01}u + 2a_{00}a_{02}v + 2a_{01}a_{02}uv}, \\ v = \lambda \cdot \frac{a_{20}\theta + a_{21}\theta u^2 + a_{22}\theta v^2 + 2a_{20}a_{21}u + 2a_{20}a_{22}v + 2a_{21}a_{22}uv}{a_{00}\theta + a_{01}\theta u^2 + a_{02}\theta v^2 + 2a_{00}a_{01}u + 2a_{00}a_{02}v + 2a_{01}a_{02}uv}. \end{cases} \quad (1)$$

where $\lambda = \exp(\beta J)$, $\theta = \exp(\beta J_1)$.

It is important to note that if there is more than one positive solution for the system (1), then there is more than one translation-invariant limiting Gibbs measure (TILGM) corresponding to each solution. We say that a phase transition occurs for the model, if the system (1) has more than one positive solution. We obtain the following result

Theorem 1. Let $k = 2$, $\lambda > 0$ and $\theta > 0$. For the HC model with one-level nearest-neighbour interactions on the Cayley tree in the case “Wand“ the following is true

- if $0 < \lambda \leq 1$ is satisfied, then there exists a unique TILGM, i.e., phase transition does not occur;
- if $\lambda > 1$ is satisfied, then there exist three TILGMs, i.e., the phase transition occurs.

Remark 1. The Theorem 1 coincides with the classical result for the HC model when $\theta = 1$ ($J_1 = 0$), i.e. if $\lambda > 1$ then there exists three distinct TILGM.

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THE ISING MODEL ON THE CLOSED CAYLEY TREE WITH BRANCHING RATIO THREE

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In 1979 Jellitto [1] introduced the zero-field Ising model on the closed Cayley tree of branching ratio two and proceeded to solve it exactly. In the work, we consider the Ising model on the closed Cayley tree of branching ratio three. We exactly solve the model, i.e., we find the critical value of the parameter such that below this value there are three limiting Gibbs measures.

To define a random field, it is useful to consider the construction of the closed Cayley tree of branching ratio three. We begin with a single vertical edge containing a pair of sites. This structure is then duplicated to the left and right, with the three top sites linked through an additional site using three edges. Similarly, the same process is applied to the three bottom sites, forming the fundamental polygons. Next, this structure is again copied to the left and right, linking the three top and three bottom sites in the same manner as before. This iterative process results in a closed tree with three upper levels and three lower levels. Denote the finite closed tree constructed at the n th stage by $\mathcal{T}^{(n)}$. The infinite closed Cayley tree is then defined as $\mathcal{T} = \lim_{n \rightarrow \infty} \mathcal{T}^{(n)}$. The infinite closed Cayley tree $\mathcal{T}$ can be decomposed into three parts:

$$\mathcal{T} = \Gamma^1 \cup \Gamma^2 \cup \mathcal{S},$$

where $\Gamma^1 = (V_1, L_1)$ represents the *upper tree* and $\Gamma^2 = (V_2, L_2)$ represents the *lower tree* and $\mathcal{S}$ is the set of edges which connects upper and lower tree.

We study binary random fields on $\mathcal{T}$, which can be viewed as the limiting measures of a spin system defined on a sequence of finite closed Cayley trees. Let $V = V_1 \cup V_2$ and $L = L_1 \cup L_2$. If $l \in L$ an edge with endpoints $x, y \in V$ then we write $l = \langle x, y \rangle$ and the endpoints are called nearest neighbors.

We assume that $\Phi = \{-1, 1\}$, and $\sigma \in \Omega = \Phi^V$ represents a configuration, i.e., $\sigma = \{\sigma_x \in \Phi : x \in V\}$.

The random field on $\mathcal{T}^{(n)} = \{V^{(n)}, L^{(n)}\}$ is represented in terms of the potential function:

$$H(\sigma) = -J_0 \sum_{\langle x, y \rangle \in \mathcal{S}} \sigma_x \sigma_y - J_1 \sum_{\langle i, j \rangle \in L_1} \sigma_i \sigma_j - J_2 \sum_{\langle l, m \rangle \in L_2} \sigma_l \sigma_m,$$

where the first summation extends over all spins on the surface sites (i.e., those corresponding to nearest-neighbor pairs on upper level 1 and lower level 1), while the second and third summations extend over the upper and lower trees of $\mathcal{T}^{(n)}$, respectively.

Several notable special cases of this system include:

- (a) To recover the system on an open tree, we set $J_0 = 0$. The resulting system corresponds to two independent Ising models on open trees.
- (b) If $J_0 \neq 0$ and $J_1 = J_2$, we obtain the closed symmetric model.

- (c) If we set $J_1 = J_2$ and $J_0 = \infty$, we obtain a spin system on the closed tree, where the upper level 1 and lower level 1 (i.e., the surface levels) have been fused.

Let $\mathbf{B}$ be the σ -algebra generated by cylindrical sets with finite base of Ω^a . For any n we let $\mathbf{B}_{V^{(n)}} = \{\sigma \in \Omega : \sigma|_{V^{(n)}} = \sigma_n\}$ denote the subalgebra of $\mathbf{B}$, where $\sigma|_{V^{(n)}}$ is restriction of σ to $V^{(n)}$ and $\sigma_n : x \in V^{(n)} \mapsto \sigma_n(x)$ an admissible configuration in $V^{(n)}$.

The resulting for $\sigma \in \mathbf{B}_{V^{(n)}}$ Gibbs distribution is then defined by

$$P(\sigma) = \frac{1}{Z^{(n)}} \exp \left(-\frac{1}{kT} H(\sigma) \right),$$

where $Z^{(n)}$ is the partition function.

Taking an advantage of structure of the closed Cayley tree, under some conditions we obtain the following system of equations

$$\begin{cases} x_n = \left(\frac{x_{n-1} + bz_{n-1} + c + bcy_{n-1}}{cx_{n-1} + bcz_{n-1} + 1 + by_{n-1}} \right)^3, \\ y_n = \left(\frac{bcx_{n-1} + cz_{n-1} + b + y_{n-1}}{cx_{n-1} + bcz_{n-1} + 1 + by_{n-1}} \right)^3, \\ z_n = \left(\frac{bx_{n-1} + z_{n-1} + bc + cy_{n-1}}{cx_{n-1} + bcz_{n-1} + 1 + by_{n-1}} \right)^3. \end{cases} \quad (1)$$

where $a = e^{-2h_0}$, $b = e^{-2h_1}$, $c = e^{-2h_2}$ $h_i = J_i/kT$, $i = 0, 1, 2$.

If the system of equations (1) has a unique solution, the corresponding limiting measure is also unique, indicating the absence of a phase transition. On the other hand, if the system has multiple solutions, this implies there is more than one limiting measures, which is characteristic of a phase transition. We obtain the following result

Theorem 1. For the Ising model on the symmetric closed Cayley tree with a branching ratio of three:

- if $0 < b \leq 0.7953$ or $b > 2 + \sqrt{3}$, a phase transition occurs,
- if $0.7953 < b \leq 2 + \sqrt{3}$, a phase transition does not occur.

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p-ADIC GIBBS MEASURES FOR THE BLUM-CAPEL MODEL

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In the present paper we study p -adic (quasi) Gibbs measures for the Blum-Kapel on a Cayley tree. Moreover, we prove that there exists at least one translation-invariant p -adic Gibbs measure for this model on a Cayley tree and we also study p -adic quasi Gibbs measures.

Let $\mathbb{Q}_p$ be the field of p -adic numbers (see e.g.[1]) and $\Phi = \{-1, 0, 1\}$. A configuration σ on $A \subset V$ is defined by the function $x \in A \rightarrow \sigma(x) \in \Phi$. The set of all configurations on A is denoted by Ω_A and $\Omega = \Omega_V$ (see e.g.[2]). The (formal) Hamiltonian of the p -adic Blum-Kapel is defined by

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y), \quad (1)$$

where $J \in B(0, p^{-1/(p-1)})$ is a constant, $\langle x, y \rangle$ stands for nearest neighbor vertices. Assume that $\mathbf{h} : V \rightarrow \mathbb{Q}_p^3$ is a mapping, i.e. $\mathbf{h}_x = (h_{1,x}, h_{2,x}, h_{3,x})$, where $h_{i,x} \in \mathbb{Q}_p$, $i \in \Phi$ and $x \in V$. Given $n \in \mathbb{N}$, we consider a p -adic probability distribution $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x}, \quad (2)$$

where $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor. If p -adic probability distributions (2) are compatibly (see [2]) then a non-Archimedean analogue of the Kolmogorov's extension theorem [3], there exists a unique p -adic *quasi Gibbs measure*.

In real case, the Blum-Kapel model was well-studied by Khatamov N.M. (see e.g. [4]).

A condition for $h_{i,x}$ ensuring the consistency of the measures $\mu_{\mathbf{h}}^{(n)}(\sigma)$ is formulated in the next theorem.

Theorem 1. The sequence of probability distributions (2) is consistent if and only if for $h_{i,x}$ the following equalities hold

$$\begin{cases} h_{+1,x} = \prod_{y \in S(x)} \frac{\lambda h_{+1,y} + \lambda^{-1} h_{-1,y} + 1}{h_{+1,y} + h_{-1,y} + 1}, \\ h_{-1,x} = \prod_{y \in S(x)} \frac{\lambda^{-1} h_{+1,y} + \lambda h_{-1,y} + 1}{h_{+1,y} + h_{-1,y} + 1}, \end{cases} \quad (3)$$

where $\lambda = \exp_p\{J\}$.

Definition 1. If $\forall x \in V$, $\mathbf{h}_x = \mathbf{h}$ then the quantities $\mathbf{h}_x$ are called translation-invariant. The measure $\mu_{\mathbf{h}}$ is called translation-invariant measure if it corresponding to translation-invariant quantities $\mathbf{h}_x$.

Theorem 2. Let $p \neq 3$, $h_{-1,x} = h_{1,x}$. Then there exist a unique translation-invariant p -adic Gibbs measures for the Blum-Kapel model on the Cayley tree of order k .

Let $\kappa_p = \text{Card}\{x \in \mathbb{F}_p : x^k \equiv \frac{p-1}{2} \pmod{p}\}$.

Theorem 3. Let $p \neq 3$, $h_{-1,x} = h_{1,x}$. Then there exist $\kappa_p + 1$ translation-invariant p -adic quasi Gibbs measures for the Blum-Kapel model on the Cayley tree of order k .

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p -Adic quasi Gibbs measures for the SOS model with an external field on the binary tree

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Let $\mathbb{Q}$ be the field of rational numbers. The completion of $\mathbb{Q}$ with respect to the p -adic norm defines the p -adic field $\mathbb{Q}_p$ (see [1]).

Let $\Gamma^k = (V, L)$ be Cayley tree (see [2]) and $\Phi = \{0, \dots, m\}$. A configuration σ on $A \subset V$ is defined by the function $x \in A \rightarrow \sigma(x) \in \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$. For simplicity, we denote $\Omega = \Omega_V$.

A Hamiltonian $H : \Omega \rightarrow \mathbb{Q}_p$ of the p -adic SOS model with an external field is defined by

$$H_n(\sigma_n) = J \sum_{\langle x, y \rangle \in L_n} |\sigma(x) - \sigma(y)|_\infty + \sum_{x \in V_n} \alpha_{\sigma(x), x}$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\alpha_x = (\alpha_{0,x}, \alpha_{1,x}, \dots, \alpha_{m,x}) \in \mathbb{Q}_p^{m+1}$ such that $\alpha_{i,x} \in B(0, p^{-1/(p-1)})$ ($i = \overline{0, m}$) is the external field, $\langle x, y \rangle$ stands for nearest neighbor vertices and $|\cdot|_\infty$ stands for usual absolute value.

We define a function $z : x \rightarrow z_x \in \mathbb{Q}_p^{m+1}$, $\forall x \in V$ and consider p -adic probability measure $\mu_z^{(n)}$ on Ω_{V_n} defined by

$$\mu_z^{(n)}(\sigma_n) = Z_{n,z}^{-1} \exp_p \{H_n(\sigma_n)\} \prod_{x \in W_n} \tilde{z}_{\sigma(x), x}. \quad (2)$$

Here $Z_{n,z}$ is the normalizing constant. We say that the p -adic probability measures (2) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega \in \Omega_{W_n}} \mu_z^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu_z^{(n-1)}(\sigma_{n-1}). \quad (3)$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations.

Theorem 1. The sequence of p -adic probability measures (2) satisfy (3) if and only if for any $x \in V$ the set of quantities $z = \{\tilde{z}_x, x \in V \setminus x_0\}$ satisfy:

$$z_{i,x} = \alpha_{i,x} \prod_{y \in S(x)} \frac{\sum_{j=0}^{m-1} \theta^{|i-j|_\infty} z_{j,y} + \theta^{m-i}}{\sum_{j=0}^{m-1} \theta^{m-j} z_{j,y} + 1}, \quad i = 0, 1, \dots, m-1, \quad (4)$$

here, $\theta = \exp_p(J)$, $\alpha_{i,x} = \exp_p\{\tilde{\alpha}_{i,x} - \tilde{\alpha}_{m,x}\}$ and $z_{i,x} = \alpha_{i,x} \frac{\tilde{z}_{i,x}}{\tilde{z}_{m,x}}$ for $i = 0, 1, \dots, m-1$.

In this case, by the p -adic analogue of the Kolmogorov theorem, there exists a unique p -adic quasi Gibbs measure μ_z on the set Ω such that $\mu_z(\{\sigma|_{V_n} \equiv \sigma_n\}) = \mu_z^{(n)}(\sigma_n)$ for all $n \in \mathbb{N}$ and $\sigma_n \in \Omega_{V_n}$ (see [2]).

The set of all p -adic quasi Gibbs measures associated with functions $z = \{\tilde{z}_x, x \in V\}$ is denoted by $\mathcal{QG}(H)$.

Definition 1.[3] If for some function z the measures $\mu_z^{(n)}$ satisfy the consistency condition, then there exists a unique p -adic probability measure, denoted by μ_z , since it depends on z . Such a measure μ_z is called *p -adic quasi Gibbs measure*.

Definition 2.[4] A *phase transition* is said to occur if there exist at least two different p -adic quasi Gibbs measures $\mu, \nu \in \mathcal{QG}(H)$ such that μ is bounded and ν is unbounded. In other words, two different functions s and h may be found defined on $\mathbb{N}$, for which corresponding measures μ_s and μ_h exist; one of these measures is bounded, while the other is unbounded.

In this work, we consider translation-invariant p -adic quasi Gibbs measures (TIpQGMs) for the three states p -adic SOS model with an external field on the Cayley tree of order two and we get the following theorem:

Theorem 2. Let $p > 3$ and $p \equiv 7(\text{mod } 24)$ or $p \equiv 13(\text{mod } 24)$. Then for the three-state p -adic SOS model with an external field on the Cayley tree of order two, a phase transition occurs.

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Gibbs Measures for the Modified Three-State SOS Model

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The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., such that from each vertex of which issues exactly $k + 1$ edges. We will denote by V the set of the vertices of the tree and by L the set of edges of the tree. The distance on this tree is defined as the number of nearest neighbour pairs of the minimal path between the vertices x and y (where path is a collection of nearest neighbour pairs, two consecutive pairs sharing at least a given vertex) and denoted by $d(x, y)$ [1,2].

We consider the three-state modified SOS model on a Cayley tree of order two, where each vertex is assigned a spin variable that takes values in the local state space $\Phi = \{-1, 0, 1\}$. For any subset $A \subset V$, a spin configuration σ_A on A is defined as a function $x \in A \mapsto \sigma(x) \in \Phi$. The set of all configurations is given by $\Omega_A = \Phi^A$, while the full configuration space is denoted as $\Omega = \Phi^V$ with $\sigma = \sigma_V$.

We define a *periodic configurations* as a configuration $\sigma \in \Omega$ that is invariant under a subgroup of shifts $K \subset G_k$ of finite index. More precisely, a configuration $\sigma \in \Omega$ is said to be K -periodic if $\sigma(xy) = \sigma(x)$ for any $x \in G_k$ and $Y \in K$. For a given periodic configuration, the index of the subgroup is called the *period of the configuration*. A configuration that is invariant under all shifts is said to be *translation-invariant*.

The modified SOS model is defined by the formal Hamiltonian

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)| \cos\left(\pi(\sigma(x) - \sigma(y))\right), \quad (1)$$

where $J \in \mathbb{R}$ is the coupling constant.

Let $h : x \mapsto h_x = (h_{-1,x}, h_{0,x}, h_{1,x}) \in \mathbb{R}^3$ be a vector function of $x \in V$. Consider a probability distribution $\mu^{(n)}$ on Ω_{V_n} :

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}), \quad (2)$$

where $\sigma_n \in \Omega_{V_n}$ and

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp\left(-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}(x), x}\right).$$

A probability distribution $\mu^{(n)}$ is said to be compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu^{(n-1)}(\sigma_{n-1}), \quad (3)$$

where $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$.

Under condition (3), there exists a unique measure μ on Φ^V such that $\forall n$ and $\sigma_n \in \Phi^{V_n}$

$$\mu(\{\sigma \mid_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian H and $x \mapsto h_x$, $x \neq x^0$.

We describe SGMs, i.e., characterize such measures with respect to any finite-index normal subgroup of G_k . Studying $G_k^{(2)}$ -periodic solution of the model (1) is reduced to solve the following system:

$$\begin{cases} z_0 = \left(\frac{t_0 + \theta t_1 + \theta^{-2}}{\theta^{-2} t_0 + \theta t_1 + 1} \right)^k, \\ z_1 = \left(\frac{\theta t_0 + t_1 + \theta}{\theta^{-2} t_0 + \theta t_1 + 1} \right)^k, \\ t_0 = \left(\frac{z_0 + \theta z_1 + \theta^{-2}}{\theta^{-2} z_0 + \theta z_1 + 1} \right)^k, \\ t_1 = \left(\frac{\theta z_0 + z_1 + \theta}{\theta^{-2} z_0 + \theta z_1 + 1} \right)^k, \end{cases} \quad (4)$$

where $G_k^{(2)} = \{x \in G_k : |x| \text{--even}\}$.

From the first and third equations in (4), we easily see that $(z_0; t_0) = (1; 1)$ is a solution of system of equations for every θ, z_1 and t_1 . In this case, from the second and fourth equations in (4), we obtain the system of equations

$$z_1 = \left(\frac{2\theta + t_1}{\theta^{-2} + 1 + \theta t_1} \right)^k, \quad t_1 = \left(\frac{2\theta + z_1}{\theta^{-2} + 1 + \theta z_1} \right)^k. \quad (5)$$

Denote

$$f(x) = \left(\frac{2\theta + x}{\theta^{-2} + 1 + \theta x} \right)^k.$$

Then the system of equations (5) becomes

$$f(f(z_1)) - z_1 = 0. \quad (6)$$

Obviously, values z_1 satisfying $f(z_1) = z_1$ satisfy (6), and for these values correspond translation-invariant Gibbs measures which obtained in [3]. The remaining solutions correspond to $G_k^{(2)}$ -periodic Gibbs measure that are not translation-invariant. In the case $k = 2$ for $G_2^{(2)}$ -periodic (non-translation-invariant) Gibbs measures, we obtain the following result after solving equation (6):

Theorem. Let $\theta_{cr} \approx 2.312345325$. If $\theta > \theta_{cr}$, then there are at least two $G_2^{(2)}$ -periodic (non-translation-invariant) Gibbs measures.

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New non-translation-invariant Gibbs Measures for the Ising Model on Cayley Tree

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Let $\tau^k = (V, L)$, $k \geq 1$, be the Cayley tree of order k , i.e. an infinite graph such that every vertex of which is incident to exactly $k + 1$ edges. Here V is the set of all vertices, L is the set of all edges of the tree τ^k . It is known that τ^k can be represented as a group G_k , which is the free product of $k + 1$ cyclic groups of the second order [1], [2].

For an arbitrary point $x^0 \in V$ we set

$$W_n = \{x \in V | d(x^0, x) = n\}, V_n = \bigcup_{m=0}^n W_m,$$

$$L_n = \{\langle x, y \rangle \in L | x, y \in V_n\},$$

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n$$

where $d(x, y)$ is the distance between the vertices x and y on the Cayley tree, i.e. the number of edges in the shortest path joining the vertices x and y .

Let $\Phi = \{-1, 1\}$ and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e. $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. Let $A \subset V$. We let Ω_A denote the space of configurations defined on the set A and taking values in Φ .

We consider the Hamiltonian of the Ising model:

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y), \quad (1)$$

where $J \in \mathbb{R}$, $\sigma(x) \in \Phi$ and $\langle x, y \rangle$ are nearest neighbors.

We are going to describe new Gibbs measures of the Ising model on the semi-infinite Cayley tree of order $k = 2$. We define the set of quantities $h = \{h_x, x \in V\}$ as follows:

(A) If on the vertex of the x , we have $h_x = h_3$, then at each vertex of $S(x)$ we put the value h_1 . If on the vertex of the x , we have $h_x = h_1$, then at one vertex of $S(x)$ we put the value h_2 , and on another vertex of $S(x)$ we put value of 0. If on the vertex of the x , we have $h_x = h_2$, then at one vertex of $S(x)$ we put the value h_3 , and on another vertex of $S(x)$ we put one of the value of 0. If on the vertex of the x , we have $h_x = 0$, then at each vertex from $S(x)$ we put the value 0.

It is easy to see that the boundary conditions defined by the **(A)** are compatible iff the following system of equations is satisfied:

$$\begin{cases} h_1 = f(h_2, \theta), \\ h_2 = f(h_3, \theta), \\ h_3 = 2f(h_1, \theta). \end{cases} \quad (2)$$

It is clear that $h_1 = h_2 = h_3 = 0$ is a solution of (2). Using the formula $f(h, \theta) = \operatorname{arctanh}(\theta \tanh h) = \frac{1}{2} \ln \frac{(1+\theta)e^{2h} + (1-\theta)}{(1-\theta)e^{2h} + (1+\theta)}$, and introducing the notation $\alpha = \frac{1-\theta}{1+\theta}$, $z_i = e^{2h_i}$, $i = 1, 2, 3$.

From equation (2), we derive the following system of equations:

$$\begin{cases} z_1 = \frac{z_2 + \alpha}{\alpha z_2 + 1}, \\ z_2 = \frac{z_3 + \alpha}{\alpha z_3 + 1}, \\ z_3 = \left(\frac{z_1 + \alpha}{\alpha z_1 + 1} \right)^2. \end{cases} \quad (3)$$

By analyzing system of equation (3), we get the following:

Theorem 1. The following results concerning the Gibbs measures for the Ising model on the Cayley tree are established: Let $k = 2$.

- if $\theta \in \left(\frac{1 - \sqrt[3]{2} + \sqrt[3]{4}}{2 + \sqrt[3]{2} - \sqrt[3]{4}}; 1 \right)$, then there are three Gibbs measures μ_0 , μ_1 and μ_2 , which two of them are non-translation-invariant Gibbs measures;
- if $\theta \in (-1; 0) \cup \left(0; \frac{1 - \sqrt[3]{2} + \sqrt[3]{4}}{2 + \sqrt[3]{2} - \sqrt[3]{4}} \right]$, then there exists a unique translation-invariant Gibbs measure μ_0 .

Remark 1. The translation-invariant measure corresponding to $(z_1, z_2, z_3) = (1, 1, 1)$ found in Theorem 1 was studied in [2],[3],[4]. The non-translation-invariant measures mentioned in the of Theorem 1 are new Gibbs measures.

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FUNCTIONAL EQUATIONS FOR THE TRIPLE MIXED-SPIN ISING MODEL ON THE CAYLEY TREE

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Mixed-spin systems, combining different spin magnitudes on a lattice, have attracted extensive attention across both experimental and theoretical studies, as they exhibit rich phenomena including compensation points, reentrant transitions, and exotic critical behavior [1,2]. In the present work, we obtain a set of system of equations for the triple mixed-spin Ising model on the Cayley tree.

Let $\Gamma^k = (V, L)$ denote a semi-infinite Cayley tree of order $k \geq 1$ with a designated root vertex $x^{(0)}$. Each vertex in the tree has exactly $k+1$ edges, except for the root $x^{(0)}$, which has exactly k edges. Here, V denotes the set of vertices and L denotes the set of edges. Two vertices x and y are said to be *nearest neighbors*, denoted by $l = \langle x, y \rangle$, if there exists an edge in L connecting them. A sequence of nearest-neighbor pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from vertex x to vertex y . The *distance* $d(x, y)$ between two vertices $x, y \in V$ is defined as the length of the shortest path connecting them. For each non-negative integer n , define the following subsets of the Cayley tree:

$$W_n = \{x \in V \mid d(x, x^{(0)}) = n\}, \quad V_n = \bigcup_{m=0}^n W_m, \quad L_n = \{\langle x, y \rangle \in L \mid x, y \in V_n\}.$$

The *set of direct successors* of a vertex $x \in W_n$ is defined as $S(x) = \{y \in W_{n+1} \mid d(x, y) = 1\}$. We define levels of the tree as follows:

$$\Gamma_i^k = \{x \in V \mid d(x^{(0)}, x) \equiv i \pmod{3}\}, \quad i = 0, 1, 2.$$

In this work, we consider spin state spaces given by $\Psi = \{-\frac{1}{2}, \frac{1}{2}\}$, $\Phi = \{-1, 0, 1\}$, $\Upsilon = \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$. The associated configuration spaces are defined by $\Omega_i = \Phi^{\Gamma_i^k}$, $i = 0, 1, 2$. Additionally, we define the *finite-volume configuration spaces* as $\Omega_{i,n} = \Phi^{\Gamma_i^k \cap V_n}$, $i = 0, 1, 2$. The full configuration space of the model is given by $\Xi = \Omega_0 \times \Omega_1 \times \Omega_2$. Elements of Ω_0 are denoted by $\sigma(x)$ for $x \in \Gamma_0^k$, and elements of Ω_1 are denoted by $s(x)$ for $x \in \Gamma_1^k$, elements of Ω_2 are denoted by $\varphi(x)$ for $x \in \Gamma_2^k$. We define the Ising model with triple mixed spins $(\frac{1}{2}, 1, \frac{3}{2})$ governed by the Hamiltonian

$$H(\xi) = -J \sum_{\langle x, y \rangle} \xi(x)\xi(y), \quad \xi \in \Xi.$$

Let us denote $h = \{h_x\}_{x \in \Gamma^k}$, where

$$h_x = \begin{cases} \tilde{h}_x, & x \in \Gamma_0^k, \\ \ddot{h}_x, & x \in \Gamma_1^k, \\ \hat{h}_x, & x \in \Gamma_2^k, \end{cases}$$

with $\tilde{h}_x = (\tilde{h}_{-\frac{1}{2},x}, \tilde{h}_{\frac{1}{2},x})$, $\ddot{h}_x = (\ddot{h}_{-1,x}, \ddot{h}_{0,x}, \ddot{h}_{+1,x})$ and $\hat{h}(x) = (\hat{h}_{-\frac{3}{2}}(x), \hat{h}_{-\frac{1}{2}}(x), \hat{h}_{\frac{1}{2}}(x), \hat{h}_{\frac{3}{2}}(x))$.

For each $n \geq 1$, the finite-volume Gibbs measure μ_n^h is defined as

$$\mu_n^h(\xi) = \frac{1}{Z_n} \exp \left\{ -\beta H_n(\xi) + \sum_{x \in W_n} h_{\xi(x)}(x) \right\}, \quad \xi \in \Xi_n := \Omega_{0,n} \times \Omega_{1,n} \times \Omega_{2,n}, \quad (1)$$

where Z_n is the partition function.

The sequence of measures $\{\mu_n^h\}$ is said to be *compatible* if, for all $n \geq 1$ and $\xi_{n-1} \in \Xi_{n-1}$, the following condition holds:

$$\sum_{\omega \in \Xi^{W_n}} \mu_n^h(\xi_{n-1} \vee \omega) = \mu_{n-1}^h(\xi_{n-1}). \quad (2)$$

Here, $\xi_{n-1} \vee \omega$ denotes the concatenation of the configurations. In this setting, there exists a unique probability measure μ on Ξ such that for all n and $\xi_n \in \Xi_n$, $\mu(\{\xi \mid \xi|_{V_n} = \xi_n\}) = \mu_n^h(\xi_n)$.

We obtain the following result

Theorem 1. The sequence of measures $\{\mu_n^h\}$, $n = 1, 2, \dots$ defined by (1) is compatible if and only if, for all $x \in V$, the following system of equations holds:

$$\left\{ \begin{array}{l} X_x = \prod_{y \in S(x)} \frac{Y_y + \theta + \theta^2 Z_y}{\theta^2 Y_y + \theta + Z_y}, \\ Y_x = \prod_{y \in S(x)} \frac{\theta^6 T_y + \theta^4 + \theta^2 U_y + V_y}{\theta^3 (T_y + 1 + U_y + V_y)}, \\ Z_x = \prod_{y \in S(x)} \frac{T_y + \theta^2 + \theta^4 U_y + \theta^6 V_y}{\theta^3 (T_y + 1 + U_y + V_y)}, \\ T_x = \prod_{y \in S(x)} \frac{X_y + \theta^3}{\theta^2 + \theta X_y}, \\ U_x = \prod_{y \in S(x)} \frac{\theta X_y + 1}{X_y + \theta}, \\ V_x = \prod_{y \in S(x)} \frac{\theta^3 X_y + 1}{\theta^2 + \theta X_y}. \end{array} \right.$$

where

$$\begin{aligned} \theta &= \exp\left(\frac{1}{2}\beta J\right), \\ X_x &= \exp(\tilde{h}_{\frac{1}{2},x} - \tilde{h}_{-\frac{1}{2},x}), \quad Y_x = \exp(h_{-1,x} - h_{0,x}), \\ Z_x &= \exp(h_{1,x} - h_{0,x}), \quad T_x = \exp(\hat{h}_{-\frac{3}{2},x} - \hat{h}_{-\frac{1}{2},x}), \\ U_x &= \exp(\hat{h}_{\frac{1}{2},x} - \hat{h}_{-\frac{1}{2},x}), \quad V_x = \exp(\hat{h}_{\frac{3}{2},x} - \hat{h}_{-\frac{1}{2},x}). \end{aligned}$$

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Periodic Gibbs measures for the Ising model with Ising-type interaction on a Cayley tree of order two.

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The Ising model has continued to provide new problems and insights for nearly a century, remaining a cornerstone in the study of statistical mechanics and phase transitions [1,2]. In the work, we study periodic splitting Gibbs measures for the Ising model with Ising-type interactions on the Cayley tree of order two.

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., such that from each vertex of which issues exactly $k + 1$ edges. We will denote by V the set of the vertices of the tree and by L the set of edges of the tree. The distance on this tree is defined as the number of nearest neighbour pairs of the minimal path between the vertices x and y (where path is a collection of nearest neighbour pairs, two consecutive pairs sharing at least a given vertex) and denoted by $d(x, y)$.

The mixed-type Ising model with Ising-type interaction is defined by the formal Hamiltonian

$$H(\sigma) = -J_1 \sum_{\langle x, y \rangle \subset V} \sigma(x)\sigma(y) - J_2 \sum_{\langle x, y \rangle \subset V} (\sigma^2(x)\sigma(y) + \sigma(x)\sigma^2(y)),$$

where the first and second sums run over nearest neighbour vertices $\langle x, y \rangle$, the spins $\sigma(x)$ take values in the set $\Phi = \{-1, 1\}$, $(J_1, J_2) \in R^2$ are the coupling constants.

As the root of the tree, we fix a vertex $x^0 \in V$. Let $V_n := \{x \in V : d(x, x^0) \leq n\}$, $W_n := \{x \in V : d(x, x^0) = n\}$, be respectively the ball and the sphere of radius n with center at x^0 . For $x \in W_n$ let $S(x) = \{y \in W_{n+1} : d(y, x) = 1\}$ be the set of direct successors of x .

For $x \in V \setminus \{x^0\} \mapsto h_x = (h_{-1,x}, h_{1,x}) \in R^2$ we define the (finite-dimensional) Gibbs distributions by the formula

$$\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x} \right\} \quad (1)$$

where Z_n is the corresponding partition function.

The probability distributions (1) are compatible if for all $\sigma_{n-1} \in \Phi^{V_{n-1}}$ one has $\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1}, \omega_n) = \mu_{n-1}(\sigma_{n-1})$. Under the condition, there exists a unique measure μ on Φ^V such that $\forall n$ and $\sigma_n \in \Phi^{V_n} : \mu(\{\sigma \mid_{V_n} = \sigma_n\}) = \mu_n(\sigma_n)$. Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian H and $x \mapsto h_x$, $x \neq x^0$.

The compatibility condition is satisfied if and only if $\forall x \in V \setminus \{x^0\}$ the following vector identity holds

$$\tilde{h}_x = \sum_{y \in S(x)} \ln \frac{\theta^2 \eta^2 \exp(\tilde{h}_y) + 1}{\exp(\tilde{h}_y) + \frac{\theta^2}{\eta^2}} \quad (2)$$

where $\theta = \exp(\beta J_1)$, $\eta = \exp(\beta J_2)$, $\tilde{h}_x = h_{1,x} - h_{-1,x}$.

It is known (see, e.g., [1]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k + 1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$.

Definition 1. Let K be a subgroup of G_k . We say that a collection of vectors $\tilde{h} = \{\tilde{h}_x : x \in G_k\}$ is K -periodic if $\tilde{h}_{yx} = \tilde{h}_x$ for all $x \in G_k$ and $y \in K$. A SGM is called K -periodic if it corresponds to a K -periodic collection $\tilde{h}$.

Let $G_k^{(2)} = \{x \in G_k : \text{the length of word } x \text{ is even}\}$. One can show that for any normal subgroup of finite index K in G_k , each K -periodic SGM for the model is either translation-invariant (TI) or $G_k^{(2)}$ -periodic. Thus, in the present work, we study $G_k^{(2)}$ -periodic SGMs in the case $k = 2$. This problem is reduced to describing the solutions of the system

$$\begin{cases} z = \frac{\theta^2 \eta^2 t^2 + 1}{t^2 + \frac{\theta^2}{\eta^2}}, \\ t = \frac{\theta^2 \eta^2 z^2 + 1}{z^2 + \frac{\theta^2}{\eta^2}}. \end{cases} \quad (3)$$

For $z = t$ the system (3) is reduced to

$$z^3 - \theta^2 \eta^2 z^2 + \frac{\theta^2}{\eta^2} z - 1 = 0,$$

which studied in [3], describes TISGMs. After some algebras, from (3), we obtain the quadratic equation

$$\theta^2 \eta^2 (\theta^2 \eta^6 + 1) z^2 + \eta^4 (\theta^4 - 1) z + \theta^2 \eta^6 + \theta^4 = 0,$$

which describes periodic SGMs.

As a result, we prove the following theorem

Theorem 1. *For the Ising model with Ising-type interaction on the Cayley tree of order two the following assertions hold: there exist $\theta_1 = \frac{2}{\sqrt{3}} - 1$, $\theta_2 = \frac{1}{\sqrt{3}}$ and*

$$\eta_1(\theta) = \sqrt[6]{\frac{-3\theta^8 - 6\theta^4 + 1 - (1 - \theta^4)\sqrt{(1 - 9\theta^4)(1 - \theta^4)}}{8\theta^6}},$$

$$\eta_2(\theta) = \sqrt[6]{\frac{-3\theta^8 - 6\theta^4 + 1 + (1 - \theta^4)\sqrt{(1 - 9\theta^4)(1 - \theta^4)}}{8\theta^6}},$$

such that

1. *For $\eta \in (\eta_1(\theta), \eta_2(\theta))$ and $\theta \in (\theta_1, \theta_2)$, there are two $G_2^{(2)}$ -periodic (not translation-invariant) Gibbs measures;*
2. *For the remaining cases, all the $G_2^{(2)}$ -periodic Gibbs measures are translation-invariant.*

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PERIODIC LIMITING GIBBS MEASURES FOR THE SYMMETRIC ISING MODEL ON THE CLOSED CAYLEY TREE

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We study periodic limiting Gibbs measures for the symmetric Ising model on the closed Cayley tree of branching ratio two. The problem can be formulated as the following system of six equations with unknowns:

$$\left\{ \begin{array}{l} x_2 = \frac{(x_1 + Bz_1 + B + B^2 y_1)^2}{(Bx_1 + B^2 z_1 + 1 + By_1)^2}, \\ y_2 = \frac{(B^2 x_1 + Bz_1 + B + y_1)^2}{(Bx_1 + B^2 z_1 + 1 + By_1)^2}, \\ z_2 = \frac{(Bx_1 + z_1 + B^2 + By_1)^2}{(Bx_1 + B^2 z_1 + 1 + By_1)^2}, \\ x_1 = \frac{(x_2 + Bz_2 + B + B^2 y_2)^2}{(Bx_2 + B^2 z_2 + 1 + By_2)^2}, \\ y_1 = \frac{(B^2 x_2 + Bz_2 + B + y_2)^2}{(Bx_2 + B^2 z_2 + 1 + By_2)^2}, \\ z_1 = \frac{(Bx_2 + z_2 + B^2 + By_2)^2}{(Bx_2 + B^2 z_2 + 1 + By_2)^2}. \end{array} \right. \quad (1)$$

We consider the mapping $W : R^6 \rightarrow R^6$ which defined by RHS of (1).

Lemma. The following sets are invariant with respect to the mapping W :

$$I_1 = \{(x_1, y_1, z_1, x_2, y_2, z_2) \in R^6 : x_1 = 1, y_1 = z_1, x_2 = 1, y_2 = z_2\},$$

$$I_2 = \{(x_1, y_1, z_1, x_2, y_2, z_2) \in R^6 : x_1 = z_1, y_1 = 1, x_2 = z_2, y_2 = 1\},$$

$$I_3 = \{(x_1, y_1, z_1, x_2, y_2, z_2) \in R^6 : x_1 = y_1, z_1 = 1, x_2 = y_2, z_2 = 1\}.$$

It is easy to verify that $(1, 1, 1, 1, 1, 1)$ is a solution of the system of equations (1) independent of the parameter B .

On the invariant set I_1 , from the system of equations (1) we obtain the following system of equations:

$$\left\{ \begin{array}{l} y_2 = \frac{(y_1 + B)^2}{(By_1 + 1)^2}, \\ y_1 = \frac{(y_2 + B)^2}{(By_2 + 1)^2}. \end{array} \right. \quad (2)$$

From the system of equations (2) we have the following equation for y_1 :

$$(y_1 - 1) (B^2 y_1^2 + (B^2 + 2B - 1)y_1 + B^2) (y_1^2 - (B^2 - 2B - 1)y_1 + 1) = 0. \quad (3)$$

The positive solutions of equation (3) are as follows, depending on the parameter B :

$$y_1^{(1)}, y_1^{(2,3)} \text{ if } B \in \left(0; \frac{1}{3}\right);$$

$$y_1^{(1)} \text{ if } B \in \left[\frac{1}{3}; 3\right];$$

$$y_1^{(1)}, y_1^{(4,5)} \text{ if } B \in \left(3; \infty\right),$$

where

$$y_1^{(1)} = 1, y_1^{(2,3)} = \frac{-B^2 - 2B + 1 \pm (1 - B)\sqrt{(B+1)(1-3B)}}{2B^2},$$

$$y_1^{(4,5)} = \frac{B^2 - 2B - 1 \pm (B-1)\sqrt{(B+1)(B-3)}}{2}.$$

Now, using the first equation in the system of equations (2), we find the positive $y_2^{(i)}$ corresponding to the $y_1^{(i)}, i = \overline{1, 5}$:

$$y_2^{(1)} = 1 = y_1^{(1)}, y_2^{(2,3)} = \frac{((1-2B)\sqrt{B+1} \pm \sqrt{1-3B})^2}{B^2(\sqrt{B+1} \pm \sqrt{1-3B})^2} = y_1^{(2,3)},$$

$$y_2^{(4,5)} = \frac{(\sqrt{B+1} \pm \sqrt{B-3})^2}{((B-2)\sqrt{B+1} \pm B\sqrt{B-3})^2} \neq y_1^{(4,5)}.$$

The solutions on the remaining invariant sets can be derived analogously to those obtained on the invariant set I_1 . By solving system (1) on the invariant sets specified in the Lemma, we arrive at the following theorem:

Theorem. Let N_{sIp} be the number of non-translation-invariant periodic limiting Gibbs measures for the symmetric Ising model on the closed Cayley tree of branching ratio two. Then the following statements hold for N_{sIp} :

$$N_{sIp} \geq \begin{cases} 2, & \text{if } B \in (0; 3 - 2\sqrt{2}), \\ 4, & \text{if } B \in [3 - 2\sqrt{2}; \frac{1}{3}) \cup (3; 3 + 2\sqrt{2}], \\ 6, & \text{if } B \in (3 + 2\sqrt{2}; \infty). \end{cases}$$

Remark. The translation-invariant limiting Gibbs measures for the symmetric Ising model on closed Cayley trees of branching ratio two were studied in [1]. On the invariant sets we considered above, a non-translation-invariant periodic limiting Gibbs measure did not arise when $B \in [\frac{1}{3}; 3]$, but it may exist outside the cases we have seen.

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Periodic Ground States for the Coupled Ising-Potts Model on the Cayley Tree of Order Two

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In this paper we consider Ising model's spins $s(x) \in I = \{-1, 1\}$ and Potts model's spins $\sigma(x) \in \Phi := \{1, 2, 3\}$, both are assigned to the vertices of Cayley tree. A coupled configuration (s, σ) on V is then defined as a function $x \in V \rightarrow (s(x), \sigma(x)) \in I \times \Phi$; the set of all configurations is $I^V \times \Phi^V$.

Consider the quotient group $G_k/G_k^* = \{H_1, \dots, H_r\}$, where G_k^* is a normal subgroup of index r with $r \geq 1$.

Definition 1. A configuration $(s(x), \sigma(x))$ is said to be G_k^* -periodic if $(s(x), \sigma(x)) = (s_i, \sigma_i)$ for all $x \in G_k$ with $x \in H_i$. A G_k -periodic configuration is said to be translation-invariant.

The following Hamiltonian is called the coupled Ising-Potts model:

$$H(s, \sigma) = -J_1 \sum_{\langle x, y \rangle \in L} s(x)s(y)\delta_{\sigma(x)\sigma(y)} - J_2 \sum_{\substack{d(x, y)=2: \\ x, y \in V}} s(x)s(y)\delta_{\sigma(x)\sigma(y)}, \quad (1)$$

where $J_1, J_2 \in \mathbb{R}$ is constant and δ_{ij} is the Kroneker's symbol.

Let M be the set of all unit balls with vertices in V and $S_1(x)$ be the set of all nearest neighboring vertices of $x \in V$. We denote the restriction of a configuration (s, σ) to the ball $b \in M$ by (s_b, σ_b) . The energy of configuration (s_b, σ_b) on b is defined by the formula

$$U(s_b, \sigma_b) = -\frac{J_1}{2} \sum_{x \in S_1(c_b)} s(x)s(c_b)\delta_{\sigma(x)\sigma(c_b)} - J_2 \sum_{\substack{d(x, y)=2: \\ x, y \in b}} s(x)s(y)\delta_{\sigma(x)\sigma(y)}, \quad (2)$$

We have the following lemma.

Lemma 1. Let $k = 2$. For each configuration (φ_b, ψ_b) , we have the following $U(\varphi_b, \psi_b) \in \{U_i : i = 1, 2, \dots, 13\}$ where $U_1 = -\frac{3}{2}J_1 - 3J_2$; $U_2 = -\frac{1}{2}J_1$; $U_3 = -J_1 - J_2$; $U_4 = -\frac{1}{2}J_1 - J_2$; $U_5 = -3J_2$; $U_6 = \frac{3}{2}J_1 - 3J_2$; $U_7 = \frac{1}{2}J_1$; $U_8 = J_1 - J_2$; $U_9 = \frac{1}{2}J_1 - J_2$; $U_{10} = -\frac{1}{2}J_1 + J_2$; $U_{11} = \frac{1}{2}J_1 + J_2$; $U_{12} = J_2$; $U_{13} = -J_2$.

Definition 2. A configuration (φ, ψ) is called a *ground state* for the Hamiltonian (1), if $U(\varphi_b, \psi_b) = \min\{U_i : i = 1, 2, \dots, 13\}$, for any $b \in M$.

We denote

$$A_\xi = \{(J_1, J_2) \in \mathbb{R}^2 : U_\xi = \min\{U_i : i = 1, 2, \dots, 13\}.$$

Calculations show that:

$$A_1 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, J_2 \geq -\frac{1}{4}J_1\};$$

$$A_2 = A_4 = A_7 = A_9 = A_{13} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, J_2 = 0\};$$

$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, J_2 = -\frac{1}{4}J_1\}; \quad A_5 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, J_2 \geq 0\};$$

$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, J_2 \geq \frac{1}{4}J_1\}; \quad A_8 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, J_2 = \frac{1}{4}J_1\};$$

$$A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, J_2 \leq -\frac{1}{4}J_1\}; \quad A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, J_2 \leq \frac{1}{4}J_1\};$$

$$A_{12} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, J_2 \leq 0\};$$

and $\bigcup_{i=1}^{13} A_i = \mathbb{R}^2$.

In this subsection we study all $G_2^{(2)}$ -periodic ground states for the coupled Ising-Potts model on the Cayley tree of order three, where $G_2^{(2)} = \{x \in G_2 : |x| - \text{is even}\}$, where $|x|$ means length of the word x .

All $G_2^{(2)}$ -periodic configurations have the following form:

$$(s(x), \sigma(x)) = \begin{cases} (s_0, \sigma_0), & \text{if } x \in G_2^{(2)}, \\ (s_1, \sigma_1), & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \quad (3)$$

where $(s_i, \sigma_i) \in I \times \Phi$, $i = 0, 1$.

Theorem 1. Let $k = 2$. Then for the coupled Ising-Potts model the following assertions hold:

i) If $(J_1, J_2) \in A_1$, then the following six translation-invariant configurations $(s(x), \sigma(x)) = (i, j)$, $i \in \{\pm 1\}$, $j \in \{1, 2, 3\}$, $\forall x \in V$ are translation-invariant ground states;

ii) If $(J_1, J_2) \in A_5$ and $k_1 \neq k_2$, $k_{1,2} = 1, 2, 3$, then the following $G_2^{(2)}$ -periodic configurations

$$(s(x), \sigma(x)) = \begin{cases} (i, 1), & \text{if } x \in G_2^{(2)}, \\ (i, j), & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \quad (s(x), \sigma(x)) = \begin{cases} (i, 2), & \text{if } x \in G_2^{(2)}, \\ (i, j_0), & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases}$$

$$(s(x), \sigma(x)) = \begin{cases} (i, 3), & \text{if } x \in G_2^{(2)}, \\ (i, j_1), & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \quad (s(x), \sigma(x)) = \begin{cases} (1, k_1), & \text{if } x \in G_2^{(2)}, \\ (-1, k_2), & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases}$$

$$(s(x), \sigma(x)) = \begin{cases} (-1, k_1), & \text{if } x \in G_2^{(2)}, \\ (1, k_2), & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases}$$

where $i = 1, -1$; $j = 2, 3$; $j_0 = 1, 3$; $j_1 = 1, 2$; are $G_2^{(2)}$ -periodic ground states;

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THE FIXED POINTS OF A TWO-VARIABLE p -ADIC DYNAMICAL SYSTEM

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Let $\mathbb{Q}$ be a field of rational numbers. For a fixed prime number p , every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$ where, $r, n \in \mathbb{Z}$, m is a positive integer, and $(n, p) = 1$ and $(m, p) = 1$, where number r is called a p -order of x and it is denoted by $\text{ord}_p(x) = r$. The p -adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The norm of $|\cdot|_p$ is non-Archimedean, meaning that, it satisfies the *strong triangle inequality*:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \quad \forall x, y \in \mathbb{Q}.$$

Completing the field of rational numbers $\mathbb{Q}$ with respect to the p -adic norm yields the p -adic field $\mathbb{Q}_p$. Ostrowski's theorem states that any possible completion of $\mathbb{Q}$ by norm is either the field of real numbers $\mathbb{R}$ or one of the p -adic fields $\mathbb{Q}_p$. Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots)$$

where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 \neq 0$, $x_j \in \{0, 1, \dots, p-1\}$, $j \in \mathbb{N}$ (see [1]). In this case $|x|_p = p^{-\gamma(x)}$.

In this paper, we study a two-variable dynamical system.

Let $G : \mathbb{Q}_p^2 \rightarrow \mathbb{Q}_p^2$ be an analytic mapping and let $x^* \in \mathbb{Q}_p^2$ be a fixed point, i.e.

$$G(x^*) = x^*.$$

Denote by $J_G(x^*)$ the Jacobian matrix of G at x^* . Let its eigenvalues be $\lambda_1, \lambda_2 \in \mathbb{C}_p$ (the p -adic complex extension).

Definition 1. ([2])

- If $\max\{|\lambda_1|_p, |\lambda_2|_p\} < 1$, then x^* is said an **attracting fixed point**;
- If $\min\{|\lambda_1|_p, |\lambda_2|_p\} > 1$, then x^* is said a **repelling fixed point**;
- If $|\lambda_1|_p < 1 < |\lambda_2|_p$ (or vice versa), then x^* is said a **saddle point**;
- If for at least one λ_i we have $|\lambda_i|_p = 1$, then the nature of the point requires a special analysis; it may be a **neutral (indifferent) fixed point**.

We consider the following map: $G : \mathbb{Q}_p^2 \rightarrow \mathbb{Q}_p^2$ is defined by $G(x, y) = (G_1(x, y), G_2(x, y))$ with

$$\begin{aligned} G_1(x, y) &= \left(\frac{\theta x + y + \eta}{x + y + \theta \eta} \right)^2, \\ G_2(x, y) &= \left(\frac{\theta y + x + \eta}{x + y + \theta \eta} \right)^2. \end{aligned} \tag{1}$$

where $\theta, \eta \in B(1, p^{-1/(p-1)}) \setminus 1$ are parameters.

It should be noted that, for each fixed point of the operator defined in the form (1), there corresponds a unique p -adic Gibbs measure for the three-state Potts model with an external field on the Cayley tree of order two[3].

Theorem 1. Let $p > 3$, $|\eta - 1|_p < |\theta - 1|_p$, $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$. Then there are four fixed points of (1). Moreover, all fixed points are saddle.

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WEAKLY PERIODIC GIBBS MEASURES FOR THE INHOMOGENEOUS ISING MODEL ON THE CAYLEY TREE OF ORDER TWO

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The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a cycles-free graph, such that from each vertex of which issues exactly $k + 1$ edges. Let $\Gamma^k = (V, L)$, where V is the set of vertices and L is the set of edges.

It is known that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k + 1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$ [1].

The Hamiltonian of the inhomogeneous Ising model [2] has the following form:

$$H(\sigma) = \sum_{\langle x, y \rangle \in L} J_{xy} \sigma(x) \sigma(y), \quad (1)$$

where summation is over all pairs of nearest neighbors and $J_{xy} \in R$, $\sigma(x) \in \{-1, 1\}$ for any $x, y \in V$. It is known that each Gibbs measure of the inhomogeneous Ising model corresponds to a set $h = \{h_x, x \in G_k\}$ of quantities satisfying the relation

$$h_x = \sum_{y \in S(x)} f(h_y, \theta_{xy}), \quad (2)$$

where $S(x)$ is a set of direct descendants, $x \in V$, $f(h, \theta) = \operatorname{arctanh}(\theta \tanh h)$, and $\theta_{xy} = \tanh(J_{xy}/T)$, holds [3]. Let $G_k/\widehat{G}_k = \{H_1, \dots, H_r\}$ be a quotient group, where $\widehat{G}_k$ is a normal divisor of index $r \geq 1$. We introduce the notation $\{x_\downarrow\} = \{y \in G_k : \langle x, y \rangle \in L\} \setminus S(x)$ for $x \in G_k$.

Definition 1. A set of quantities $h = \{h_x, x \in G_k\}$ is said to be $\widehat{G}_k$ periodic if $h_{xy} = h_x$ for any $x \in G_k$ and $y \in \widehat{G}_k$ (here, $\widehat{G}_k$ is a normal subgroup of index $r \geq 1$). In this paper, we give a more general definition of the Gibbs measure. For $x \in G_k$, we set $\{x_\downarrow\} = \{y \in G_k : \langle x, y \rangle \in L\} \setminus S(x)$. Let $G_k/\widehat{G}_k = \{H_1, \dots, H_r\}$ be a quotient group.

Definition 2. A set of quantities $h = \{h_x, x \in G_k\}$ is said to be $\widehat{G}_k$ weakly periodic if $h_x = h_{ij}$ for $x \in H_i$ and $x_\downarrow \in H_j$ for any $x \in G_k$.

We note that the weakly periodic set of h coincides with an ordinary periodic one (see definition 1) if the quantity h_x is independent of $x_\downarrow$.

Definition 3. A Gibbs measure μ is said to be $\widehat{G}_k$ (weakly) periodic if it corresponds to the $\widehat{G}_k$ -(weakly) periodic set of h . The G_k -periodic measure is said to be translation invariant.

We note that weakly periodic Gibbs measures depend on the choice of the normal divisor.

Let $A \subset \{1, 2, \dots, k + 1\}$ and $H_A = \{x \in G_k : \sum_{i \in A} w_x(a_i) \text{ is even}\}$, $G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\}$, where $w_x(a_i)$ is the number of letters of $x \in G_k$,

we consider the quotient group $G_k/H_A = \{H_A, G_k \setminus H_A\}$.

By virtue of (2), the H_A -weakly periodic set h then has the form

$$h_x = \begin{cases} h_1, & x_{\downarrow} \in H_A, & x \in H_A, \\ h_2, & x_{\downarrow} \in G_k \setminus H_A, & x \in H_A, \\ h_3, & x_{\downarrow} \in H_A, & x \in G_k \setminus H_A, \\ h_4, & x_{\downarrow} \in G_k \setminus H_A, & x \in G_k \setminus H_A. \end{cases} \quad (3)$$

The (3) $h_j, j = \overline{1, 4}$, satisfy the system of equations.

Let $A = \{1\}$. We see the θ_{xy} in this case as follows:

$$\theta_{xy} = \begin{cases} \theta_1, & x \in G_2^{(2)}, & y \in G_2 \setminus G_2^{(2)}, \\ \theta_2, & x \in G_2 \setminus G_2^{(2)}, & y \in G_2^{(2)}. \end{cases}$$

And from equation (2) we have:

$$\begin{cases} h_1 = f(h_1, \theta_1) + f(h_3, \theta_1), \\ h_1 = f(h_1, \theta_2) + f(h_3, \theta_2), \\ h_2 = 2f(h_1, \theta_2), \\ h_2 = 2f(h_1, \theta_1), \\ h_3 = 2f(h_4, \theta_1), \\ h_3 = 2f(h_4, \theta_2), \\ h_4 = f(h_4, \theta_1) + f(h_2, \theta_1), \\ h_4 = f(h_4, \theta_2) + f(h_2, \theta_2), \end{cases} \quad (4)$$

Theorem. Let $k = 2, |A| = 1$. Then all H_A -weakly periodic Gibbs measures for the $G_2^{(2)}$ - inhomogeneous Ising model are translation-invariant.

Remark. We studied the H_A -weakly periodic Gibbs measures for the $G_2^{(2)}$ - inhomogeneous Ising model. Translation-invariant Gibbs measures was studied for the inhomogeneous Ising model in [3].

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GIBBS MEASURES FOR (2, 3)-STATE ISING-POTTS MODEL

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We consider present paper the Ising-Potts model on a Cayley tree. We find at least 7 translation-invariant Gibbs measures under some conditions for (2, 3)-state Ising-Potts model on the binary tree.

The Ising-Potts model is applicable to a diverse array of complex systems, encompassing multicomponent alloys, disordered magnetic structures such as spin glasses, magnetic systems exhibiting phase coexistence, biological networks, neural architectures, and socio-dynamic models.

Let $\Gamma^k = (V, L)$ be a Cayley tree of order k [1] and $\Phi = \{-1, 1\}$, $\Psi = \{1, 2, \dots, q\}$ ($q \geq 3$). A configuration (σ, s) on $A \subset V$ is then defined as a function $x \in A \mapsto (\sigma(x), s(x)) \in \Phi \times \Psi$. The set of all configurations on A is denoted by $\Omega_A = (\Phi \times \Psi)^A$. One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $(\sigma_{n-1}, s_{n-1}) \in \Omega_{V_{n-1}}$ and $(\sigma_{[n]}, s_{[n]}) \in \Omega_{W_n}$ we define their concatenations by

$$((\sigma_{n-1}, s_{n-1}) \vee (\sigma_{[n]}, s_{[n]}))(x) = \begin{cases} (\sigma_{n-1}, s_{n-1})(x), & \text{if } x \in V_{n-1}, \\ (\sigma_{[n]}, s_{[n]})(x), & \text{if } x \in W_n. \end{cases} \quad (1)$$

where $(\sigma, s)(x) := (\sigma(x), s(x))$. It is clear that $(\sigma_{n-1}, s_{n-1}) \vee (\sigma_{[n]}, s_{[n]}) \in \Omega_{V_n}$.

The Hamiltonian of the Ising-Potts model is given by

$$H_n(\sigma_n, s_n) = -\alpha \sum_{\langle x, y \rangle \in L_n} J_{x, y}^{(I)} \sigma(x) \sigma(y) - (1 - \alpha) \sum_{\langle x, y \rangle \in L_n} J_{x, y}^{(P)} \delta_{s(x) s(y)}, \quad (2)$$

where $\sigma \in \Phi$, $s \in \Psi$, $\alpha \in [0, 1]$.

Note that in [2], the homogeneous Ising-Potts model was studied, however, the hamiltonian is different from (2).

Let us consider a probability measure μ_n on Ω_{V_n} defined by

$$\mu_n(\sigma_n, s_n) = Z_n^{-1} \exp \left\{ -\beta H_n(\sigma_n, s_n) + \sum_{x \in W_n} h_{\sigma(x), s(x), x} \right\}, \quad (3)$$

where $\beta = \frac{1}{T}$, $T > 0$ - temperature, $\mathbf{h}_x = \{(h_{-1,1,x}, \dots, h_{-1,q,x}, h_{1,1,x}, \dots, h_{1,q,x}) \in \mathbb{R}^{2q}, x \in V\}$ is a collection of vectors, Z_n^{-1} is the normalizing factor.

A probability distribution μ_n is said to be consistent if for all $n \geq 1$ and $(\sigma_{n-1}, s_{n-1}) \in \Omega_{V_{n-1}}$, we have

$$\sum_{(\sigma_{[n]}, s_{[n]}) \in \Omega_{W_n}} \mu_n((\sigma_{n-1}, s_{n-1}) \vee (\sigma_{[n]}, s_{[n]})) = \mu_{n-1}(\sigma_{n-1}, s_{n-1}). \quad (4)$$

By the Kolmogorov theorem, there exists a unique measure μ on the set Ω_V such that, for all n and $(\sigma_n, s_n) \in \Omega_{V_n}$,

$$\mu(\{(\sigma, s)|_{V_n} = (\sigma_n, s_n)\}) = \mu_n(\sigma_n, s_n).$$

Let $I = \{\mathbf{z} = (\underbrace{z_1, \dots, z_1}_q, 1, z_2, \dots, z_2) \in \mathbb{R}^{2q-1}\}$, where $z_{i,j,x} = \exp\{h_{i,j,x} - h_{-1,q,x}\}$, $i = -1, 1, j = 1, 2, \dots, q$.

Theorem. Let $k = 2$, $J_P = \frac{\alpha}{1-\alpha} J_I$ and $a = \exp\{J\beta\alpha\}$, $\alpha \in (0, 1)$. For the translation-invariant Gibbs measures of the $(2, 3)$ -state Ising-Potts model, on the invariant set I , the following assertions hold

- If $a \in (0, 1) \cup (1, B_1^c)$ then there exists a unique TIGM;
- If $a \in [B_1^c, B_2^c)$ then there are three TIGMs;
- If $a = B_2^c$ then there are four TIGMs;
- If $a \in (B_2^c, B_3^c)$ then there are five TIGMs;
- If $a \in [B_3^c, +\infty)$ then there are seven TIGMs.

where $B_1^c \approx 2,010$ $B_2^c \approx 3,828$ $B_3^c \approx 4,921$.

Remark. In [3], the authors proved that there exist at most 7 translation-invariant Gibbs measures for the 3-state Potts model. We prove that there exist at least 7 translation-invariant Gibbs measures $(2, 3)$ -state Ising-Potts model on the binary tree.

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DYNAMICAL ANALYSIS OF A THREE-DIMENSIONAL DISCRETE-TIME DIABETES MODEL

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Let $W : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^3$ be the operator defined by

$$\begin{cases} x' = g_0 - g_1x - cxy + x, \\ y' = \frac{s_1x^2}{s_2 + x^2} z - ky + y, \\ z' = (-d_0 + r_1x - r_2x^2)z + z, \end{cases} \quad (1)$$

where all parameters are positive.

Interpreting (1) as a biological model, we let x , y , and z denote the plasma glucose concentration, the insulin concentration, and the functional β -cell mass (i.e., the β -cells capable of proper insulin secretion), respectively. The constant g_0 represents the average daily glucose influx, including hepatic glucose production and dietary intake. The term g_1x models insulin-independent glucose uptake, primarily by brain and nerve cells, while cxy represents insulin-dependent uptake, mainly by fat and muscle tissues; here, c is the insulin sensitivity coefficient. Insulin secretion from β -cells is assumed to follow a Hill-type function of coefficient 2, activated by elevated glucose levels, where s_1 denotes the secretory capacity per β -cell. The insulin clearance rate is denoted by k . The net growth of functional β -cell mass is assumed to depend quadratically on glucose: moderate glucose promotes proliferation, while excessive glucose induces apoptosis, resulting in a net loss. Here, d_0 is the zero-glucose death rate, and r_1, r_2 are constants [1],[2].

We denote

$$A = \frac{r_1 + \sqrt{r_1^2 + 4r_2(1 - d_0)}}{2r_2}, \quad B = \frac{1 - g_1}{c}, \quad C = \frac{(1 - g_1)k}{s_1c}.$$

Lemma. If

$$0 < g_1 < 1, \quad 0 < k \leq 1, \quad 0 < d_0 \leq 1, \quad g_0/g_1 \leq A, \quad r_1^2 \leq 4r_2d_0 \quad (2)$$

then the operator (1) maps the set

$$\Omega = \{(x, y, z) \in \mathbb{R}_+^3 \mid g_0 \leq x \leq A, \quad 0 \leq y \leq B, \quad 0 \leq z \leq C\}$$

into itself.

Theorem 1. The operator W , as defined in (1), has no periodic points of period $p \geq 2$ in the set Ω .

Theorem 2. Assume that (2) holds and that $r_1^2 < 4r_2d_0$. Then the trajectory of W converges to the fixed point $\left(\frac{g_0}{g_1}, 0, 0\right)$ for any initial state $(x^{(0)}, y^{(0)}, z^{(0)}) \in \Omega$, i.e.,

$$\lim_{n \rightarrow \infty} W^n(x^{(0)}, y^{(0)}, z^{(0)}) = \left(\frac{g_0}{g_1}, 0, 0\right).$$

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Non-uniqueness of p -adic Markov chain with countable state on treesSattarov I.A.^{1,2}¹V.I. Romanovskiy Institute of mathematics, Tashkent, Uzbekistan;²Namangan State University, Namangan, Uzbekistan;

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Note that the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p -adic norm defines the field of p -adic numbers $\mathbb{Q}_p$. Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form $x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots)$, where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0$, $0 \leq x_j \leq p-1$ (see [1]). In this case norm of the p -adic number $x \in \mathbb{Q}_p$ defined by $|x|_p := p^{-\gamma(x)}$.

Let m_p denote the set of all bounded sequences in the field of p -adic numbers $\mathbb{Q}_p$. Then it is not difficult to see that the set m_p is a subspace of the linear space $\mathbb{Q}_p^{\mathbb{N}}$. We can define the norm in m_p as $\|\mathbf{z}\| = \sup_{i \in \mathbb{N}} |z_i|_p$.

Tree. A tree is a connected graph without cycles. Let $\mathcal{T} = (V, L)$ be a tree, where V is the set of vertices and L is the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $b = \langle x, y \rangle \in L$ connecting them. Let $A \subset V$. Denote $\partial A = \{x \in V \setminus A : \exists y \in A, \langle x, y \rangle\}$.

We consider a system with nearest-neighbor interactions on an *infinite tree* in which each vertex has at least two edges, and where the spins assigned to the vertices take values from the set of natural numbers $\mathbb{N}$. A configuration σ_A on $A \subset V$ is then defined as a function $x \in A \mapsto \sigma_A(x) \in \mathbb{N}$. The set of all configurations is $\mathbb{N}^A$.

We always assume that for each edge $\langle x, y \rangle \in L$ a matrix $\mathbb{P}_{\langle x, y \rangle} = (P_{\langle x, y \rangle}(i, j))_{i, j \in \mathbb{N}}$ satisfies the following conditions

$$\sup_{i, j \in \mathbb{N}} |P_{\langle x, y \rangle}(i, j)|_p < \infty, \quad \sum_{j \in \mathbb{N}} P_{\langle x, y \rangle}(i, j) = 1, \quad (1)$$

where, $P_{\langle x, y \rangle}(i, j)$ is the probability that the y -end of the edge $\langle x, y \rangle$ is in state j when the x -end is in state i .

Let $\mathbf{z} : L \rightarrow m_p$ be a vector-valued function, i.e., $\mathbf{z}(x, y) = \{z_i(x, y)\}_{i \in \mathbb{N}}$ be a vector in m_p . For any finite connected nonempty set $\Lambda \subset V$ (and $\bar{\Lambda} = \Lambda \cup \partial \Lambda$) we consider the p -adic probability distribution $\mu_{\bar{\Lambda}}^{\mathbf{z}}$ on $\mathbb{N}^{\bar{\Lambda}}$ defined by

$$\mu_{\bar{\Lambda}}^{\mathbf{z}}(\sigma_{\bar{\Lambda}} = \varsigma) = \frac{1}{Z_{\bar{\Lambda}}} \prod_{x \in \partial \Lambda} z_{\varsigma(x)}(x_{\Lambda}, x) \prod_{\substack{\langle x, y \rangle \in L: \\ \langle x, y \rangle \cap \Lambda \neq \emptyset}} P_{\langle x, y \rangle}(\varsigma(x), \varsigma(y)), \quad (2)$$

where x_{Λ} denotes the unique neighbor of $x \in \partial \Lambda$ belonging to Λ , and $Z_{\bar{\Lambda}}$ is the normalizing factor, i.e.,

$$Z_{\bar{\Lambda}} = \sum_{\varphi_{\bar{\Lambda}}} \prod_{x \in \partial \Lambda} z_{\varphi(x)}(x_{\Lambda}, x) \prod_{\substack{\langle x, y \rangle \in L: \\ \langle x, y \rangle \cap \Lambda \neq \emptyset}} P_{\langle x, y \rangle}(\varphi(x), \varphi(y)).$$

We say that the p -adic probability distributions $\mu_{\bar{\Lambda}}^{\mathbf{z}}$ are compatible if for all $\varsigma \in \mathbb{N}^{\bar{\Lambda}}$:

$$\sum_{\varphi \in \mathbb{N}^{\partial \Lambda}} \mu_{\bar{\Lambda}}^{\mathbf{z}}(\sigma_{\Lambda} = \varsigma, \sigma_{\partial \Lambda} = \varphi) = \mu_{\bar{\Lambda}}^{\mathbf{z}}(\sigma_{\Lambda} = \varsigma). \quad (3)$$

In this case, according to the p -adic version of the Kolmogorov's extension theorem (see [2]), there exists a unique p -adic probability distribution $\mu^{\mathbf{z}}$ on $\mathbb{N}^V$ such that, for any finite connected nonempty set $A \subset V$ and for all $\varsigma \in \mathbb{N}^A$, $\mu^{\mathbf{z}}(\{\sigma|_A = \varsigma\}) = \mu_A^{\mathbf{z}}(\sigma_A = \varsigma)$. Such a distribution $\mu^{\mathbf{z}}$ on $\mathbb{N}^V$ is called a p -adic Markov chain with transition matrices $(\mathbb{P}_{\langle x, y \rangle})_{\langle x, y \rangle \in L}$, corresponding to the function $\mathbf{z} : L \rightarrow m_p$.

Without loss of generality, we set hereafter $z_1(x, y) \equiv 1$ for all $\langle x, y \rangle \in L$. Then the following statement describes conditions on $\mathbf{z}(x, y)$ guaranteeing compatibility of (2).

Theorem 1. *p -Adic probability distributions $\mu_A^{\mathbf{z}}$ in (2) are compatible if and only if for any $\langle x, y \rangle \in L$, and for all $i \in \mathbb{N}$, the following equations hold*

$$z_i(x, y) = \prod_{v \in \partial\{y\} \setminus \{x\}} \frac{1 + \sum_{j=2}^{\infty} (z_j(y, v) - 1) P_{\langle y, v \rangle}(i, j)}{1 + \sum_{j=2}^{\infty} (z_j(y, v) - 1) P_{\langle y, v \rangle}(1, j)}. \quad (4)$$

Now we examine the conditions on the parameters and on $\mathbb{P}_{\langle x, y \rangle}$ for the non-uniqueness of the solutions of the equation (4) on the Cayley tree of order $k \geq 1$. Suppose the matrix $\mathbb{P}_{\langle x, y \rangle}$, $\langle x, y \rangle \in L$ in the system of equations (4) satisfies the condition

$$P_{\langle x, y \rangle}(2, 1) = P_{\langle x, y \rangle}(i, 1), \quad \forall i > 2, \quad (5)$$

and we assume further that $P_{\langle x, y \rangle}(1, 1)$ and $P_{\langle x, y \rangle}(2, 1)$ are independent on $\langle x, y \rangle$, that is

$$1 - \alpha \equiv P_{\langle x, y \rangle}(2, 1), \quad 1 - \beta \equiv P_{\langle x, y \rangle}(1, 1). \quad (6)$$

Theorem 2. *If (5), (6) are satisfied, α, β are p -adic integers, and there exists $\gamma \in \{0, 1, 2, \dots\}$ such that*

$$\begin{aligned} k(\beta - \alpha) + 1 &\neq 0, \quad k(\beta - \alpha) + 1 \equiv 0 \pmod{p^{2\gamma+1}}, \\ k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) &\equiv 0 \pmod{p^\gamma}, \\ k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) &\not\equiv 0 \pmod{p^{\gamma+1}}, \end{aligned}$$

then the equation (4) has at least two solutions.

A quasi-phase transition in the context of p -adic Markov (or Gibbs) measures occurs when more than one distinct measure (solution) exists that satisfies all the required compatibility and consistency conditions. As a corollary of Theorem 2, we have the following.

Theorem 3. *If the conditions of Theorem 2 are satisfied then for the matrix $\mathbb{P}_{\langle x, y \rangle}$ on the Cayley tree of order $k \geq 1$, there are at least two p -adic Markov chains, (i.e., a quasi-phase transition occurs).*

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σ -Chekli filtrlangan o'lchovli fazoga misol**Sattarov I., Abduqaxxorov A.**

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Bizga bo'sh bo'lmagan X to'plam va uning ba'zi qism to'plamlaridan tashkil topgan $\mathfrak{A}$ to'plamlar sistemasi berilgan bo'lsin.

1-Ta'rif. $\mathfrak{A}$ to'plamlar sistemasi X to'plamning ba'zi qism to'plamlaridan tashkil topgan σ -algebra deyiladi, agar u quyidagi shartlarni qanoatlantirsa:

$$X \in \mathfrak{A};$$

$$A \in \mathfrak{A} \implies A^c \in \mathfrak{A};$$

$$(A_j)_{j \in N} \subset \mathfrak{A} \implies \bigcup_{j \in N} A_j \in \mathfrak{A}.$$

1-Misol. (i) $\{X, \emptyset\}$ X to'plam yordamida hosil qilish mumkin bo'lgan eng kichik σ -algebra bo'ladi.

(ii) $\mathfrak{P}(X)$ - X ning barcha qism to'plamlari sistemasi bo'lsin. U holda bu sistema X yordamida hosil qilish mumkin bo'lgan eng katta σ -algebra bo'ladi.

Endi biz uchun muhim bo'lgan tushuncha – o'lchov tushunchasini kiritamiz. Biga bo'sh bo'lmagan X to'plam va uning ba'zi qism to'plamlaridan tashkil topgan $\mathfrak{A}$ σ -algebra berilgan bo'lsin.

2-Ta'rif. X to'plamda aniqlangan μ o'lchov deb, quyidagi ikki shartni qanoatlantiruvchi $\mu : \mathfrak{A} \longrightarrow [0; \infty)$ akslantirishga aytiladi:

$$\mu(\emptyset) = 0 \tag{1}$$

$$\mu\left(\bigcup_{j \in N} A_j\right) = \sum_{j \in N} \mu(A_j) \quad \forall (A_j)_{j \in N} \subset \mathfrak{A}, \quad A_j \cap A_i = \emptyset, i \neq j. \tag{2}$$

2-Misol. a) $(X, \mathfrak{A})$ o'lchovli fazo bo'lsin. U holda

$$\mu = \begin{cases} |A|, & \text{agar } A \text{ chekli} \\ +\infty, & \text{agar } A \text{ cheksiz} \end{cases}$$

akslantirish o'lchov bo'ladi. Bunday o'lchov sanoq o'lchovi deyiladi.

b)

$$\mu([a; b]) = b - a$$

akslantirish ham o'lchov bo'ladi. Ushbu yo'l bilan aniqlangan o'lchov Lebeg o'lchovi deb ataladi.

3-Ta'rif. X bo'sh bo'lmagan to'plam, $\mathfrak{A}$ uning ba'zi qism to'plamlaridan tashkil topgan σ -algebra va μ unda aniqlangan o'lchov bo'lsin. U holda $(X, \mathfrak{A}, \mu)$ sistemaga o'lchovli fazo deyiladi.

Biz endilikda σ -chekli filtrlangan o'lchovli fazo haqida tushunchaga ega bo'lamiz va unga misol keltiramiz. Buning uchun biz avvalo filtr va σ -chekli fazolar haqida bilib olishimiz kerak.

Bizga $(X, \mathfrak{A}, \mu)$ o'lchovli fazo berilgan bo'lsin.

4-Ta'rif. $(\mathfrak{A}_k)_{k \in I}$, σ -algebralar ketma-ketligi filtr deyiladi agar $\forall k, m \in I, m > k$, $\mathfrak{A}_k \subset \mathfrak{A}_m$ shart o'rinli bo'lsa, ya'ni:

$$\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \mathfrak{A}_3 \dots \subset \mathfrak{A}$$

shart bajarilsa.

3-Misol. $(\Omega, \mathcal{F})$ o'lchovli fazo, $\Omega = \{1, 2, 3, 4\}$ va $\mathcal{F} = 2^\Omega$ bo'lsin. $(\mathcal{F}_n)_{n \geq 0}$ σ -algebralar ketma-ketligini quyidagicha aniqlaylik:

$$\begin{aligned}\mathcal{F}_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}_1 &= \sigma(\{\{1, 2\}, \{3, 4\}\}) = \{\emptyset, \{1, 2\}, \{3, 4\}, \Omega\}, \\ \mathcal{F}_2 &= \sigma(\{\{1\}, \{2\}, \{3\}, \{4\}\}).\end{aligned}$$

U holda $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$, so $(\mathcal{F}_n)$ filtrlangan sistema bo'ladi.

5-Ta'rif. Agar $(X, \mathcal{A}, \mu)$ o'lchovli fazoda o'suvchi $(A_i)_{i \in N} \subset \mathcal{A}, A_i \uparrow X$ to'plamlar ketma-ketligi mavjud bo'lib, $\forall i, \mu(A_i) < \infty$ shart bajarilsa, bu fazo σ -chekli o'lchovli fazo deyiladi.

Bizga $(X, \mathcal{A}, \mathcal{A}_i, \mu)$ filtrlangan o'lchovli fazo berilgan bo'lsin.

6-Ta'rif. Agar $(X, \mathcal{A}_0, \mu)$ σ -chekli fazo tashkil qilsa, berilgan $(X, \mathcal{A}, \mathcal{A}_i, \mu)$ fazoga filtrlangan σ -chekli o'lchovli fazo deyiladi.

Biz $\mathcal{R}$ da σ -algebralar ketma-ketligini quyidagicha aniqlaylik:

$$\mathcal{B}_k = \sigma\left(\left[\frac{n}{2^k}; \frac{n+1}{2^k}\right)\right), k \in N, n \in Z$$

μ o'lchov sifatida esa yuqorida keltirilgan Lebeg o'lchovini olaylik.

Teorema. $(\mathcal{R}, \mathcal{B}, \mathcal{B}_k, \mu)$ fazo filtrlangan o'lchovli fazoning barcha shartlarini qanoatlantiradi. $\mathcal{B} = \bigcup_{k \in N} \mathcal{B}_k$ sistema esa Borel σ -algebrasi bo'ladi.

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EXACT SOLUTIONS FOR THE ISING MODEL ON THE CLOSED CAYLEY TREE WITH BRANCHING RATIO THREE

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Since their introduction in [1], closed Cayley trees have played a pivotal role in the modeling of various physical systems and phenomena. Notably, it was demonstrated in [1] that the use of closed Cayley trees leads to more accurate results compared to open Cayley trees [2], [3]. In the work, we consider the Ising model on the closed Cayley tree of branching ratio three. We exactly solve the model, i.e., we find the critical value of the parameter such that below this value there are three limiting Gibbs measures.

To define a random field, it is useful to consider the construction of the closed Cayley tree of branching ratio three. We begin with a single vertical edge containing a pair of sites. This structure is then duplicated to the left and right, with the three top sites linked through an additional site using three edges. Similarly, the same process is applied to the three bottom sites, forming the fundamental polygons. Next, this structure is again copied to the left and right, linking the three top and three bottom sites in the same manner as before. This iterative process results in a closed tree with three upper levels and three lower levels. Denote the finite closed tree constructed at the n th stage by $\mathcal{T}^{(n)}$. The infinite closed Cayley tree is then defined as $\mathcal{T} = \lim_{n \rightarrow \infty} \mathcal{T}^{(n)}$. The infinite closed Cayley tree $\mathcal{T}$ can be decomposed into three parts:

$$\mathcal{T} = \Gamma^1 \cup \Gamma^2 \cup \mathcal{S},$$

where $\Gamma^1 = (V_1, L_1)$ represents the *upper tree* and $\Gamma^2 = (V_2, L_2)$ represents the *lower tree* and $\mathcal{S}$ is the set of edges which connects upper and lower tree.

We study binary random fields on $\mathcal{T}$, which can be viewed as the limiting measures of a spin system defined on a sequence of finite closed Cayley trees. Let $V = V_1 \cup V_2$ and $L = L_1 \cup L_2$. If $l \in L$ an edge with endpoints $x, y \in V$ then we write $l = \langle x, y \rangle$ and the endpoints are called nearest neighbors.

We assume that $\Phi = \{-1, 1\}$, and $\sigma \in \Omega = \Phi^V$ represents a configuration, i.e., $\sigma = \{\sigma_x \in \Phi : x \in V\}$.

The random field on $\mathcal{T}^{(n)} = \{V^{(n)}, L^{(n)}\}$ is represented in terms of the potential function:

$$H(\sigma) = -J_0 \sum_{\langle x, y \rangle \in \mathcal{S}} \sigma_x \sigma_y - J_1 \sum_{\langle i, j \rangle \in L_1} \sigma_i \sigma_j - J_2 \sum_{\langle l, m \rangle \in L_2} \sigma_l \sigma_m,$$

where the first summation extends over all spins on the surface sites (i.e., those corresponding to nearest-neighbor pairs on upper level 1 and lower level 1), while the second and third summations extend over the upper and lower trees of $\mathcal{T}^{(n)}$, respectively.

Let $\mathbf{B}$ be the σ -algebra generated by cylindrical sets with finite base of Ω^a . For any n we let $\mathbf{B}_{V^{(n)}} = \{\sigma \in \Omega : \sigma|_{V^{(n)}} = \sigma_n\}$ denote the subalgebra of $\mathbf{B}$, where $\sigma|_{V^{(n)}}$ is restriction of σ to $V^{(n)}$ and $\sigma_n : x \in V^{(n)} \mapsto \sigma_n(x)$ an admissible configuration in $V^{(n)}$.

The resulting for $\sigma \in \mathbf{B}_{V^{(n)}}$ Gibbs distribution is then defined by

$$P(\sigma) = \frac{1}{Z^{(n)}} \exp \left(-\frac{1}{kT} H(\sigma) \right),$$

where $Z^{(n)}$ is the partition function.

Taking an advantage of structure of the closed Cayley tree, under some conditions we obtain the following system of equations

$$\begin{cases} x_n = \left(\frac{x_{n-1} + bz_{n-1} + c + bcy_{n-1}}{cx_{n-1} + bcz_{n-1} + 1 + by_{n-1}} \right)^3, \\ y_n = \left(\frac{bcx_{n-1} + cz_{n-1} + b + y_{n-1}}{cx_{n-1} + bcz_{n-1} + 1 + by_{n-1}} \right)^3, \\ z_n = \left(\frac{bx_{n-1} + z_{n-1} + bc + cy_{n-1}}{cx_{n-1} + bcz_{n-1} + 1 + by_{n-1}} \right)^3. \end{cases} \quad (1)$$

where $a = e^{-2h_0}$, $b = e^{-2h_1}$, $c = e^{-2h_2}$ $h_i = J_i/kT$, $i = 0, 1, 2$.

If the system of equations (1) has a unique solution, the corresponding limiting measure is also unique, indicating the absence of a phase transition. On the other hand, if the system has multiple solutions, this implies there is more than one limiting Gibbs measures (LGM), which is characteristic of a phase transition. It is evident that the following sets are invariant under the operator $W : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, as defined by the RHS of (1):

$$I_1 = \{(x, y, z) \in \mathbb{R}^3 : y = z, x = 1\},$$

$$I_2 = \{(x, y, z) \in \mathbb{R}^3 : x = z, y = 1\},$$

$$I_3 = \{(x, y, z) \in \mathbb{R}^3 : x = y, z = 1\}.$$

Let $I = \bigcup_{j=1}^3 I_j$. Summarizing, we have the following result

Theorem 1. For the Ising model on the closed Cayley tree with a branching ratio of three in the case of an invariant set I has:

- three TILGMs if $b < \frac{1}{2}$, $c < \frac{1}{2}$, and $b > \frac{2c-1}{c-2}$;
- one TILGM if $b \leq \frac{1}{2}$, $c \leq \frac{1}{2}$, and $b \leq \frac{2c-1}{c-2}$;
- three TILGMs if $b > \frac{1}{2}$, $c < \frac{1}{2}$, and $b \leq \frac{2c-1}{c-2}$;
- five TILGMs if $b > \frac{1}{2}$, $c > \frac{1}{2}$, and $b \leq \frac{2c-1}{c-2}$;
- seven TILGMs if $b > \frac{1}{2}$, $c > \frac{1}{2}$, and $b > \frac{2c-1}{c-2}$.

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BIFURCATION ANALYSIS OF A PHYTOPLANKTON-ZOOPLANKTON MODEL WITH TOXIN-MEDIATED INTERACTIONS

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Bifurcation theory provides a powerful framework for analyzing how qualitative changes in system dynamics arise as parameters vary. In predator-prey systems, an important phenomenon is the Neimark-Sacker bifurcation, the discrete analogue of the Hopf bifurcation. This occurs when a fixed point loses stability as a pair of complex-conjugate eigenvalues cross the unit circle in the complex plane, leading to the emergence of a closed invariant curve. Ecologically, such a transition represents a shift from a steady coexistence state to sustained quasiperiodic oscillations, which can be interpreted as persistent and irregular bloom cycles.

In the present work, we investigate a discrete-time phytoplankton-zooplankton model that incorporates Holling type III ($\frac{u^2}{\gamma^2+u^2}$) grazing response and Holling type II ($\frac{u}{\gamma+u}$) toxin release, reflecting the saturating nature of these biological interactions.

$$\begin{cases} u^{(1)} = u(2-u) - \frac{u^2v}{\gamma^2+u^2}, \\ v^{(1)} = \frac{\beta u^2v}{\gamma^2+u^2} + (1-r)v - \frac{\theta uv}{\gamma+u}, \end{cases} \quad (1)$$

where r, β, γ, θ – are positive parameters, u and v denote the scaled densities of phytoplankton and zooplankton, respectively.

Theorem. Let $\bar{E} = (\bar{u}, \bar{v})$ be a positive fixed point of system (1), and $\beta = \beta_0$ be such that $\Delta(\bar{u}) = 1$. If β varies within a sufficiently small neighborhood of β_0 , then system (1) undergoes a Neimark-Sacker bifurcation at $\bar{E}$, where $\Delta(\bar{u})$ denotes the determinant of the Jacobian matrix evaluated at $\bar{E}$.

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2-ADIC ISING MAPPING AND ITS DYNAMICS

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In [1], it was established that if $p \equiv 1 \pmod{4}$, then the dynamical system generated by the Ising mapping

$$f(x) = \left(\frac{ax + 1}{x + a} \right)^2$$

is topologically conjugate to a symbolic dynamical system with two symbols. In [2], the dynamics of the 2-adic Ising mapping were thoroughly analyzed for the case $a \not\equiv 3 \pmod{4}$. The authors proved that the mapping is regular whenever $|a|_2 > 1$ or $|a-1|_2 < \frac{1}{2}$, while for $|a|_2 < 1$ the trajectory of every initial point is contained in a 2-periodic cycle.

In the present work, we study the 2-adic Ising mapping in the case $a \equiv 3 \pmod{4}$.

For each $a \in \mathbb{Q}_2$, $r > 0$ we denote

$$B_r(a) = \{x \in \mathbb{Q}_2 : |x - a|_2 < r\}, \quad S_r(a) = \{x \in \mathbb{Q}_2 : |x - a|_2 = r\}.$$

We notice that $B_r(a)$ and $S_r(a)$ are clopen, compact subsets of $\mathbb{Q}_2$. Recall that $\mathbb{Z}_2 := B_2(0)$ and $\mathbb{Z}_2^* := S_1(0)$ are the set of all 2-adic integers and 2-adic units, respectively. One can check that for every $n \geq 1$ it hold:

$$S_{\frac{1}{2^n}}(1) = 1 + 2^n + 2^{n+1}\mathbb{Z}_2, \quad B_{\frac{1}{2^n}}(1) = 1 + 2^{n+1}\mathbb{Z}_2.$$

We notice that the dynamical system $(\mathbb{Q}_2, f)$ can be restricted to the open ball $B_{\frac{1}{4}}(1)$, i.e., $(B_{\frac{1}{4}}(1), f)$ defines a well-posed dynamical system. It is easy to see that $x \in B_{\frac{1}{4}}(1)$ if and only if $x = 1 + x_3 \cdot 2^3 + x_4 \cdot 2^4 + \dots$, where $x_i \in \{0, 1\}$, $i \geq 3$. It follows that $B_{\frac{1}{4}}(1)$ can be represented as a countable union of pairwise disjoint spheres centered at 1, i.e., $B_{\frac{1}{4}}(1) = \bigcup_{n=3}^{\infty} S_{\frac{1}{2^n}}(1)$. Below we show that each sphere $S_{\frac{1}{2^n}}(1)$ is invariant under iteration of the function f . This observation allows us to decompose the dynamical system $(B_{\frac{1}{4}}(1), f)$ into a disjoint union of simpler dynamical systems $(S_{\frac{1}{2^n}}(1), f)$.

Theorem. Let $a \equiv 3 \pmod{16}$. Then $(B_{\frac{1}{2^{n+1}}}(1 + 2^n), f)$ and $(B_{\frac{1}{2^{n+1}}}(1 + 2^n + 2^{n+1}), f)$ are minimal systems if and only if one of the following situations occur:

(A) $a \equiv 3 \pmod{32}$ and $n = 3$;

(B) $a \equiv 19 \pmod{32}$ and $n > 3$.

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FREE ENERGY OF p -ADIC ISING MODEL

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Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p , every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$ where, $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p . The p -adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

This norm is non-Archimedean, i.e., it satisfies the strong triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the p -adic norm defines the p -adic field $\mathbb{Q}_p$ (see [1]).

Let $\Gamma^k = (V, L)$ be a Cayley tree of order $k \geq 1$, where V is the set of vertices, and L is the set of edges (see [2]).

Let $\Phi = \{-1, 1\}$. A configuration σ on $A \subset V$ is define by the function $x \in A \rightarrow \sigma(x) \in \Phi$. The set of all configurations on A is denoted by Ω_A and $\Omega = \Phi^V$.

For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\varphi^{(n)} \in \Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1} \vee \varphi^{(n)})(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \varphi^{(n)}(x), & \text{if } x \in W_n. \end{cases}$$

A formal p -adic Hamiltonian $H : \Omega \rightarrow \mathbb{Q}_p$ of Ising model is defined as

$$H(\sigma) = N \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y), \quad (1)$$

where N is a positive integer.

Let $h : x \rightarrow h_x \in \mathbb{Q}_p \setminus \{0\}$ be a p -adic function on V . We consider p -adic probability distributions $\mu_h^{(n)}$ on Ω_{V_n} , which is defined as

$$\mu_h^{(n)}(\sigma_n) = Z_{n,h}^{-1} \rho^{H_n(\sigma_n)} \prod_{x \in W_n} h_{\sigma(x), x}, \quad n = 1, 2, \dots, \quad (2)$$

where $Z_{n,h}$ is the normalizing constant and $\rho \in \mathbb{Q}_p \setminus \{-1, 0, 1\}$.

A p -adic probability measures $\mu_h^{(n)}$ is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \varphi) = \mu_h^{(n-1)}(\sigma_{n-1}).$$

In this case, by the p -adic analogue of Kolmogorov theorem [3], there exists a unique measure μ_h on the set Ω such that $\mu_h(\{\sigma|_{V_n} \equiv \sigma_n\}) = \mu_h^{(n)}(\sigma_n)$ for all n and $\sigma_{n-1} \in \Omega_{V_{n-1}}$. Such a mesure is called p -adic generalized Gibbs measure.

If $h_x = h, \forall x \in V$, then the corresponding measure μ_h is called translation-invariant measure.

Translation-invariant p -adic generalized Gibbs measures for Ising model on the Cayley tree were studied in [4-7].

The p -adic free energy defined as

$$F(h, \theta) = \lim_{n \rightarrow \infty} \frac{1}{|V_n|} \ln |Z_{n,h}|_p,$$

where $\theta = \rho^{2N}$.

Theorem 1. Let $F(h, \theta)$ be free energy of the translation-invariant p -adic generalized Gibbs measures for Ising model on the Cayley tree of order k . Then the free energy $F(h, \theta)$ is a constant. Moreover,

$$F(h, \theta) = \begin{cases} \ln |\theta - 1|_p, & \text{if } |\theta|_p = 1, \quad h \neq 1; \\ 0, & \text{if } |\theta|_p = 1, \quad h = 1; \\ -\frac{1}{2} \ln |\theta|_p, & \text{if } |\theta|_p < 1; \\ \frac{1}{2} \ln |\theta|_p, & \text{if } |\theta|_p > 1, \quad |h|_p = |\theta|_p^{-k}; \\ \frac{2k-1}{2} \ln |\theta|_p, & \text{if } |\theta|_p > 1, \quad |h|_p = |\theta|_p^k. \end{cases}$$

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Система функциональных уравнений для модели Поттса-Изинга со скрытой марковской структурой в машинном обучении

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В данной работе исследуются гамильтониан связывающим две слоя модели Поттса и Изинга: скрытые спины $s(x) \in \{1, 2, 3\}$ и наблюдаемые спины $\sigma(x) \in \{\pm 1\}$ на дереве Кэли $[1, 2]$.

Дерево Кэли $\Gamma^k = (V, L)$ с фактором ветвления $k \geq 1$ - связный бесконечный граф, в котором каждая вершина имеет ровно $k + 1$ соседей. Граф Γ^k ациклический, то есть не содержит циклов $[1]$.

Фиксируем вершину $x_0 \in V$, которую рассматриваем как корень дерева. Вершина $y \in V$ называется прямым потомком $x \in V$, если x - предпоследняя вершина на единственном пути от корня x_0 к y , то есть $d(x_0, y) = d(x_0, x) + 1$ и $d(x, y) = 1$. Множество прямых потомков x обозначается $S(x)$.

Определим $W_n = \{x \in V \mid d(x, x_0) = n\}$,

$$V_n = \{x \in V \mid d(x, x_0) \leq n\}, \quad L_n = \{\langle x, y \rangle \in L \mid x, y \in V_n\}. \quad (1)$$

Для $x \in W_n$ множество $S(x)$ имеет вид

$$S(x) = \{y \in W_{n+1} \mid \langle x, y \rangle \in L\}. \quad (2)$$

В данной работе рассматриваются спины Поттса $s(x) \in I = \{1, 2, 3\}$, назначенные вершинам дерева Кэли. Конфигурация s - функция $x \mapsto s(x)$, множество всех конфигураций обозначается $\Omega = I^V$.

Гамильтониан, учитывающий взаимодействия Поттс-Изинга и связь между скрытыми и наблюдаемыми состояниями, задается формулой

$$H(s, \sigma) = -J \sum_{\langle x, y \rangle} (\delta_{s(x)s(y)} - \sigma(x)\sigma(y)) - \sum_{x \in V} p(\sigma(x) \mid s(x)), \quad (4)$$

где $J \in \mathbb{R}$ - параметр взаимодействия и δ -символ Кронеккера. Здесь первый член отражает локальные взаимодействия скрытых состояний, а второй - вероятность наблюдения $\sigma(x)$ при скрытом состоянии $s(x)$.

Определим конечномерное распределение вероятностной меры μ в объеме V_n как

$$\mu_n(s_n, \sigma_n) = Z_n^{-1} \exp \left(-\beta H_n(s_n, \sigma_n) + \sum_{x \in W_n} h_{s(x), \sigma(x), x} \right), \quad (6)$$

где $\beta = 1/T$, $T > 0$ - температура, Z_n - нормализующий множитель, а

$$h_x = (h_{1,-1,x}, h_{1,1,x}, h_{2,-1,x}, h_{2,1,x}, h_{3,-1,x}, h_{3,1,x}) \in \mathbb{R}^6, \quad x \in V. \quad (7)$$

Гамильтониан H_n задается как

$$H_n(s_n, \sigma_n) = -J \sum_{\langle x, y \rangle \in L_n} (\delta_{s(x)s(y)} - \sigma(x)\sigma(y)) - \sum_{x \in V_n} p(\sigma(x) \mid s(x)).$$

Определение. Семейство распределений (6) называется совместимым, если для всех $n \geq 1$ и s_{n-1}, σ_{n-1} выполняется

$$\sum_{w_n, \omega_n \in I^{W_n}} \mu_n(s_{n-1} \vee w_n, \sigma_{n-1} \vee \omega_n) = \mu_{n-1}(s_{n-1}, \sigma_{n-1}), \quad (8)$$

где $\vee$ - операция объединения конфигураций.

По теореме Колмогорова существует единственная мера μ на $I^V \times I_1^V$, согласующаяся с μ_n , где $I_1 = \{\pm 1\}$. Такая мера называется *сплиттинговой мерой Гиббса* (СМГ) для гамильтониана (4) и векторные функции (7).

Теорема. Распределения вероятностей $\mu(s_n, \sigma_n)$, $n \in \mathbb{N}$, в (6) совместимы тогда и только тогда, когда для любого $x \in V \setminus \{x^0\}$ выполняется следующее уравнение:

$$\begin{aligned} z_{1,-1,x} &= \prod_{y \in S(x)} \frac{\theta^2 + az_{1,-1,y} + \theta^{-1}bz_{2,-1,y} + \theta cz_{2,1,y} + \theta^{-1}dz_{3,-1,y} + \theta fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1}fz_{3,1,y}}, \\ z_{2,-1,x} &= \prod_{y \in S(x)} \frac{\theta + \theta^{-1}az_{1,-1,y} + bz_{2,-1,y} + \theta^2 cz_{2,1,y} + \theta^{-1}dz_{3,-1,y} + \theta fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1}fz_{3,1,y}}, \\ z_{2,1,x} &= \prod_{y \in S(x)} \frac{\theta^{-1} + \theta az_{1,-1,y} + \theta^2 bz_{2,-1,y} + cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1}fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1}fz_{3,1,y}}, \\ z_{3,-1,x} &= \prod_{y \in S(x)} \frac{\theta + \theta^{-1}az_{1,-1,y} + \theta^{-1}bz_{2,-1,y} + \theta cz_{2,1,y} + dz_{3,-1,y} + \theta^2 fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1}fz_{3,1,y}}, \\ z_{3,1,x} &= \prod_{y \in S(x)} \frac{\theta^{-1} + \theta az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta^2 dz_{3,-1,y} + fz_{3,1,y}}{1 + \theta^2 az_{1,-1,y} + \theta bz_{2,-1,y} + \theta^{-1}cz_{2,1,y} + \theta dz_{3,-1,y} + \theta^{-1}fz_{3,1,y}}, \end{aligned}$$

где

$$z_{\epsilon,\alpha,x} = \exp(h_{\epsilon,\alpha,x} - h_{1,1,x}), \quad \epsilon = 1, 2, 3; \alpha = \pm 1,$$

$$\theta = \exp(J\beta),$$

$$a = \exp[\beta(p(-1|1) - p(1|1))], \quad b = \exp[\beta(p(-1|2) - p(1|1))], \quad c = \exp[\beta(p(1|2) - p(1|1))],$$

$$d = \exp[\beta(p(-1|3) - p(1|1))], \quad f = \exp[\beta(p(1|3) - p(1|1))].$$

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3-SHO‘BA: DIFFERENSIAL TENGLAMALAR VA MATEMATIK FIZIKA TENGLAMALARI.

Recursion formulas for Lauricella functions

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We introduce the following notation

$$\mathbf{a} := (a_1, \dots, a_m), \quad \mathbf{b} := (b_1, \dots, b_m), \quad \mathbf{c} := (c_1, \dots, c_m), \quad \mathbf{z} := (z_1, \dots, z_m);$$

$$\mathbf{k} := (k_1, \dots, k_m), \quad |\mathbf{k}| := k_1 + \dots + k_m; \quad |\mathbf{k}_j| := k_1 + \dots + k_j, \quad 1 \leq j < m; \quad |\mathbf{k}_0| := 0;$$

A symbols $\mathbf{e}_j := (0, \dots, 1, \dots, 0)$ denote the vectors with j th component equal to 1 and the others equal to 0 ($1 \leq j \leq m$).

Lauricella's functions are read as follows:

$$\begin{aligned} F_A^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &:= \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|} (b_1)_{k_1} \dots (b_m)_{k_m}}{(c_1)_{k_1} \dots (c_m)_{k_m}} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_m^{k_m}}{k_m!}; \\ F_B^{(m)} \left[\begin{matrix} \mathbf{a}, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] &:= \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a_1)_{k_1} \dots (a_m)_{k_m} (b_1)_{k_1} \dots (b_m)_{k_m}}{(c)_{|\mathbf{k}|}} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_m^{k_m}}{k_m!}; \\ F_C^{(m)} \left[\begin{matrix} a, b; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &:= \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|} (b)_{|\mathbf{k}|}}{(c_1)_{k_1} \dots (c_m)_{k_m}} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_m^{k_m}}{k_m!}; \\ F_D^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] &:= \sum_{|\mathbf{k}|=0}^{\infty} \frac{a_{|\mathbf{k}|} (b_1)_{k_1} \dots (b_m)_{k_m}}{(c)_{|\mathbf{k}|}} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_m^{k_m}}{k_m!}, \end{aligned}$$

where a, b, c, a_j, b_j, c_j are independent of z_j ($j = 1, \dots, m$) and $(a)_k$ is the Pochhammer symbol, which is expressed in terms of the gamma function $\Gamma(s)$ as $(\lambda)_k = \Gamma(\lambda + k)/\Gamma(\lambda)$. We shall call a, b, c, a_j, b_j, c_j ($j = 1, \dots, m$) the parameters of the Lauricella's hypergeometric functions; they are arbitrary complex numbers; $z_1, \dots, z_m$ are complex variables; $k_1, \dots, k_m$ are nonnegative integer numbers: $k_1 \geq 0, \dots, k_m \geq 0$.

Theorem. The following recursive formulas for the Lauricella hypergeometric functions are valid:

$$\begin{aligned} F_A^{(m)} \left[\begin{matrix} a + n, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= F_A^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] + \sum_{j=1}^m \frac{b_j z_j}{c_j} \sum_{k=1}^n F_A^{(m)} \left[\begin{matrix} a + k, \mathbf{b} + \mathbf{e}_j; \\ \mathbf{c} + \mathbf{e}_j; \end{matrix} \mathbf{z} \right]; \\ F_A^{(m)} \left[\begin{matrix} a - n, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= F_A^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] - \sum_{j=1}^m \frac{b_j z_j}{c_j} \sum_{k=0}^{n-1} F_A^{(m)} \left[\begin{matrix} a - k, \mathbf{b} + \mathbf{e}_j; \\ \mathbf{c} + \mathbf{e}_j; \end{matrix} \mathbf{z} \right]. \end{aligned}$$

$$\begin{aligned}
F_A^{(m)} \left[\begin{matrix} a, \mathbf{b} + n\mathbf{e}_j; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= F_A^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] + \frac{az_j}{c_j} \sum_{k=1}^n F_A^{(m)} \left[\begin{matrix} a+1, \mathbf{b} + k\mathbf{e}_j; \\ \mathbf{c} + \mathbf{e}_j; \end{matrix} \mathbf{z} \right]; \\
F_A^{(m)} \left[\begin{matrix} a, \mathbf{b} - n\mathbf{e}_j; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= F_A^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] - \frac{az_j}{c_j} \sum_{k=0}^{n-1} F_A^{(m)} \left[\begin{matrix} a+1, \mathbf{b} - k\mathbf{e}_j; \\ \mathbf{c} + \mathbf{e}_j; \end{matrix} \mathbf{z} \right]; \\
F_A^{(m)} \left[\begin{matrix} a, \mathbf{b} + n\mathbf{e}_j; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= \sum_{k=0}^n \binom{n}{k} \frac{(a)_k}{(c_j)_k} z_j^k F_A^{(m)} \left[\begin{matrix} a+k, \mathbf{b} + k\mathbf{e}_j; \\ \mathbf{c} + k\mathbf{e}_j; \end{matrix} \mathbf{z} \right]; \\
F_A^{(m)} \left[\begin{matrix} a, \mathbf{b} - n\mathbf{e}_j; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= \sum_{k=0}^n \binom{n}{k} \frac{(a)_k}{(c_j)_k} (-z_j)^k F_A^{(m)} \left[\begin{matrix} a+k, \mathbf{b}; \\ \mathbf{c} + k\mathbf{e}_j; \end{matrix} \mathbf{z} \right]; \\
F_A^{(m)} \left[\begin{matrix} a; \mathbf{b}; \\ \mathbf{c} - n\mathbf{e}_j; \end{matrix} \mathbf{z} \right] &= F_A^{(m)} \left[\begin{matrix} a; \mathbf{b}; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] + a \sum_{i=1}^m b_i z_i \sum_{k=1}^n \frac{F_A^{(m)} \left[\begin{matrix} a+1; \mathbf{b} + \mathbf{e}_i; \\ \mathbf{c} + (2-k)\mathbf{e}_j; \end{matrix} \mathbf{z} \right]}{(c_j - k)(c_j - k + 1)}; \\
F_B^{(m)} \left[\begin{matrix} \mathbf{a} + n\mathbf{e}_j, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] &= F_B^{(m)} \left[\begin{matrix} \mathbf{a}, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] + \frac{b_j z_j}{c} \sum_{k=1}^n F_B^{(m)} \left[\begin{matrix} \mathbf{a} + k\mathbf{e}_j, \mathbf{b} + \mathbf{e}_j; \\ c+1; \end{matrix} \mathbf{z} \right]; \\
F_B^{(m)} \left[\begin{matrix} \mathbf{a} + n\mathbf{e}_j, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] &= \sum_{k=0}^n \frac{(b_j)_k}{(c)_k} z_j^k F_B^{(m)} \left[\begin{matrix} \mathbf{a} + k\mathbf{e}_j, \mathbf{b} + k\mathbf{e}_j; \\ c+k; \end{matrix} \mathbf{z} \right]; \\
F_B^{(m)} \left[\begin{matrix} \mathbf{a}, \mathbf{b}; \\ c-n; \end{matrix} \mathbf{z} \right] &= F_B^{(m)} \left[\begin{matrix} \mathbf{a}, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] + \sum_{j=1}^m a_j b_j z_j \sum_{k=1}^n \frac{F_B^{(m)} \left[\begin{matrix} \mathbf{a} + \mathbf{e}_j, \mathbf{b} + \mathbf{e}_j; \\ c+2-k; \end{matrix} \mathbf{z} \right]}{(c-k)(c-k+1)}; \\
F_C^{(m)} \left[\begin{matrix} a+n, b; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] &= F_C^{(m)} \left[\begin{matrix} a, b; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] + b \sum_{j=1}^m \frac{z_j}{c_j} \sum_{k=1}^n F_C^{(m)} \left[\begin{matrix} a+k, b+1; \\ \mathbf{c} + \mathbf{e}_j; \end{matrix} \mathbf{z} \right]; \\
F_C^{(m)} \left[\begin{matrix} a, b; \\ \mathbf{c} - n\mathbf{e}_j; \end{matrix} \mathbf{z} \right] &= F_C^{(m)} \left[\begin{matrix} a, b; \\ \mathbf{c}; \end{matrix} \mathbf{z} \right] + ab z_j \sum_{k=0}^{n-1} \frac{F_C^{(m)} \left[\begin{matrix} a+1, b+1; \\ \mathbf{c} + (1-k)\mathbf{e}_j; \end{matrix} \mathbf{z} \right]}{(c_j - k)(c_j - k - 1)}; \\
F_D^{(m)} \left[\begin{matrix} a+n, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] &= F_D^{(m)} \left[\begin{matrix} a, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] + \frac{1}{c} \sum_{j=1}^m b_j z_j \sum_{k=1}^n F_D^{(m)} \left[\begin{matrix} a+k, \mathbf{b} + \mathbf{e}_j; \\ c+1; \end{matrix} \mathbf{z} \right]; \\
F_D^{(m)} \left[\begin{matrix} a+n, \mathbf{b}; \\ c; \end{matrix} \mathbf{z} \right] &= \sum_{k_1=0}^{n-|\mathbf{k}_0|} \dots \sum_{k_m=0}^{n-|\mathbf{k}_{m-1}|} \frac{1}{(c)_{|\mathbf{k}|}} \times \\
&\quad \times \prod_{j=1}^m \left[\binom{n-|\mathbf{k}_{j-1}|}{k_j} (b_j)_{k_j} z_j^{k_j} \right] F_D^{(m)} \left[\begin{matrix} a+|\mathbf{k}|, \mathbf{b} + \mathbf{k}; \\ c+|\mathbf{k}|; \end{matrix} \mathbf{z} \right].
\end{aligned}$$

Proof of the theorem is carried out using the method described in [1].

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TIP O'ZGARISH CHIZIG'I SILLIQ BO'LMAGAN PARABOLO-GIPERBOLIK TENGLAMA UCHUN TO'RTINCHI NOLOKAL MASALA

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Ushbu ishda tip o'zgarish chizig'i silliq bo'lmagan parabolo-giperbolik tenglama uchun qo'yilgan integral shartli masalaning bir qiymatli yechilishi tadqiq qilingan.

xOy tekisligining $x \geq 0, y \geq 0$ bo'lganda $x = 1, y = 1$ to'g'ri chiziqlar bilan, $x \geq 0, y \leq 0$ bo'lganda $x = 0, x - y = 1$ hamda $x \leq 0, y \geq 0$ bo'lganda esa $y = 0, x - y = -1$ to'g'ri chiziqlar bilan chegaralangan D sohada

$$u_{xx} - \frac{1}{2}(1 - \operatorname{sign}(xy))u_{yy} - \frac{1}{2}(1 + \operatorname{sign}(xy))u_y = 0 \quad (1)$$

tenglamani qaraylik,

$$D_0 = D \cap \{(x, y) : x > 0, y > 0\}, \quad D_1 = D \cap \{(x, y) : y < 0, x + y > 0\},$$

$$D_2 = D \cap \{(x, y) : y < 0, x + y < 0\}, \quad D_3 = D \cap \{(x, y) : x < 0, x + y > 0\},$$

$$D_4 = D \cap \{(x, y) : x < 0, x + y < 0\};$$

OA_1, OA_2, OB_1, OB_2 lar mos ravishda $x = 0, y = 0$ chiziqlarning kesmalari, bu yerda $O(0, 0), A_1(0, 1), B_1(1, 0), A_2(0, -1), B_2(-1, 0), E(1/2, -1/2), F(-1/2, 1/2), I = \{(x, y) : x + y = 0, (-1/2) < x < (1/2)\}$.

(1) tenglama D_0 va $D_1 \cup D_2 (D_3 \cup D_4)$ sohalarda mos ravishda parabolik va giperbolik tipga tegishli bo'lib, u D_0 sohada

$$u_{xx} - u_y = 0, \quad (x, y) \in D_0, \quad (2)$$

$D_1 \cup D_2$ va $D_3 \cup D_4$ sohalarda

$$u_{xx} - u_{yy} = 0, \quad (x, y) \in D_1 \cup D_2 \cup D_3 \cup D_4 \quad (3)$$

ko'rinishlarda yoziladi; OB_1 va OA_1 - (1) tenglamaning tip o'zgarish chiziqlari bo'lib, OB_1 - (1) tenglama uchun xarakteristika bo'ladi. OA_1 esa xarakteristika bo'lmaydi.

D sohada quyidagi masalani qaraylik.

BS_3 masala. Quyidagi shartlarni qanoatlantiruvchi $u(x, y)$ funksiya topilsin:

- 1) D_0, D_1, D_2, D_3, D_4 sohalarda (1) tenglamaning regular yechimi bo'lsin;
- 2) $u(x, y) \in C(\bar{D}) \cap C_{x,y}^{2,1}(D_0) \cap C_{x,y}^{2,2}(D_1 \cup D_2 \cup D_3 \cup D_4 \setminus I)$ sinfga tegishli bo'lsin;
- 3) quyidagi shartlarni qanoatlantirsin:

$$u(1, y) = \alpha(y) \int_0^l u(x, y) dx + \varphi(y), \quad 0 \leq y \leq 1; \quad (4)$$

$$u_x(0, y) = f_1(y), \quad -1 < y < 0; \quad (5)$$

$$u(0, y) = -u(0, -y) + f_2(y), \quad -1 \leq y \leq 0; \quad (6)$$

$$u_y(x, 0) = g_1(x), \quad -1 < x < 0; \quad (7)$$

$$u(x, 0) = u(-x, 0) + g_2(x), \quad -1 \leq x \leq 0 \quad (8)$$

shartlarni, OB_1 va OA_1 tip o'zgarish chiziqlarida esa

$$u_y(x, +0) = u_y(x, -0), \quad 0 < x < 1,$$

$$u_x(+0, y) = u_x(x, -0), \quad 0 < y < 1$$

ulash shartlarini bajarsin, bu yerda $\alpha(y)$, $\varphi(y)$, $f_1(y)$, $f_2(y)$, $g_1(x)$, $g_2(x)$ - berilgan uzluksiz funksiyalar, $\alpha(0) \neq 2$.

Natijada ba'zi hisoblashlarni bajarib, $\nu_2(y)$ noma'lum funksiyaga nisbatan 2-tur Fredgolim integral tenglamasiga ega bo'lamiz:

$$\nu_2(y) = \int_y^1 \nu_2(t) K_1(y, t) dt + \int_0^y \nu_2(t) [K_1(y, t) + K_2(y, t)] dt + \omega_6(y)$$

$$\nu_2(y) = \int_0^1 \nu_2(t) K_3(y, t) dt + \omega_6(y), \quad (9)$$

$$K_3(y, t) = \begin{cases} K_1(y, t) + K_2(y, t), & 0 < t < y \\ K_1(y, t), & y < t < 1 \end{cases}$$

$$\text{Bu yerda } K_1(y, t) = \int_0^y K_\eta(\eta, t) (G_2(0, y; 0, \eta) - G_2(0, y; 1, \eta)) d\eta + \int_0^1 K_\xi(\xi, t) G_2(0, y; \xi, 0) d\xi$$

$$K_2(y, t) = G_2(0, y; 0, t) - G_2(0, y; 1, t)$$

Integral tenglamaning yadrosi $K_3(y, t)$.

Masala yechimining yagonaligini ekstremum prinsipi yordamida isbotlaymiz. Buning uchun bir jinsli nolokal masalani qaraymiz, ya'ni

$$\varphi(y) \equiv f_1(y) \equiv f_2(y) \equiv g_1(x) \equiv g_2(x) \equiv 0. \quad (10)$$

Teorema. Agar (10) shart bajarilsa, u holda nolokal masalaning $u(x, y)$ yechimi o'qzining musbat maksimumiga (manfiy minimumiga) yopiq $\bar{D}_0$ sohaning $\overline{OA_1} \cup \overline{A_0B_1}$ chegarasida erishadi.

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АНАЛОГ ЗАДАЧИ ДИРИХЛЕ ДЛЯ НАГРУЖЕННОГО УРАВНЕНИЯ СМЕШАННОГО ТИПА С ДРОБНЫМИ ПРОИЗВОДНЫМИ В ПРЯМОУГОЛЬНОЙ ОБЛАСТИ

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Отметим, что дифференциальные уравнения в частных производных, содержащие производные дробного порядка, имеют важные приложения в различных областях [1]–[6] и полученные результаты, безусловно, представляют большое значение из-за их новизны. Если в различных частях области задания дифференциальное уравнение в частных производных, содержащее дробную производную, принадлежит различным типам, то такое дифференциальное уравнение следует называть уравнением смешанного типа в этой области.

В данной работе изучена аналог задачи Дирихле для нагруженного уравнения смешанного типа с дробными производными в прямоугольной области.

Рассмотрим уравнение

$$Lu = \begin{cases} u_{xx} - {}_c D_{0y}^\alpha u + \mu_1 u(x, 0) = 0 & \text{при } y > 0, \quad 1 < \alpha \leq 2, \\ u_{xx} - {}_c D_{0y}^\beta u + \mu_2 u(x + y, 0) = 0 & \text{при } y < 0, \quad 1 < \beta \leq 2 \end{cases}$$

в прямоугольной области $\Omega = \{(x, y) : 0 < x < 1, -p < y < q\}$, где $p, q, \mu_j (j = 1, 2)$ – заданные действительные числа, причем

$$p > 0, \quad q > 0, \quad \mu_j \geq 0 \quad (j = 1, 2),$$

а ${}_c D_{0y}^\gamma$ – оператор дробного порядка в смысле Капуто [2], [7, с. 38], и оно имеет вид:

$${}_c D_{0y}^\gamma u = \frac{1}{\Gamma(2 - \gamma)} \int_0^y (y - t)^{1-\gamma} \frac{\partial^2 u(x, t)}{\partial t^2} dt, \quad 1 < \gamma \leq 2.$$

Введем обозначения: $J = \{(x, y) : 0 < x < 1, y = 0\}$, $\Omega_1 = \Omega \cap \{(x, y) : x > 0, y > 0\}$, $\Omega_2 = \Omega \cap \{(x, y) : x > 0, y < 0\}$, $\Omega = \Omega_1 \cup \Omega_2 \cup J$.

Класс C_θ^k . Через $C_\theta^k(M)$ обозначим класс функций, непрерывных вместе со своими частными производными по θ до k -го порядка включительно в области M .

Задача AD. Найти функцию $u(x, y)$ со свойствами:

$$u(x, y) \in C(\overline{\Omega}) \cap C_x^2(\Omega_1 \cup \Omega_2 \cup J), \quad {}_c D_{0y}^\gamma u \in C(\Omega_1 \cup \Omega_2), \quad (1)$$

$$y^{1-\alpha} u_y(x, y) \in C(\Omega_1 \cup J), \quad (-y)^{1-\beta} u_y(x, y) \in C(\Omega_2 \cup J);$$

$$Lu(x, y) = 0, \quad (x, y) \in \Omega_1 \cup \Omega_2;$$

$$u(0, y) = 0, \quad u(l, y) = 0, \quad -p \leq y \leq q;$$

$$u(x, -p) = \psi(x), \quad u(x, q) = \varphi(x), \quad 0 \leq x \leq l;$$

$$\lim_{y \rightarrow +0} y^{1-\alpha} u_y(x, y) = \lim_{y \rightarrow -0} (-y)^{1-\beta} u_y(x, y), \quad (x, 0) \in J,$$

где $\psi(x), \varphi(x)$ – заданные достаточно гладкие функции, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$.

Теорема 1. Если существует решение задачи AD, то оно единственно тогда и только тогда, когда при всех $n \in N$ выполнены условия

$$\Delta_n(p, q) = pE_{\alpha,1}(\lambda_n q^\alpha)E_{\beta,2}(\lambda_n p^\beta) + qE_{\beta,1}(\lambda_n p^\beta)E_{\alpha,2}(\lambda_n q^\alpha) \neq 0, \quad \lambda_n = \left(\frac{\pi n}{l}\right)^2.$$

Теорема 2. Если функции $\varphi(x)$ и $\psi(x)$ удовлетворяют условиям $\varphi(x), \psi(x) \in C^3[0, l]$, $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0$, $\psi(0) = \psi(l) = \psi''(0) = \psi''(l) = 0$ и $\Delta_n(p, q) \neq 0$ при всех $n = \overline{1, n_0}$, то существует единственное решение задачи AD и оно определяется рядом $u(x, y) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} u_n(y) \sin \frac{\pi n}{l} x$. При этом функция $u(x, y)$ принадлежит классу (1).

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Об одном методе решения краевой задачи для вязко-трансзвукового уравнения в полугораниченной области

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Дифференциальные уравнения в частных производных третьего порядка рассматриваются при решении задач теории нелинейной акустики и в гидродинамической теории космической плазмы, фильтрации жидкости в пористых средах [1].

В работе [2], учитывая свойства вязкости и теплопроводности газа, из системы Навье-Стокса было получено уравнение третьего порядка с кратными характеристиками, содержащее вторую производную по времени

$$u_{xxx} + u_{yy} - \frac{\nu}{y}u_y = u_x u_{xx}, \quad \nu = \text{const.}$$

Это уравнение при $\nu = 1$ описывает осесимметричный поток, а при $\nu = 0$ описывает плоско - параллельный поток [3].

В области $D^+ = \{(x, y) : 0 < x < +\infty, 0 < y < q\}$ рассмотрим уравнение:

$$L[u] \equiv u_{xxx} + u_{yy} + \mu u = 0, \quad (1)$$

где $q \in R$ и для него исследуем следующую задачу.

Задача А. Найти регулярное решение уравнения (1) в области D^+ из класса $C_{x,y}^{3,2}(D^+) \cap C_{x,y}^{2,1}(D^+ \cup \Gamma_1)$, имеющего ограниченные первой производной по переменной y равномерно по x и второй производной по x , при $x \rightarrow +\infty$ и $u_y \in L_2(D^+)$, удовлетворяющее краевым условиям:

$$u(x, 0) = u(x, q) = 0, \quad 0 \leq x < +\infty, \quad (2)$$

$$\begin{cases} au(0, y) + bu_{xx}(0, y) = \psi_1(y), \\ u_x(0, y) = \psi_2(y), \\ \lim_{x \rightarrow +\infty} u(x, y) = 0, \end{cases} \quad 0 \leq y \leq q, \quad (3)$$

где $\Gamma_1 = \partial D^+$ - граница области D^+ , $a, b \in R$ а также $a^2 + b^2 \neq 0$, а $\psi_i(y)$, $i = \overline{1, 2}$ - заданные достаточно гладкие функции, причем

$$\psi_{in}(0) = \psi_{in}(q) = 0, \quad i = \overline{1, 2}. \quad (4)$$

Отметим, что аналогичная задача исследована в работах [4-6] для уравнения $u_{xxx} - u_{yy} = 0$.

Теорема 1. Если задача А имеет решение, то при выполнении условий $ab \leq 0$, $\mu \leq 0$ оно единственно.

Доказательство. Предположим обратное, пусть задача А имеет два различных решений: $u_1(x, y)$ и $u_2(x, y)$. Тогда функция $u(x, y) = u_1(x, y) - u_2(x, y)$ удовлетворяет уравнению (1) с однородными краевыми условиями. Докажем, что $u(x, y) \equiv 0$ в области D^+ .

В области D^+ справедливо тождество

$$u L[u] \equiv \frac{\partial}{\partial x} \left(u u_{xx} - \frac{1}{2} u_x^2 \right) + \frac{\partial}{\partial y} (u u_y) - u_y^2 + \mu u^2 = 0. \quad (5)$$

Интегрируя тождество (5) по области $D_d^+ = \{(x, y) : 0 < x < d, 0 < y < q\}$, учитывая однородные краевые условия и требуя $a \neq 0$ получим

$$\frac{1}{2} \lim_{d \rightarrow +\infty} \int_0^q u_x^2(d, y) dy - \frac{b}{a} \int_0^q u_{xx}^2(0, y) dy + \iint_{D^+} u_y^2(x, y) dx dy - \mu \iint_{D^+} u^2(x, y) dx dy = 0.$$

Пусть $\mu = 0$, тогда по условия теоремы 1, т.е. $ab \leq 0$, получим $u_y(x, y) = 0$, отсюда имеем $u(x, y) = f(x)$. Пологая $y = 0$ и учитывая что $u(x, 0) = 0$, имеем $f(x) = 0$. Следовательно, $u(x, y) \equiv 0$, $(x, y) \in D^+$. Если $\mu < 0$, то из четвертого слагаемого, получим $u(x, y) \equiv 0$, $(x, y) \in D^+$. В силу последнего, получим $u_1(x, y) = u_2(x, y)$.

В случае $b \neq 0$ аналогично получаем равенство $u(x, y) \equiv 0$ в D^+ .

Теорема 1 доказана.

Теорема 2. Если функции $\psi_i(y) \in C^2(0 \leq y \leq q)$, $i = \overline{1, 2}$ и выполняется условия (4), то решение задачи A существует.

Теорема 2 была доказана с применением метода разделения переменных. Решение рассматриваемой задачи получено в форме бесконечного ряда, при этом установлена равномерная сходимость и возможность почленного дифференцирования при выполнении определенных условий на заданные функции. В процессе обоснования равномерной сходимости доказано отсутствие проблемы "малого знаменателя".

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INVERSE SCATTERING FOR THE NEGATIVE ORDER MATRIX MODIFIED KORTEWEG-DE VRIES EQUATION WITH A SELF-CONSISTENT SOURCE

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We analyze the following problem

$$\left\{ \begin{array}{l} \rho_x(x, t) = v(x, t)v_t(x, t) + v_t(x, t)v(x, t) \\ v_{xt}(x, t) + v(x, t)\rho(x, t) + \rho(x, t)v(x, t) + v(x, t) = 2 \sum_{n=1}^{2N} (f_{1,n} \otimes \hat{f}_{2,n}^T + f_{2,n} \otimes \hat{f}_{1,n}^T), \\ Lf_k = \xi_k f_k, \quad k = 1, 2, \dots, 2N, \end{array} \right. \quad (1)$$

under the initial condition

$$v(x, t)|_{t=0} = v_0(x), \quad (2)$$

where $v(x, t)$ and $\rho(x, t)$ are $(m \times m)$ square matrices and the matrix norm is defined as $\|H\| = \max_i \sum_{j=1}^N |H_{ij}|$. Besides, let the matrix $v_0(x)$ is a real and symmetric $(m \times m)$ matrix having the following properties:

- $\int_{-\infty}^{\infty} (1 + |x|) \|v_0(x)\| dx < \infty$
- Matrix operator $L(0) = -iJ \frac{d}{dx} - U_0(x)$ has exactly $2N$ simple eigenvalues

$$\lambda_1(0), \lambda_2(0), \dots, \lambda_{2N}(0).$$

$$\text{Here } U_0(x) = \begin{pmatrix} 0 & iv_0(x) \\ iv_0(x) & 0 \end{pmatrix}.$$

Let matrix functions $v(x, t)$ and $\rho(x, t)$ belong to the class

$$\Lambda = \left\{ \begin{array}{l} v \in C^2, \quad \rho \in C^{1,0}; \quad \lim_{x \rightarrow \pm\infty} \rho(x, t) = 0, \quad t \in \mathbb{R}; \\ \int_{-\infty}^{+\infty} [(1 + |x|) (\|v(x, t)\| + \|v_x(x, t)\| + \|v_t(x, t)\| + \|v_{xt}(x, t)\|)] dx < \infty \end{array} \right\}.$$

Furthermore, $f_n = \begin{pmatrix} f_{1,n} \\ f_{2,n} \end{pmatrix}$ are column eigenvector functions that corresponds to the eigenvalue λ_n and $\hat{f}_n(\lambda_n, x, t) = \sigma f_n(-\lambda_n, x, t)$ corresponds to the eigenvalue $-\lambda_n$, which are normalized by the following conditions

$$\int_{-\infty}^{\infty} \hat{f}_n^T(x, t) f_n(x, t) dx = a_n^2(t), \quad n = 1, 2, \dots, 2N. \quad (3)$$

The system is equivalent to the following Lax equation

$$L_t = [L, B] + \frac{1}{2\lambda} \begin{pmatrix} 0 & \sum_{n=1}^{2N} [J, f_n \otimes \hat{f}_n^T] \\ -\sum_{n=1}^{2N} [J, f_n \otimes \hat{f}_n^T] & 0 \end{pmatrix},$$

where

$$L(t) = -iJ \frac{d}{dx} - U(x, t),$$

$$B(t) = -\frac{i}{2\lambda} \left(\frac{1}{2}J + PJ \right) + \frac{U_t}{2\lambda}.$$

Here $U(x, t) = i \begin{pmatrix} 0_m & v(x, t) \\ v(x, t) & 0_m \end{pmatrix}$ and $P(x, t) = \begin{pmatrix} \rho(x, t) & 0_m \\ 0_m & \rho(x, t) \end{pmatrix}$ are $2m \times 2m$ matrices $J = \begin{bmatrix} I_m & 0_m \\ 0_m & -I_m \end{bmatrix}$, I_m is the unit matrix, $\sigma = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ and $a_n^2(t)$, $n = \overline{1, N}$ are nonzero continuous scalar functions.

The work is devoted to find the solution $\{v(x, t), \rho(x, t), f_n(x, t), n = 1, \dots, 2N\}$ to the problem (1)-(3) in the class Λ by IST method.

Theorem. If $\{v(x, t), \rho(x, t), f_n(x, t), n = 1, \dots, 2N\}$ solves the problem (1)-(3), then the scattering data for the operator $L(t)$ evolve in time t by the formulas

$$\frac{\partial R(\xi, t)}{\partial t} = \frac{i}{2\xi} R(\xi, t), \quad \xi \in \mathbb{R},$$

$$\frac{d\xi_j}{dt} = 0, \quad \frac{d\Gamma_j(t)}{dt} = \left(\frac{i}{2\xi_j} + a_j^2(t) \right) \Gamma_j(t), \quad j = \overline{1, N}.$$

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Initial-boundary value problem for a fourth-order differential equation with variable coefficients

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In this work, an initial-boundary value problem is considered for a fourth-order partial differential equation with variable coefficients and a source term in a rectangular domain. Mixed boundary value problems for partial differential equations with variable coefficients have been studied in the literature [1–2], while problems for equations with source terms have been considered in [3–5].

In the domain $\Omega = \{(x, t) : 0 < x < p, 0 < t < \beta\}$ we consider the following equation:

$$Lu \equiv t u_{tt} + a^2 u_{xxxx} + b u_t + \mu u(x, d) = 0$$

here, $0 < b < 1$, $0 < d < \beta$, μ -are given constants

Problem 1. Find a function $u(x, t)$ in the domain Ω that satisfies the following conditions:

1)

$$u(x, t) \in C_{x,t}^{2,1}(\bar{\Omega}) \cap C_{x,t}^{4,2}(\Omega) \quad (1)$$

2)

$$Lu(x, t) = 0 \quad (2)$$

3) The function $u(x, t)$ satisfies the initial conditions

$$u(x, 0) = \varphi(x), \lim_{t \rightarrow 0} t^b u_t(x, t) = \psi(x), \quad 0 \leq x \leq p, \quad (3)$$

and the boundary conditions

$$u(0, t) = u(p, t) = 0, \quad u_{xx}(0, t) = u_{xx}(p, t) = 0, \quad 0 \leq t \leq \beta, \quad (4)$$

Here, $\varphi(x)$, $\psi(x)$ are given continuous functions.

A function $u(x, t)$ that satisfies conditions (1)–(4) is called a regular solution of the problem (1)–(4).

The existence of a unique solution to this problem has been proven. The existence of the solution was demonstrated using the method of separation of variables. The solution of the problem was found in the form of a series expansion in terms of the eigenfunctions of the corresponding spectral problem. In the closed domain, the series representing the solution, as well as the series composed of its second and fourth-order derivatives with respect to t and x , were shown to converge absolutely and uniformly depending on the initial function. The uniqueness of the solution follows from the completeness of the eigenfunctions. The stability of the solution was also established.

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A CONTROL PROBLEM FOR A PSEUDO-PARABOLIC EQUATION WITH MULTIPLE INVOLUTION

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Consider a bounded domain $\Omega \subset \mathbb{R}^n$ in the form of an open rectangular parallelepiped defined by

$$\Omega := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 < x_i < p_i, \ i = 1, \dots, n\},$$

where each $p_i > 0$ denotes the length of the domain along the x_i -axis. The boundary of Ω is denoted by $\partial\Omega$.

Consider the family of mappings $S_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $S_i x = (x_1, \dots, x_{i-1}, p_i - x_i, x_{i+1}, \dots, x_n)$, where each S_i reflects the i -th coordinate of the vector x with respect to the midpoint of the interval $(0, p_i)$. It is straightforward to verify that each S_i is an involution, i.e., $S_i^2 = I$, where I denotes the identity transformation on $\mathbb{R}^n$.

Let us consider all possible products of mappings S_i , i.e., $S_{ji} = S_j S_i$, or $S_{jik} = S_j S_i S_k, \dots$. The total number of such mappings, taking into account the identity mapping $S_0 x = x$, is equal to 2^n . To number such mappings, we will use the binary number system, namely, if $0 \leq j < 2^n$ in the binary number system, the representation $j \equiv (j_n \dots j_1)_2 = j_1 + 2j_2 + \dots + 2^{n-1} j_n$, where j_k takes one of the values 0 or 1. Therefore, introducing the vector $j = (j_1, \dots, j_n)$, we can consider mappings of the type $S_j \equiv S_1^{j_1} \dots S_n^{j_n}$ corresponding to the index j .

Using these mappings, we define a nonlocal operator L_n acting on a function $U(x)$ by

$$L_n U(x) = \sum_{j=0}^{2^n-1} a_j \Delta U(S_j x),$$

where $a_0, \dots, a_{2^n-1}$ are given real coefficients, and Δ denotes the standard Laplace operator. We refer to L_n as a nonlocal analogue of the Laplacian due to its dependence on spatial reflections.

It is known that the operator L_n is defined as follows for $n = 2$:

$$L_2 U(x_1, x_2) = a_0 \Delta U(x_1, x_2) + a_1 \Delta U(p_1 - x_1, x_2) + a_2 \Delta U(x_1, p_2 - x_2) + a_3 \Delta U(p_1 - x_1, p_2 - x_2).$$

In this paper, we investigate the following pseudo-parabolic equation with involution:

$$u_t(x, t) - L_n u(x, t) - L_n u_t(x, t) = h(x) \nu(t), \quad x \in \Omega, \quad t > 0, \quad (1)$$

with the homogeneous Dirichlet boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (2)$$

and the initial condition

$$u(x, 0) = 0, \quad x \in \overline{\Omega}, \quad (3)$$

where, $h(x)$ is a given function, and $\nu(t)$ is a control function. We say that function $\nu \in L_2(\mathbb{R}_+)$ is an *admissible control* if it satisfies the pointwise constraint $|\nu(t)| \leq 1$ for almost every $t \geq 0$.

The present study is devoted to the analysis of the following thermal control problem:

Control Problem. For given a measurable function $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}$, find an admissible control function $\nu \in L_2(\mathbb{R}_+)$ such that the solution $u(x, t)$ of the mixed problem (1)-(3) satisfies the integral condition

$$\int_{\Omega} \rho(x) u(x, t) dx = \phi(t), \quad \forall t \geq 0, \quad (4)$$

where $\rho(x)$ is weight function.

Initial studies on time-optimal control for heat conduction in bounded n -dimensional domains were presented in [1], where estimates for the minimal time required to reach a prescribed average temperature were established. In [2], boundary control problems for pseudo-parabolic equations were further explored, where the construction of a suitable control function was achieved to ensure the heating of an inhomogeneous thin plate up to a prescribed temperature distribution. In recent years, increasing attention has been paid to mixed problems for parabolic-type equations involving involution. Various inverse problems for such equations have been analyzed, for instance, in [3].

For any constant $M > 0$, we define the class $W(M)$ as the set of functions $\phi \in W_2^2(\mathbb{R})$ satisfying the conditions

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \leq M, \quad \phi(t) = 0 \quad \text{for } t \leq 0.$$

We now state the main result of this paper.

Theorem. There exists constant $M > 0$ such that for any function $\phi \in W(M)$, the solution $\nu(t)$ of the equation (4) exists and satisfies condition $|\nu(t)| \leq 1$.

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EVASION DIFFERENTIAL GAME ESCAPE FROM MULTIPLE PURSUERS IN A DIFFERENTIAL GAME WITH INTEGRAL RESTRICTIONS

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This research focuses on analyzing an evasion differential game in which a single evader attempts to avoid capture by multiple pursuers, subject to integral constraints on the players' control functions. The control resources of both pursuers and the evader vary linearly with time. The game is formulated in the Euclidean space $\mathbb{R}^n$. A constructive scheme for the evader's strategy is presented that ensures successful escape. Since the extension to $\mathbb{R}^n$ is straightforward, it suffices to analyze the case in $\mathbb{R}^2$. The dynamics of the pursuers x_i and the evader y are described by the following system:

$$\begin{cases} \dot{x}_i = u_i, & x_i(0) = x_{i0}, \quad i = 1, 2, \dots, M, \\ \dot{y} = v, & y(0) = y_0, \end{cases} \quad (1)$$

where $x_i, x_{i0}, u_i, y, y_0, v \in \mathbb{R}^2$, $x_{i0} \neq y_0$, and u_i and v represent the control variables of the pursuers and evader, respectively.

Definition 1. The measurable control functions $u_i(t)$ and $v(t)$, $t \geq 0$, are called admissible if they satisfy the following integral constraints:

$$\int_0^t |u_i(s)|^2 ds \leq \rho_i^2 t, \quad i = 1, \dots, M; \quad \int_0^t |v(s)|^2 ds \leq \sigma^2 t.$$

Definition 2. A mapping $(t, x_1, \dots, x_M, y, u_1, \dots, u_M) \rightarrow V(t, x_1, \dots, x_M, y, u_1, \dots, u_M)$, $V : [0; +\infty) \times \mathbb{R}^{4M+2} \rightarrow \mathbb{R}^2$, is termed a strategy of the evader if the initial value problem (1) possesses a unique solution $(x_1(t), \dots, x_M(t), y(t))$ with $v = V(t, x_1, \dots, x_M, y, u_1, \dots, u_M(t))$ for all admissible controls $u_1(t), \dots, u_M(t)$ of the pursuers.

Here, the pursuers may apply arbitrary admissible controls, while the evader adheres to a prescribed strategy.

Definition 3. Evasion in game (1) is said to be achievable if there exists a strategy V for the evader such that, regardless of the pursuers' controls, $x_i(t) \neq y(t)$ holds for all $t \geq 0$, $i = 1, 2, \dots, M$.

Let

$$\mathcal{J}_k = \bigcup_{j=k}^{m_0} [\tau_j, \tau'_j], \quad \mathcal{J}_{m_0+1} \neq \emptyset, \quad k = 1, 2, \dots, m_0.$$

where m_0 is a positive integer.

We define the function

$$r : [0, T] \rightarrow \{0, 1, 2, \dots, m_0\}$$

by the following equation

$$r(t) = \begin{cases} 0, & t \in [0, T] \setminus \mathcal{J}_1, \\ k, & t \in [\tau_k, \tau'_k] \setminus \mathcal{J}_{k+1}, \quad k = 1, 2, \dots, m_0, \end{cases}$$

for $m_0 > 1$, $k = 1, 2, \dots, (m_0 - 1)$.

This function has the following properties:

- 1) $r(t) = k$, $\tau_k \leq t < \tau'_k$, if $\tau_k \leq \tau'_{k+1}$.
- 2) $r(t) = k$, $\tau_k \leq t < \tau_{k+1}$, if $\tau_{k+1} < \tau'_k$.

We construct a strategy for the evader. Let $u_i(t)$, $i = 1, \dots, M$, be arbitrary admissible controls of pursuers:

$$T = T_0 + \frac{2a_1}{\alpha},$$

$$\text{where } T_0 = \frac{1}{\alpha} \max_{1 \leq i \leq M} |y_2^0 - x_{i,2}^0|$$

$$v(t) = V_0(t) = (0, \alpha + U(t)), \quad t \in [0, T] \setminus \mathcal{J}_1, \quad (2)$$

$$v(t) = V_r(t) = (V_{r1}(t), \alpha + U(t)), \quad t \in [0, T] \cap \mathcal{J}_1, \quad (3)$$

$$v(t) = (0, U(t)), \quad t > T, \quad (4)$$

where

$$r = r(t), \quad V_k(t) = (V_{k1}(t), \alpha + U(t)), \quad \tau_k < t < \tau'_k, \quad k \in I = \{1, 2, \dots, m\},$$

$$V_{k1}(t) = \begin{cases} \alpha + |u_{k1}(t)|, & y_1(\tau_k) \geq x_k(\tau_k) \\ -(\alpha + |u_{k1}(t)|), & y_1(\tau_k) < x_{k1}(\tau_k) \end{cases}, \quad U(t) = \left(\sum_{i=1}^M u_{i2}^2(t) \right)^{1/2}.$$

Theorem. If

$$\rho_1^2 + \dots + \rho_M^2 \leq \sigma^2,$$

then evasion is possible in game (1).

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APPLICATIONS OF EMDEN–FOWLER EQUATIONS

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This work is devoted to the study of Emden–Fowler equations and their applications in various areas of mathematical physics. Such equations frequently appear in the modeling of heat conduction, astrophysical processes, and hydrodynamic problems. In particular, nonlinear equations of the form

$$y''(x) + \frac{k}{x}y'(x) + \lambda y^n(x) = 0, \quad x > 0, \quad (1)$$

where $k, \lambda \in \mathbb{R}$, $n > 1$, are called Emden–Fowler equations.

Definition. A solution of the Emden–Fowler equation is understood as a twice continuously differentiable function $y(x)$ that satisfies (1) on $(0, +\infty)$ along with the prescribed initial conditions.

Emden–Fowler equations find applications in describing stellar equilibrium states, in the study of polynomial nonlinear oscillations, as well as in stability problems.

Theorem. Let $n > 1$, $\lambda > 0$, $k > -1$. Then equation (1) with initial conditions $y(0) = a > 0$, $y'(0) = 0$ admits a positive decreasing solution on $[0, +\infty)$.

Sketch of the proof. By employing the method of integral inequalities and comparison function techniques, it is shown that the solution $y(x)$ does not vanish for finite x and decreases monotonically. Moreover, as $x \rightarrow +\infty$, the following estimate holds: $[y(x) = \mathcal{O}\left(x^{-\frac{2}{n-1}}\right)]$.

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YUQORI TOQ TARTIBLI TENGLAMA UCHUN CHEGARAVIY MASALA

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$\Omega = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ sohada quyidagi chegaraviy masalani o'rganamiz:

$$L[u] = (-1)^n \frac{\partial^{2n+1} u}{\partial x^{2n+1}} - \frac{\partial^2 u}{\partial y^2} = f(x, y). \quad (1)$$

A masala. (1) tenglamaning, $u(x, y) \in C_{x,y}^{2n+1,2}(\Omega) \cap C_{x,y}^{2n,1}(\bar{\Omega})$ sinfga tegishli, shunday yechimi topilsinki, u quyidagi shartlarni qanoatlantirsin:

$$u_y(x, 0) = 0, \quad u_y(x, 1) = 0,$$

$$\frac{\partial^i u}{\partial x^i}(0, y) = 0, \quad i = 0, 1, \dots, n,$$

$$\frac{\partial^j u}{\partial x^j}(1, y) = 0, \quad j = 0, 1, \dots, n-1,$$

bu yerda $f(x, y)$ - yetarlicha silliq funksiya.

Teorema 1. Agar A masala yechimga ega bo'lsa, u holda yechim yagona bo'ladi.

A masalaning yechimini quyidagi qator ko'rinishida izlaymiz:

$$u(x, y) = \sum_{k=1}^{\infty} X_k(x) \cos(\pi k y). \quad (2)$$

Teorema 2. $k \rightarrow +\infty$ bo'lganda $X_k(x)$ funksiyalar uchun quyidagi baholar o'rinli:

$$|X_k^{(j)}(x)| \leq \frac{M}{k^{\frac{j-2n}{2n+1}}} |f_k(\tau)|, \quad j = 0, 1, \dots, 2n,$$

bu yerda $|f_k(\tau)| = \max_{0 \leq \xi \leq 1} |f_k(\xi)|$.

Teorema 3. Agar quyidagi shartlar bajarilsa:

$$1. \quad f(x, y), \quad \frac{d^2 f}{dx^2} \in C[0, 1];$$

$$2. \quad f_y(x, 0) = f_y(x, 1) = 0,$$

u holda (2) qator A masalaning yechimi bo'ladi.

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RIMAN-LIUUVILL MA'NOSIDAGI HOSILA QATNASHGAN BUZILISHGA EGA OLTINCHI TARTIBLI TENGLAMA UCHUN ARALASH MASALA

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Bizga $\Omega = \Omega_x \times \Omega_y$, $\Omega_x = \{0 < x < 1\}$, $\Omega_y = \{0 < y < 1\}$ sohada

$$D_{0x}^\alpha u(x, y) + x^\beta \frac{\partial^6 u(x, y)}{\partial y^6} = 0, \quad (1)$$

tenglama berilsin. Bu yerda $0 < \alpha < 1$, $-\alpha < \beta \in R$,

$$D_{0x}^\alpha u(x, y) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{u(\tau, y) d\tau}{(x-\tau)^\alpha}$$

- Riman-Liuvill ma'nosidagi kasr tartibli hosila. Quyidagi masalani qaraylik

H masala. Ω sohada berilgan (1) tenglamani va quyidagi shartlarni bajaruvchi $u(x, y)$ funksiya topilsin:

$$D_{0x}^\alpha u(x, y) \in C(\Omega), \quad D_{0x}^{\alpha-1} u(x, y) \in C(\bar{\Omega}), \quad \frac{\partial^5 u(x, y)}{\partial y^5} \in C(\bar{\Omega}_y), \quad \frac{\partial^6 u(x, y)}{\partial y^6} \in C(\Omega),$$

$$A_1 \times B_1(u(x, y)) = 0, \quad A_2 \times B_2(u(x, y)) = 0, \quad (2)$$

$$A_1 = a_{ij}, \quad i = \overline{1, 3}; \quad j = \overline{1, 6}, \quad A_2 = b_{ij}, \quad i = \overline{1, 3}; \quad j = \overline{1, 6};$$

$$B_1(u(x, y)) = a_j = \frac{\partial^j}{\partial y^j} u(x, 0), \quad j = \overline{0, 5}, \quad B_2(u(x, y)) = b_j = \frac{\partial^j}{\partial y^j} u(x, 1), \quad j = \overline{0, 5}$$

$$D_{0x}^{\alpha-1} u(0, y) = \varphi(y),$$

bu yerda $\text{rank} A_1 = \text{rank} A_2 = 3$, $\varphi(y)$ - yetarlicha silliq funksiya. H masalaning yechimini quyidagi qator ko'rinishida izlaymiz

$$u(x, y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y), \quad (3)$$

bu yerda

$$\begin{cases} Y_n^{(6)}(y) = \lambda_n Y_n(y), \\ A_1 \times B_1(Y_n(y)) = 0, \quad A_2 \times B_2(Y_n(y)) = 0. \end{cases} \quad (4)$$

$Y_n(y)$ funksiyalar masalaning xos funksiyalari, $X_n(x)$ funksiyalar quyidagi masalaning yechimi:

$$\begin{cases} D_{0x}^\alpha X_n(x) = -\lambda_n x^\beta X_n(x), \\ D_{0x}^{\alpha-1} X_n(0) = \varphi_n = \int_0^1 \varphi(y) Y_n(y) dy, \quad n = 1, 2, \dots \end{cases}$$

Lemma 1. Agar

$$\begin{vmatrix} a_{10} & a_{12} & a_{13} \\ a_{20} & a_{22} & a_{23} \\ a_{30} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{10} & a_{11} & a_{14} \\ a_{20} & a_{21} & a_{24} \\ a_{30} & a_{31} & a_{34} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} = \begin{vmatrix} a_{10} & a_{12} & a_{15} \\ a_{20} & a_{22} & a_{25} \\ a_{30} & a_{32} & a_{35} \end{vmatrix},$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{10} & a_{11} & a_{15} \\ a_{20} & a_{21} & a_{25} \\ a_{30} & a_{31} & a_{35} \end{vmatrix}$$

shartlar bajarilsa, quyidagi differensial operator

$$L(Y) = \frac{d^6}{dy^6} Y,$$

o'z-o'ziga qo'shma bo'ladi. Bu yerda L operatorning aniqlanish sohasi quyidagicha:

$$D(L) = (Y : Y^{(6)}(y) \in C(0, 1), Y^{(5)}(y) \in C[0, 1]) \cap (4).$$

Faraz qilaylik $\lambda > 0$ bo'lsin. U holda quyidagilar o'rinli

Teorema 1. Agar $\varphi(y) \in A(L)$; $L(\varphi) \in L_2[0, 1]$ shartlar bajarilsa, u holda $\varphi(y)$ funksiyani (4) masalaning $Y_n(y)$ xos funksiyalari orqali tekis va absolyut yaqinlashuvchi Furre qatoriga yoyish mumkin.

Endi (3) qatorni H masalaning yechimi bo'lishi uchun quyidagi teorema o'rinli.

Teorema 2. Agar quyidagi shartlar:

- 1) $\varphi(y) \in D(L)$;
 - 2) $\varphi^{(6)}(y) \in D(L)$, $\varphi^{(12)}(y) \in L_2(0, 1)$
- bajarilsa, u holda H masalaning yechimi mavjud.

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Vaqt bo'yicha Kaputo operatori qatnashgan balka tebranish tenglamasi uchun Koshi masalasi

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Kasr tartibli differensial tenglamalar nazariyasi matematikaning eng faol rivojlanayotgan yo'nalishlaridan biriga aylandi. Ushbu nazariya klassik butun tartibli hosila tushunchasini umimlashtirib, modellashtirilayotgan jarayonlarning xotira va nolokal xususiyatlarini hisobga olish imkonini beradi. Ushbu ishda vaqt bo'yicha Kaputo operatori qatnashgan balka tebranish tenglamasi uchun Koshi masalasi tadqiq etilgan.

$\Omega = \{(x, t) : x \in R, t > 0\}$ sohada ushbu

$${}^C D_{0t}^\alpha V(x, t) + \frac{\partial^4 V(x, t)}{\partial x^4} = 0 \quad (1)$$

kasr tartibli differensial tenglamaning

$$V(x, 0) = \varphi(x), \lim_{t \rightarrow 0} {}^C D_{0t}^{\alpha/2} V(x, t) = 0 \quad (2)$$

boshlang'ich shartlarni qanoatlantiruvchi $V(x, y) \in M$ yechimini topilsin, bu yerda ${}^C D_{0t}^\alpha$ Kaputo kasr tartibli hosilasi bo'lib quyidagicha aniqlanadi:

$${}^C D_{0t}^\alpha f(x, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} f(x, \tau) d\tau,$$

$1 < \alpha < 2$, $\varphi(x)$ berilgan uzluksiz funksiya, $M - \Omega$ sohada t bo'yicha 2 marta, x bo'yicha 4 marta differensiallanuvchi, $\bar{\Omega} = \{(x, t) : x \in R, t \geq 0\}$ soha chegarasida yaqinida t bo'yicha 1 marta, x bo'yicha 2 marta differensiallanuvchi funksiyalar fazosi.

Teorema. $V(x, t) \in M$ funksiya (1)-(2) masalaning yechimi bo'lishi uchun

$$U(x, t) = V(x, t) + i I_{0,t}^{\alpha/2} V_{xx}(x, t) \quad (3)$$

funksiya

$${}^C D_{0t}^{\alpha/2} U(x, t) - i U_{xx}(x, t) = 0, (x, t) \in \Omega \quad (4)$$

tenglamaning

$$U(x, 0) = \varphi(x), x \in \mathbb{R} \quad (5)$$

boshlang'ich shartini qanoatlantiruvchi yechimi bo'lishi zarur va yetarli, bu yerda i mavhum birlik.

Bu teormadan quyidagi natija kelib chiqadi.

1-natija. Aytaylik $U \in M$ (4)-(5) masalaning yechimi bo'lsin, bu yerda $\varphi(x)$ haqiqiy funksiya. U holda $V(x, t) = \operatorname{Re} U(x, t)$ funksiya (1) - (2) masalaning yechimi bo'qladi. Bu yerda $\operatorname{Re} U(x, t)$ $U(x, t)$ funksiyaning haqiqiy qismini anglatadi.

(1) - (2) masalani yechish uchun 1- natijadan foydalanamiz. U holda (18)- boshlang'ich shartni qanoatlantiruvchi ushbu

$${}^C D_{0t}^{\alpha/2} U(x, t) - iU_{xx}(x, t) = 0$$

bir o'qilchovli vaqt bo'qiyicha kasr tartibli Shredinger tenglamasining yechimini topish masalasiga kelamiz. Bu masalaning yechimi [1] ishda quyidagi ko'rinishda aniqlangan:

$$U(x, t) = \int_{-\infty}^{\infty} \varphi(\xi) G_{\alpha}(x - \xi, t) d\xi,$$

bu yerda

$$G_{\alpha}(x - \xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-\xi)} E_{\frac{\alpha}{2},1}(-ik^2 t^{\frac{\alpha}{2}}) dk = \frac{1}{2} \frac{1}{\sqrt{it^{\frac{\alpha}{4}}}} M_{\frac{\alpha}{4}}\left(\frac{|x-\xi|}{\sqrt{it^{\frac{\alpha}{4}}}}\right).$$

1-natijadan (1)- (2) masalaning yechimi quyidagi ko'rinishga keladi:

$$V(x, t) = \int_{-\infty}^{\infty} \varphi(\xi) G_{1\alpha}(x - \xi, t) d\xi,$$

bu yerda

$$\begin{aligned} G_{1\alpha}(x - \xi, t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(x - \xi) E_{\alpha,1}(-k^4 t^{\alpha}) dk \\ &= \frac{1}{2t^{\frac{\alpha}{4}}} \sum_{n=0}^{\infty} \frac{\left(-\frac{|x-\xi|}{t^{\frac{\alpha}{4}}}\right)^n \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}{n! \Gamma\left(-\frac{\alpha}{4}n + 1 - \frac{\alpha}{4}\right)}. \end{aligned}$$

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Iteratsiyalangan Dalmber operatori qatnashgan spektral parametrli tenglama uchun Gursa masalasi

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Ushbu maqolada iteratsiyalangan Dalmber operatori qatnashgan spektral parametrli tenglama uchun Gursa masalasini ko'rib chiqamiz. Bunday masalani Riman usuli bilan yechish ilgari o'rganilmagan.

Tekislikda (x, y) ; $y = 0$, $x + y = 0$, $x - y = 1$ chiziqlar bilan chegaralangan sohani Ω bilan belgilab va bu sohada quyidagi tenglamani qaraymiz:

$$L(u) \equiv \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right)^m u(x, y) - \lambda u(x, y) = f(x, y), \quad m \in N, \quad (1)$$

bu yerda $u(x, y)$ noma'lum funksiya va $\lambda \in R \setminus \{0\}$, $f(x, y)$ esa uzluksiz funksiya.

Gursa masalasi. (1) tenglamani qanoatlantiruvchi $u(x, y) \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ sinfga tegishli va quyidagi chegaraviy shartlarni qanoatlantiruvchi $u(x, y)$ funksiya topilsin

$$\begin{aligned} u|_{OK} = \varphi_0(x), \quad \frac{\partial u}{\partial n_1} \Big|_{OK} = \varphi_1(x), \quad \frac{\partial^2 u}{\partial n_1^2} \Big|_{OK} = \varphi_2(x), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \Big|_{OK} = \varphi_{m-1}(x), \\ u|_{KN} = \psi_0(x), \quad \frac{\partial u}{\partial n_2} \Big|_{KN} = \psi_1(x), \quad \frac{\partial^2 u}{\partial n_2^2} \Big|_{KN} = \psi_2(x), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_2^{m-1}} \Big|_{KN} = \psi_{m-1}(x). \end{aligned} \quad (2)$$

$$OK = \left\{ (x, y) : x + y = 0, 0 \leq x \leq \frac{1}{2} \right\}, KN = \left\{ (x, y) : x - y = 1, \frac{1}{2} \leq x \leq 1 \right\},$$

(Ω_0 —soha Ω sohani ichki qismida bo'lib, $CEKF$ to'g'ri to'rtburchakdan iborat, bu yerda ixtiyoriy $C(x_0, y_0) \in \Omega$ nuqta, CE va CF kesmalar esa karakteristik chiziqlarga parallell holda joylashgan), bu yerda $\varphi_0(x)$, $\psi_0(x)$, $\varphi_1(x)$, $\psi_1(x)$, $\varphi_2(x)$, $\psi_2(x)$, ..., $\varphi_{m-1}(x)$, $\psi_{m-1}(x)$ berilgan funksiyalar bo'lib, yetarlicha silliq bo'lgan funksiyalar, hamda

$$\begin{aligned} \varphi_0\left(\frac{1}{2}\right) = \psi_0\left(\frac{1}{2}\right), \quad -\varphi_1\left(\frac{1}{2}\right) = \psi_1\left(\frac{1}{2}\right), \quad \varphi_2\left(\frac{1}{2}\right) = \psi_2\left(\frac{1}{2}\right), \quad \dots, \\ (-1)^{m-1} \varphi_{m-1}\left(\frac{1}{2}\right) = \psi_{m-1}\left(\frac{1}{2}\right), \end{aligned} \quad (3)$$

shartlarni qanoatlantiradi. n_1 va n_2 lar xarakteristika chiziqlariga o'tkazilgan normallar bo'lib,

$$\frac{\partial u}{\partial n_1} \Big|_{OK} = \frac{\partial u}{\partial x} \cos \alpha_1 + \frac{\partial u}{\partial y} \cos \beta_1, \quad \frac{\partial^2 u}{\partial n_1^2} \Big|_{OK} = \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_1 + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_1 \cos \beta_1 +$$

$$+ \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_1, \dots, \left. \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \right|_{OK} = \frac{\partial^{m-1} u}{\partial x^{m-1}} \cos^{m-1} \alpha_1 + 2 \frac{\partial^{m-1} u}{\partial x \partial y} \cos^{m-2} \alpha_1 \cos \beta_1 + \dots +$$

$$+ \frac{\partial^{m-1} u}{\partial y^{m-1}} \cos^{m-1} \beta_1.$$

$$\left. \frac{\partial u}{\partial n_2} \right|_{NK} = \frac{\partial u}{\partial x} \cos \alpha_2 + \frac{\partial u}{\partial y} \cos \beta_2, \quad \left. \frac{\partial^2 u}{\partial n_2^2} \right|_{NK} = \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_2 + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_2 \cos \beta_2 +$$

$$+ \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_2, \dots, \left. \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \right|_{NK} = \frac{\partial^{m-1} u}{\partial x^{m-1}} \cos^{m-1} \alpha_2 + 2 \frac{\partial^{m-1} u}{\partial x \partial y} \cos^{m-2} \alpha_2 \cos \beta_2 + \dots +$$

$$+ \frac{\partial^{m-1} u}{\partial y^{m-1}} \cos^{m-1} \beta_2$$

bu yerda $\alpha_1 = n_1 \wedge x = \frac{5\pi}{4}$, $\beta_1 = n_1 \wedge y = \frac{3\pi}{4}$, $\alpha_2 = n_2 \wedge x = \frac{7\pi}{4}$, $\beta_2 = n_2 \wedge y = \frac{5\pi}{4}$.

$$\nu(x, y; x_0, y_0) = [(x - x_0)^2 - (y - y_0)^2]^{m-1} e_{m,-m}^{m,m} \left[\frac{\lambda}{4^m} ((x - x_0)^2 - (y - y_0)^2)^m \right]. \quad (4)$$

Riman funksiyasidan foydalanib (1) Gursa masalasining yechimini oshkor ko‘rinishida quyidagicha yozish mumkin.

$$u(x_0, y_0) = \psi_0 \left(\frac{x_0 + y_0 + 1}{2} \right) - \int_0^{x_0+y_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \frac{\partial}{\partial \xi} [(\xi - x_0 - y_0)^{m-k} \times \right.$$

$$\times (1 - x_0 + y_0)^{m-k} e_{m,m}^{m-k+1, m-k+1} \left[\frac{\lambda}{4^m} ((\xi - x_0 - y_0)(1 - x_0 + y_0))^m \right] \left. \psi_{m-k}^{(m-k)} \left(\frac{\xi + 1}{2} \right) \right] d\xi -$$

$$- \int_{x_0-y_0}^1 \left[\sum_{k=1}^m \left(-\frac{\sqrt{2}}{2} \right)^{m-k} (-x_0 - y_0)^{m-k} (\eta - x_0 + y_0)^{m-k} e_{m,m}^{m-k+1, m-k+1} \left[\frac{\lambda}{4^m} ((-x_0 - y_0) \times \right.$$

$$\times (\eta - x_0 + y_0))^m \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) \right] d\eta - \iint_{\Omega_0} [(x - x_0)^2 - (y - y_0)^2]^{m-1} e_{m,-m}^{m,m} \left[\frac{\lambda}{4^m} ((x - x_0)^2 - \right.$$

$$\left. -(y - y_0)^2) \right] f(x, y) dx dy.$$

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NONLOCAL BOUNDARY VALUE PROBLEM FOR A SYSTEM OF PARABOLIC TYPE EQUATIONS WITH TWO DEGENERATE LINES

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In this study, we examine a nonlocal boundary value problem for a system of partial differential equations in which the direction of time changes with respect to space. Similar types of equations have been investigated by N. Kislov, S.G. Pyatkov, K.S. Fayazov[1], I.E. Egorov, S. Z. Djamalov[2], I.O. Khajiev, and others. The well-posedness of nonlocal boundary value problems for various classes of differential and operator-differential equations has been analyzed from different perspectives in the works of A.A. Dezin, V.K. Romanko, Y.I. Yurchuk, and many other researchers.

The existence and uniqueness of solutions to nonlocal boundary value problems for nonclassical equations have been studied in the works of A. Ashyralyev and O. Yildirim, F. Zouyed, F. Rebbani, N. Boussetila, A.I. Kozhanov [3], K.B. Sabitov, S.Z. Djamalov, A.I. Shadrina, and several other researchers.

Let $\Omega = \Omega_0 \times Q$, $\Omega_0 = \Omega_1 \times \Omega_2$, $\Omega_1 = \{-1 < x < 1\}$, $\Omega_2 = \{-1 < y < 1\}$, $Q = (0; T)$. Consider the system of equations

$$\begin{cases} u_{1t} + \text{sign}(x)u_{1xx} + \text{sign}(y)u_{1yy} + a_1u_1 + b_1u_2 = 0, \\ u_{2t} + \text{sign}(x)u_{2xx} + \text{sign}(y)u_{2yy} + a_2u_2 + b_2u_1 = 0, \end{cases} \quad (1)$$

in the domain $\Omega \cap \{x, y \neq 0\}$, where a_j, b_j are some constants and $b_2 \neq 0$, $(a_1 - a_2)^2 + 4b_1b_2 > 0$.

Problem. Find a solution of the system of equations (1) satisfying the following conditions:

nonlocal

$$u_j(x, y, t)|_{t=0} + \alpha u_j(x, y, t)|_{t=T} = \varphi_j(x, y), \quad (x, y) \in \bar{\Omega}_0, \quad j = 1, 2, \quad (2)$$

boundary

$$u_j(x, y, t)|_{\partial\Omega_0} = 0, \quad t \in [0; T], \quad (3)$$

and gluing

$$\begin{aligned} \frac{\partial^i u_j(x, y, t)}{\partial x^i} \Big|_{x=-0} &= \frac{\partial^i u_j(x, y, t)}{\partial x^i} \Big|_{x=+0}, \quad (y, t) \in \bar{\Omega}_2 \times \bar{Q}, \\ \frac{\partial^i u_j(x, y, t)}{\partial y^i} \Big|_{y=-0} &= \frac{\partial^i u_j(x, y, t)}{\partial y^i} \Big|_{y=+0}, \quad (x, t) \in \bar{\Omega}_1 \times \bar{Q}, \quad i = 0, 1, \end{aligned} \quad (4)$$

conditions, where α is some constant, $\varphi_j(x, y)$ given sufficiently smooth functions.

In this study, we investigate the correctness of the considered problem depending on the parameter α , derive a representation of the solution, and establish an a priori estimate for it. In the case of well-posedness, the conditional correctness of the problem is also demonstrated.

Let us introduce the notation

$$\begin{aligned} u_1(x, y, t) &= \frac{a_1 + \kappa_2}{b_2(\kappa_1 - \kappa_2)} e^{\kappa_1 t} v_1(x, y, t) + \frac{a_1 + \kappa_1}{b_2(\kappa_1 - \kappa_2)} e^{\kappa_2 t} v_2(x, y, t), \\ u_2(x, y, t) &= \frac{1}{(\kappa_1 - \kappa_2)} e^{\kappa_1 t} v_1(x, y, t) + \frac{1}{(\kappa_1 - \kappa_2)} e^{\kappa_2 t} v_2(x, y, t), \end{aligned} \quad (5)$$

where κ_1, κ_2 – roots of a quadratic equation

$$\kappa^2 + (a_1 + a_2)\kappa + a_1a_2 - b_1b_2 = 0.$$

Theorem 1. *Let $\alpha = 0$. Then for any generalized solution to the problem (1)-(4) for $t \in (0, T)$ the following inequalities hold:*

$$\begin{aligned} \|u_1(x, y, t)\|_0 &\leq \left| \frac{a_1 + \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right| e^{\kappa_1 t} \omega(t, \kappa_1) + \left| \frac{a_1 + \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right| e^{\kappa_2 t} \omega(t, \kappa_2), \\ \|u_2(x, y, t)\|_0 &\leq \frac{1}{|\kappa_1 - \kappa_2|} e^{\kappa_1 t} \omega(t, \kappa_1) + \frac{1}{|\kappa_1 - \kappa_2|} e^{\kappa_2 t} \omega(t, \kappa_2), \end{aligned}$$

where

$$\begin{aligned} \omega(t, \kappa_i) &= 2(|b_2| \|\varphi_1\|_0 + |a_1 + \kappa_i| \|\varphi_2\|_0)^{1 - \frac{t}{T}} \times \\ &\times \left((|b_2| \|u_1(x, y, T)\|_0 + |a_1 + \kappa_i| \|u_2(x, y, T)\|_0) e^{-\kappa_i T} \right)^{\frac{t}{T}}. \end{aligned}$$

Theorem 2. *Let $\alpha \neq 0$. Then for any solution to the problem (1)-(4) for $t \in (0, T)$ the following inequalities hold:*

$$\begin{aligned} \|u_1(x, y, t)\|_0^2 &\leq 2 \left(\frac{a_1 + \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right)^2 e^{2\kappa_1 t} \|v_1(x, y, t)\|_0^2 + 2 \left(\frac{a_1 + \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right)^2 e^{2\kappa_2 t} \|v_2(x, y, t)\|_0^2, \\ \|u_2(x, y, t)\|_0^2 &\leq \frac{2}{(\kappa_1 - \kappa_2)^2} e^{2\kappa_1 t} \|v_1(x, y, t)\|_0^2 + \frac{2}{(\kappa_1 - \kappa_2)^2} e^{2\kappa_2 t} \|v_2(x, y, t)\|_0^2. \end{aligned}$$

where

$$\|v_j(x, y, t)\|_0^2 \leq C (1 + \alpha^{-2}) e^{-2\kappa_i T} \|\bar{\varphi}_i(x, y)\|_0^2, \quad i = 1, 2,$$

C – some constant depending on α, T ,

$$\begin{aligned} \|\bar{\varphi}_1(x, y)\|_0 &= |a_1 + \kappa_1| \|\varphi_2(x, y)\|_0 + |b_2| \|\varphi_1(x, y)\|_0, \\ \|\bar{\varphi}_2(x, y)\|_0 &= |a_1 + \kappa_2| \|\varphi_2(x, y)\|_0 + |b_2| \|\varphi_1(x, y)\|_0. \end{aligned}$$

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PROPERTIES OF THE SOLUTION OF THE NONHOMOGENEOUS RAYLEIGH-STOKES TYPE FRACTIONAL DIFFERENTIAL EQUATION

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In recent years, fractional differential equations have become an important subject of study. Compared to ordinary equations, they are more suitable for describing complex processes. For example, they can take into account the memory effect of a system and explain anomalous diffusion. One such model is the Rayleigh-Stokes type fractional differential equation.

The classical Rayleigh-Stokes equation describes how the velocity of a viscous fluid changes with time. In its fractional form, the past states of the fluid are also considered. As a result, the solution is expressed not by simple functions but by special ones, such as the Mittag-Leffler and Fox-Wright functions.

The Caputo fractional derivative of order $\rho \in (0, 1)$ is defined as follows (see, for example [1])

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \quad t > 0.$$

In this work, we study properties of the solution the Cauchy problem for the nonhomogeneous Rayleigh-Stokes type fractional differential equation. Let us consider the following problem

$$\begin{cases} y'(t) + \lambda(1 + \gamma D_t^\rho)y(t) = f(t), & \gamma > 0, \quad 0 < t < T; \\ y(0) = y_0. \end{cases} \quad (1)$$

where λ, γ are constants. The Caputo derivative is particularly suitable for physical applications because it allows the formulation of initial conditions in terms of classical integer-order derivatives, which have direct physical interpretations.

The solution to this problem is given in [2], where the solution is found using the Fox-Wright function. Let us remember the solution of equation (1) obtained in our previous work.

The Cauchy problem for the nonhomogeneous Rayleigh-Stokes type fractional differential equation (1) has the solution

$$y(t) = y_0 A_\rho(\lambda, t) + \int_0^t B_\rho(\lambda, t - \tau) f(\tau) d\tau,$$

where

$$\begin{aligned} A_\rho(\lambda, t) &= 1 - \lambda \int_0^t B_\rho(\lambda, \tau) d\tau \\ &= \int_0^\infty e^{-rt} \frac{\gamma}{\pi} \frac{\lambda^2 r^{\rho-1} \sin \rho\pi}{(-r + \lambda \gamma r^\rho \cos \rho\pi + \lambda)^2 + (\lambda \gamma r^\rho \sin \rho\pi)^2} dr \end{aligned}$$

and

$$B_\rho(\lambda, t) = \int_0^\infty e^{-rt} \frac{\gamma}{\pi} \frac{\lambda r^\rho \sin \rho\pi}{(-r + \lambda \gamma r^\rho \cos \rho\pi + \lambda)^2 + (\lambda \gamma r^\rho \sin \rho\pi)^2} dr.$$

We present the properties of the solutions. The properties of $B_\rho(\lambda, t)$ are considered in [3], [4].

Theorem. Let $A_\rho(\lambda, t)$ be a solution of the Cauchy problem for the Rayleigh-Stokes equation (1). Then

1. $A_\rho(\lambda, 0) = 1$, $0 < A_\rho(\lambda, t) < 1$, $t > 0$,
2. $D_t A_\rho(\lambda, t) < 0$, $t \geq 0$,
3. $A_\rho(\lambda_k, t) \geq C(\rho, \gamma, \lambda_1)$, $k \geq 1$,

$$\text{where } C(\rho, \gamma, \lambda_1) = \frac{\gamma \sin \rho\pi}{4\pi} \int_0^\infty \frac{r^{\rho-1} e^{-rT}}{\frac{r^2}{\lambda_1} + \gamma^2 r^{2\rho} + 1} dr.$$

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КРАЕВАЯ ЗАДАЧА С НЕСИММЕТРИЧНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ ДЛЯ УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА, СОДЕРЖАЩЕГО ТРЕТЬЮ ПРИЗВОДНУЮ ПО ВРЕМЕНИ

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Краевые задачи для уравнений четвертого порядка, содержащих третью производную по времени, мало изучены [1-5].

В области $D = \{(x, y) : 0 < x < p, 0 < y < q\}$, рассмотрим уравнение

$$u_{xxxx} - u_{yyy} + au = g(x, y). \quad (1)$$

Задача А. Найти решение уравнения (1) в области D из класса $C_{x,y}^{4,3}(D) \cap C_{x,y}^{3,2}(\bar{D})$, удовлетворяющее следующим краевым условиям:

$$u(0, y) = u_x(p, y) = u_{xx}(0, y) = u_{xx}(p, y) = 0, \quad (2)$$

$$u_y(x, 0) = \psi_1(x); \quad u_{yy}(x, 0) = \psi_2(x); \quad u_{yy}(x, q) = \psi_3(x), \quad (3)$$

где $\psi_i(y)$, $i = \overline{1, 3}$, $g(x, y)$ – заданные достаточно гладкие функции, а также выполняются следующие условия согласования:

$$\psi_i(0) = \psi'_i(p) = \psi''_i(0) = \psi'''_i(p) = \psi_i^{(4)}(0) = 0, \quad i = \overline{1, 3}, \quad g(0, y) = 0. \quad (4)$$

Теорема. Если задача А имеет решение, то при выполнении условия $a \geq 0$, оно единственно.

Доказательство. Предположим, обратное. Пусть задача А имеют два решения $u_1(x, y)$ и $u_2(x, y)$. Тогда функция $u(x, y) = u_1(x, y) - u_2(x, y)$ удовлетворяет однородному уравнению (1) с однородными краевыми условиями. Докажем, что $u(x, y) \equiv 0$ в $\bar{D}$.

В области D справедливо тождество

$$\frac{\partial}{\partial x} (u u_{xxx} - u_x u_{xx}) + u_{xx}^2 + \frac{\partial}{\partial y} \left(-u u_{yy} + \frac{1}{2} u_y^2 \right) + a u^2 = 0. \quad (5)$$

Интегрируя тождество (5) по области D , имеем

$$\begin{aligned} & \int_0^q [u(p, y) u_{xxx}(p, y) - u_x(p, y) u_{xx}(p, y)] dy - \\ & - \int_0^q [u(0, y) u_{xxx}(0, y) - u_x(0, y) u_{xx}(0, y)] dy + \\ & + \int_0^p \int_0^q u_{xx}^2 dx dy + \int_0^p (-u(x, q) u_{yy}(x, q) + u(x, 0) u_{yy}(x, 0)) dx + \\ & + \frac{1}{2} \int_0^p (u_y^2(x, q) - u_y^2(x, 0)) dx + a \int_0^p \int_0^q u^2(x, y) dx dy = 0. \end{aligned}$$

Учитывая однородные краевые условия, получим

$$\int_0^p \int_0^q u_{xx}^2 dx dy + \frac{1}{2} \int_0^p u_y^2(x, q) dx + a \int_0^p \int_0^q u^2(x, y) dx dy = 0$$

Если $a > 0$, из третьего слагаемого, получим $u(x, y) \equiv 0$, $(x, y) \in \bar{D}$. При $a = 0$, из первого слагаемого, т.е. $u_{xx}(x, y) = 0$. Отсюда $u(x, y) = xf_1(y) + f_2(y)$. Учитывая краевые условия (2), получим $f_1(y) = f_2(y) = 0$, тогда $u(x, y) \equiv 0$ в $\bar{D}$.

Теорема доказана.

Замечание. Отметим, что при нарушении условий теоремы, т.е. $a < 0$ однородная задача для однородного уравнения (1) может иметь нетривиальное решение. Например уравнение

$$u_{xxxx}(x, y) - u_{yyyy}(x, y) + \left(a - \left(\frac{\pi}{2p}(2n-1) \right)^4 \right) u(x, y) = 0,$$

с однородными условиями (2)-(3) имеет нетривиальное решение,

$$u(x, y) = \sqrt{\frac{2}{p}} \cdot \left(e^{\sqrt[3]{a}y} \cos \left(\frac{\sqrt{3}}{2} \sqrt[3]{a}q - \frac{\pi}{6} \right) + 2e^{-\frac{1}{2}\sqrt[3]{a}y} \cos \left(\frac{\sqrt{3}}{2} \sqrt[3]{a}(y-q) + \frac{\pi}{6} \right) + \right. \\ \left. + e^{\frac{\sqrt{3}}{2}(3q-y)} \sin \left(\frac{\sqrt{3}}{2} \sqrt[3]{a}y \right) \right) \sin \left(\frac{\pi(2n-1)}{2p}x \right).$$

Здесь a корни уравнения,

$$\Delta = \sqrt{3}e^{-\frac{1}{2}\sqrt[3]{a}q} \left(\frac{1}{2}e^{\frac{3}{2}\sqrt[3]{a}q} + \sin \left(\frac{\sqrt{3}}{2} \sqrt[3]{a}q - \frac{\pi}{6} \right) \right) = 0.$$

Существование решения поставленной задачи доказано методом Фурье и построением функции Грина.

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CONTROL PROBLEM FOR A HEAT EQUATION WITH INTERFACE CONDITIONS

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In this paper, we consider the following heat equation in the domain $\Omega = (0, l) \times (0, T)$:

$$\frac{\partial u(x, t)}{\partial t} - \frac{\partial}{\partial x} \left(k(x) \frac{\partial u(x, t)}{\partial x} \right) = h(x) \mu(t), \quad (x, t) \in \Omega,$$

with the homogeneous Dirichlet boundary conditions

$$u(0, t) = u(l, t) = 0, \quad t \geq 0,$$

and the interface conditions at $x = x_0 \in (0, l)$:

$$u(x_0^-, t) = u(x_0^+, t), \quad a^2 u_x(x_0^-, t) = b^2 u_x(x_0^+, t),$$

together with the initial condition

$$u(x, 0) = 0, \quad x \in [0, l].$$

Here $h(x)$ is a given function, $\mu(t)$ is a control function, and the coefficient $k(x)$ is defined as

$$k(x) = \begin{cases} a^2, & 0 < x < x_0, \\ b^2, & x_0 < x < l. \end{cases}$$

If the control function $\nu(t) \in L_2(\mathbb{R}_+)$ satisfies the condition $|\nu(t)| \leq 1$ on the half-line $\overline{\mathbb{R}_+}$, we say that it is admissible.

Assume that the weight function $\rho \in L_2([0, l])$ satisfies the conditions:

$$\rho(x) \geq 0, \quad \int_0^l \rho(x) dx = 1, \quad \rho_k \geq 0,$$

where ρ_k are Fourier coefficients of function $\rho(x)$.

The present study is devoted to the analysis of the following thermal control problem:

Control Problem. Assume that the function $\phi(t)$ is given. Then, find the control function $\mu(t)$ from the condition

$$\int_0^l \rho(x) u(x, t) dx = \phi(t), \quad \forall t \geq 0. \quad (1)$$

Initial studies on time-optimal control for heat conduction in bounded n -dimensional domains were presented in [1], where estimates for the minimal time required to reach a prescribed average temperature were established. Boundary control problem related to fast heating of the rod when the conductivity of the rod is different was studied in [2] and the existence of an admissible control function was proved. The time-varying bang-bang property of time optimal controls for heat equation and its applications is studied in [3].

For any constant $M > 0$, we define the class $W(M)$ as the set of functions $\phi \in W_2^2(\mathbb{R})$ satisfying the conditions

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \leq M, \quad \phi(t) = 0 \quad \text{for } t \leq 0.$$

We now state the main result of this paper.

Theorem. *There exists constant $M > 0$ such that for any function $\phi \in W(M)$, the solution $\nu(t)$ of the equation (1) exists and satisfies condition $|\nu(t)| \leq 1$.*

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О СУЩЕСТВОВАНИИ ОГРАНИЧЕННЫХ РЕШЕНИЙ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА ВОЛЬТЕРРА С БЕСКОНЕЧНЫМ ПРЕДЕЛОМ

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Рассмотрим систему интегро-дифференциальных уравнений типа Вольтерра вида

$$\frac{dx}{dt} = Ax + f \left(t, x, \int_{-\infty}^t R(t, s) \varphi(t, s, x(s)) ds \right), \quad (1)$$

где $x = (x_1, x_2, \dots, x_n)$ – n -мерный вектор, A – $(n \times n)$ -мерная постоянная матрица, $f(t, x, y) = (f_1, f_2, \dots, f_n)$, $\varphi(t, s, x) = (\varphi_1, \varphi_2, \dots, \varphi_n)$ – n -мерные вектор-функции, $R(t, s)$ – $(n \times n)$ -мерная матричная функция.

Предположим, что выполняются следующие условия:

1) действительные части всех собственных чисел $\lambda_j(A)$, $j = \overline{1, n}$ матрицы A отличны от нуля: $\operatorname{Re} \lambda_j(A) \neq 0$, $j = 1, 2, \dots, n$;

2) векторные функции $f(t, x, y)$, $\varphi(t, s, x)$ и матричная функция $R(t, s)$ определены, непрерывны и ограничены для всех

$$t \in R = (-\infty, \infty), s \in R, x \in D \subset E_n, y \in D_1 \subset E_n, \quad (2)$$

кроме того, функция $f(t, x, y)$ удовлетворяет условию Липшица по переменным x и y с постоянной Липшица K_1 и K_2 , а функция $\varphi(t, s, x)$ удовлетворяет этому условию по переменной x с постоянной Липшица K_3 , т.е. выполняются следующие неравенства для всех $t, s, x, y, \bar{x}, \bar{y}$ из области (2):

$$\|f(t, x, y) - f(t, \bar{x}, \bar{y})\| \leq K_1 \|x - \bar{x}\| + K_2 \|y - \bar{y}\|,$$

$$\|\varphi(t, s, x) - \varphi(t, s, \bar{x})\| \leq K_3 \|x - \bar{x}\|;$$

3) ядро $R(t)$ удовлетворяет условию $\int_{-\infty}^t \|R(t - \tau)\| d\tau \leq R_0 < \infty$;

4) $\sup_t \|f(t, 0, 0)\| = M < \infty$, $\sup_t \|\varphi(t, s, 0)\| = N < \infty$.

Следуя работе [1] систему уравнений (1) при помощи замены переменных вида

$$\frac{dx}{dt} = Ax(t) + u(t) \quad (3)$$

можно привести к интегральному уравнению

$$u(t) = f \left(t, \int_{-\infty}^{\infty} G(t - \tau) u(\tau) d\tau, \int_{-\infty}^t R(t - \tau) \int_{-\infty}^{\infty} G(t - \tau) u(s) ds d\tau \right), \quad (4)$$

где $G(\tau - s)$ – функция Грина задачи об ограниченных решениях системы уравнений (3).

Доказывается следующая теорема.

Теорема. Пусть для системы интегро-дифференциальных уравнений (1) выполняются условия 1) – 4). Тогда при достаточных малых постоянных Липшица K_i , $i = \overline{1, 3}$ существует единственное непрерывное и ограниченное на всей оси $R = (-\infty, \infty)$ решение системы интегральных уравнений (4) и, следовательно, такое же решение системы уравнений (1).

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О ПРИБЛИЖЕННОМ РЕШЕНИИ МНОГОТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Рассматривается система интегро-дифференциальных уравнений типа Вольтерра вида

$$\frac{dx}{dt} = P(t)x + f(t, x) + \int_0^t \varphi(t, s, x(s)) ds \quad (1)$$

при линейных многоточечных краевых условиях

$$\sum_{i=0}^p A_i x(t_i) = d, \quad 0 = t_0 < t_1 < \dots < t_{p-1} < t_p = T, \quad (2)$$

где x, f, φ, d – точки n -мерного евклидова пространства E_n ; $P(t)$ – $(n \times n)$ -мерная матричная функция, A_i – $(n \times n)$ -мерные постоянные матрицы, $T > 0$ – некоторая постоянная.

Предположим, что векторные функции f и φ , а также матричная функция $P(t)$ определены и непрерывны в области

$$t \in [0, T], \quad 0 \leq s \leq t \leq T, \quad x \in D, \quad D \subset E_n.$$

Постоянные матрицы A_i такие, что выполняется условие

$$\det \left[\sum_{i=0}^p A_i t_i \right] \neq 0.$$

Для исследования решений краевой задачи (1), (2) применим численно-аналитический метод последовательных приближений [1, 2], согласно которому решение исходной задачи ищется как предел следующей последовательности функции $x_m(t, x_0)$:

$$\begin{aligned} x_m(t, x_0) = & x_0(t, x_0) + \int_0^t \Omega_\tau^t(P) \left[f(\tau, x_{m-1}(\tau, x_0)) + \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0)) ds \right] d\tau + \\ & + \frac{t}{T} H \left(d - \sum_{i=0}^p A_i \Omega_0^{t_i}(P) \right) - \\ & - \frac{t}{T} H \left(\sum_{i=0}^p \int_0^{t_i} \Omega_\tau^{t_i}(P) \left[f(\tau, x_{m-1}(\tau, x_0)) + \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0)) ds \right] d\tau \right), \quad (3) \end{aligned}$$

где

$$x_0(t, x_0) = \Omega_0^t(P)x_0, \quad m = 1, 2, \dots, \quad H = \left(\sum_{i=0}^p \frac{A_i}{T} t_i \right)^{-1},$$

где $\Omega_0^t(P)$ – матрицант линейной однородной системы $\frac{dx}{dt} = P(t)x$.

Получены условия сходимости при $m \rightarrow \infty$ последовательных приближений вида (3) к предельной функции $x^*(t, x_0)$ и изучены связь этой предельной функции с точным решением исходной краевой задачи (1), (2).

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ON THE CONTROL PROBLEM ASSOCIATED WITH HIGER EVEN-ORDER PARABOLIC EQUATION

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In this paper, we consider the following parabolic equation of even order

$$\frac{\partial u(x, t)}{\partial t} + (-1)^m \frac{\partial^{2m} u(x, t)}{\partial x^{2m}} = 0, \quad x \in (0, l), \quad t > 0, \quad m \in \mathbb{N},$$

subject to the boundary conditions

$$u(0, t) = \mu(t), \quad \left. \frac{\partial^{2k} u(x, t)}{\partial x^{2k}} \right|_{x=0} = 0, \quad k = 1, \dots, m-1, \quad t \geq 0,$$

and

$$\left. \frac{\partial^{2k} u(x, t)}{\partial x^{2k}} \right|_{x=l} = 0, \quad k = 0, \dots, m-1, \quad t \geq 0,$$

with the initial condition

$$u(x, 0) = 0, \quad x \in [0, l],$$

where $\mu(t)$ is a given control function.

If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \leq 1$ on the half-line $t \geq 0$, we say that it is admissible.

The present study is devoted to the analysis of the following control problem:

Control Problem. Assume that the function $\phi(t)$ is given. Then, find the control function $\mu(t)$ from the condition

$$\int_0^l u(x, t) dx = \phi(t), \quad \forall t \geq 0. \quad (1)$$

Initial studies on time-optimal control for heat conduction in bounded n -dimensional domains were presented in [1], where estimates for the minimal time required to reach a prescribed average temperature were established. The minimal time control problem for a fourth-order parabolic equation in a square domain was further investigated in [2], providing estimates for the optimal time. In [3], the analyzed epitaxial growth models of nanoscale thin films described by fourth-order parabolic equations, establishing existence, uniqueness, and regularity of solutions within appropriate functional spaces.

For any constant $M > 0$, we define the class $W(M)$ as the set of functions $\phi \in W_2^2(\mathbb{R})$ satisfying the conditions

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \leq M, \quad \phi(t) = 0 \quad \text{for } t \leq 0.$$

We now state the main result of this paper.

Theorem. *There exists constant $M > 0$ such that for any function $\phi \in W(M)$, the solution $\nu(t)$ of the equation (1) exists and satisfies condition $|\nu(t)| \leq 1$.*

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ON THE GENERALIZED CONTINUITY OF A FUNCTION AT THE POINTS OF CONVERGENCE OF ITS SPECTRAL DECOMPOSITION ASSOCIATED WITH SOME SINGULAR ELLIPTIC OPERATORS

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Denote by the symbol R_N^+ the half-space of points (x', x_N) , where $x' = (x_1, x_2, \dots, x_{N-1}) \in R^{N-1}$ and $y = x_N > 0$. Let a bounded domain Ω^+ with a smooth boundary Γ be located in the half-space R_N^+ and be adjacent to the hyperplane $x_N = 0$. The part of the boundary Γ which is located in R_N^+ , denote by Γ^+ , and the part of the boundary Γ lying on the hyperplane $x_N = 0$ denote by Γ^0 .

In the Hilbert space $L_{2,k}(\Omega^+)$, consisting of functions whose square modulus is summable over the domain ω^+ with weight y_k , we consider the operator L defined by the differential expression

$$L = - \sum_{i=1}^{N-1} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y^2} - \frac{k}{y} \frac{\partial}{\partial y} + q(x', y) = -\Delta_B + q, \quad k > 0, \quad (1)$$

on the domain of definition $D(L) = C_{0,+}^\infty(\Omega^+ \cup \Gamma^0)$. With respect to the potential $q(x', y)$, we will assume that it has the form $q(x', y) = q(r) = a(r)/r$, where the function $a(r) \in C^\infty(0, \infty)$ and satisfies for $r > 0$ the estimate $0 \leq a(r) \leq C_\tau r^{\tau-1}$, where $\tau > 0$ for $N > 2$ and $\tau > 1/2$ in the case $N = 2$, $r^2 = x_1^2 + \dots + x_{N-1}^2 + y^2$.

The operator L , considered in the space $L_{2,k}(\Omega^+)$, is symmetric and positive. According to the classical Friedrichs theorem, it admits at least one positive self-adjoint extension in $L_{2,k}(\Omega^+)$. Let $\hat{L}$ denote an arbitrary positive self-adjoint extension of the operator L with a discrete spectrum.

Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ denote the sequence of eigenvalues of the operator $\hat{L}$ and let $\{u_n\}_{n=1}^\infty$ be a complete orthonormal system of the corresponding eigenfunctions in $L_{2,k}(\Omega^+)$. The spectral decomposition of an arbitrary function $f \in L_{2,k}(\Omega)$ has the form

$$E_\lambda f(x) = \sum_{\lambda_n < \lambda} f_n u_n(x) \quad (2)$$

and their Riesz means of order s are defined by the equality

$$E_\lambda^s f(x) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x) = \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s dE_t f(x). \quad (3)$$

In the present work, we study the behavior of a function f in a neighborhood of a point at which its spectral expansion or its Riesz means converge. Typically, to ensure the convergence of the spectral expansions (2) or the Riesz means (3) at a given point, it is sufficient to require that the function be smooth in a neighborhood of that point (see [3]).

The average value of order $\alpha > 0$ of a function $f(x)$ over a hemisphere of radius r centered at the point x is defined as follows:

$$S_r^\alpha f(x) = \frac{1}{\omega_N(k, \alpha) r^{N+k}} \int_{|y| < r, y_N > 0} \left(1 - \frac{|y|^2}{r^2}\right)^{\alpha-1} f(x+y) y_N^k dy, \quad (4)$$

where

$$\omega_N(k, \alpha) = \int_{|y| < 1, y_N > 0} (1 - |y|^2)^{\alpha-1} y_N^k dy.$$

The main results of this article are as follows.

Theorem 1. *If the spectral expansion of a function $f \in L_{2,k}(\Omega^+)$ converges at a point $x \in \Gamma^0$, then for $\alpha > \frac{1}{2}$ the following equality holds:*

$$\lim_{r \rightarrow 0} S_r^\alpha f(x) = f(x).$$

Theorem 1 considers the partial Riesz sum, similar theorems were proved in [6,7].

Theorem 2. *Suppose that the numbers $\alpha > 0$ and $s \geq 0$ satisfy the inequality*

$$s - \alpha < \frac{N + k - 3}{2}. \quad (5)$$

Let the expansion of a function $f \in L_{2,k}(\Omega^+)$ at some point $x \in \Gamma^0$ be summable by Riesz means of order s :

$$\lim_{\lambda \rightarrow \infty} E_\lambda^s f(x) = f(x). \quad (6)$$

Then the equality

$$\lim_{r \rightarrow 0} S_r^\alpha f(x) = f(x) \quad (7)$$

holds.

Similar theorems are proved for the Laplace and B -elliptic operators in the works [4]. Also in the work [5] for a B -elliptic operator with a Bessel operator in one of the variables it is proved that if the spectral decomposition of an arbitrary function at some point of a given hypersurface is summed by Riesz means, then its average value over a hemisphere with the center at the specified point has generalized smoothness.

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KASR TARTIBLI INTEGRO-DIFFERENSIAL OPERATORLARNING MELLIN FAZOSIDAGI IFODALANISHI

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Kasr tartibli integro-differensial operatorlar tushunchasi giperbolik va aralash tipdagi xususiy hosilali differensial tenglamalar uchun chegaraviy masalalarni o'rganish bilan uzviy bog'langan. Bunga o'xshash ishlarni [1], [2] lardagi ustozlarimiz bajarishgan. Ular kabi $\varphi(x)$ funksiyadan $g(x)$ funksiya bo'yicha $|\alpha|$ kasr tartibli integro-differensial operatorlarni quyidagicha kiritamiz:

$$D_{a,x;g(x)}^\alpha \varphi(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} \varphi(t) g'(t) dt, & \text{agar } \alpha > 0 \text{ bo'lsa,} \\ \varphi(x), & \text{agar } \alpha = 0 \text{ bo'lsa,} \\ \frac{1}{g'(x)} \frac{d}{dx} D_{a,x;g(x)}^{\alpha-1} \varphi(x), & \text{agar } 0 < \alpha < 1 \text{ bo'lsa,} \end{cases} \quad (1)$$

bu yerda $\varphi(x) \in L_1(a, b)$, $g(x)$ monoton va uzluksiz hosilaga ega funksiya bo'lib, $g'(x) \neq 0$; $\Gamma(\cdot)$ - Eylerning gamma funksiyasi [1]. Bizga ma'lumki $\varphi(x)$ ($x > 0$) funksiyaning Mellin integral almashtirishi

$$\varphi^*(s) = M^{-1}(\varphi(x) : s) = \int_0^\infty x^{s-1} \varphi(x) dx \quad (2)$$

formula bilan, teskari Mellin integral almashtirishi esa

$$\varphi(x) = M^{-1}\{\varphi^*(s) : x\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \varphi^*(s) x^s ds \quad (3)$$

formula bilan aniqlanadi [2]. Agar funksiya bilan uning Mellin integral almashtirishi orasidagi bog'lanishni " $\leftrightarrow$ " kabi belgilasak quyidagi umumiy formula o'rinli bo'ladi:

$$x^\alpha \varphi(x) \leftrightarrow \varphi^*(s + \alpha) \quad (4)$$

Teorema. Agar 1) $\alpha < 0$ va $x^{s-\alpha-1} \varphi(x) \in L(0, \infty)$ yoki 2) $0 < \alpha < 1$, $\varphi(x) \in C^1(0, 1)$, $x^{s-\alpha-1} \varphi(x) \in L(0, \infty)$, $x = 0$ va $x = \infty$ da $x^{s-1} D_{0,x;x}^{\alpha-1} \varphi(x) = 0$ bo'lsa, u holda

$$D_{0,x;x}^{\alpha-1} \varphi(x) \leftrightarrow \Gamma \left[\begin{matrix} 1 + \alpha - s \\ 1 - s \end{matrix} \right] \varphi^*(s - \alpha), \quad s < 1 + \alpha, \quad (5)$$

tenglik o'rinli bo'ladi.

Isbot. 1) $\alpha < 0$ bo'lsin u holda ta'rifga ko'ra

$$D_{0,x;x}^{\alpha-1} \varphi(x) \leftrightarrow \int_0^\infty x^{s-1} D_{0,x;x}^\alpha \varphi(x) dx = \frac{1}{\Gamma(-\alpha)} \int_0^\infty x^{s-1} \int_0^x (x-t)^{-\alpha-1} \varphi(t) dt dx$$

ifodani hosil qilamiz[3]. Bu ifodaga Dirixle formulasini qo'llab, ichki integralda $x - t = zx$ almashtirish bajarib

$$D_{0,x;x}^{\alpha-1}\varphi(x) \leftrightarrow \frac{1}{\Gamma(-\alpha)} \cdot \frac{\Gamma(1+\alpha-s)\Gamma(-\alpha)}{\Gamma(1-s)} \int_0^\infty y^{s-\alpha-1}\varphi(y)dy = \Gamma\left[\begin{matrix} 1+\alpha-s \\ 1-s \end{matrix}\right] \varphi^*(s-\alpha)$$

munosabatni hosil qilamiz. Demak, (5) formula o'rinli ekan. 2) $0 < \alpha < 1$ bo'lsin. Teorema shartiga ko'ra $\varphi(x) \in C^1(0,1)$ bo'lgani uchun $D_{0,x;x}^\alpha\varphi(x)$ kasr tartibli hosila mavjud va uni $D_{0,x;x}^\alpha\varphi(x) = \frac{d}{dx}D_{0,x;x}^{\alpha-1}\varphi(x)$ ko'rinishida yozib olish mumkin. Bu ifodaga Mellin integral almashtirishini qo'llaymiz va hosil bo'lgan ifodani bo'laklab integrallaymiz:

$$D_{0,x;x}^\alpha\varphi(x) \leftrightarrow \int_0^\infty x^{s-1}d(D_{0,x;x}^{\alpha-1}\varphi(x))dx = (1-s)x^{s-1}D_{0,x;x}^{\alpha-1}\varphi(x)|_0^\infty + (1-s)\int_0^\infty x^{s-2}D_{0,x;x}^{\alpha-1}\varphi(x)dx =$$

Bu ifodadan, teorema shartiga ko'ra,

$$D_{0,x;x}^\alpha\varphi(x) \leftrightarrow (1-s)\int_0^\infty x^{s-1}(x^{-1}D_{0,x;x}^{\alpha-1}\varphi(x))dx$$

Ifoda kelib chiqadi. Bundan (4) formulaga ko'ra, (5) formulaning to'g'riligi kelib chiqadi.

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О точных решениях автономных дифференциальных уравнений

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В работе показано, что нелинейные автономные обыкновенные дифференциальные уравнения второго порядка можно свести к Абелевым уравнениям первого рода.

Рассмотрим следующее общее автономное дифференциальное уравнение второго порядка:

$$y''_x + \sum_{i=1}^n g_i(y)(y'_x)^i + g_0(y) = 0, \quad g_0 \cdots g_n \neq 0, \quad y = y(x), \quad n = 1, 2, \quad (1)$$

где $g_i(y), i = 0, \dots, n$ - непрерывные функции от y . В частности, при $n = 1$ имеем уравнение Льева. Применяя преобразование (M) :

$$(M): \quad y'_x = \frac{1}{p(y)},$$

приходим к уравнению типа (Ab-1):

$$n = 1: \quad p_y = -g_0 p^2 + g_1 p, \quad (2)$$

$$n = 2: \quad p'_y = g_0 p^3 + g_1 p^2 + g_2 p. \quad (3)$$

Далее рассмотрим уравнение Абеля первого рода:

$$y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y, \quad y = y(x). \quad (4)$$

Тогда, учитывая решаемое в замкнутой форме уравнение Абеля, включающее произвольные функции, и используя преобразование, предложенное Камке, приходим к справедливости следующей теоремы.

Теорема 1. Если выполнено следующее соотношение:

$$\left(\frac{f_3}{f_2}\right)'_x = \frac{c}{(1+c)^2} f_2 - \frac{f_3 f_1}{f_2}, \quad c \neq \pm 1,$$

тогда уравнение Абеля (4) ($f_2 \neq 0$) имеет общее неявное решение вида:

$$\left[1 + \frac{f_2(x)}{(1+c)y(x)f_3(x)}\right] \left[1 + \frac{cf_2(x)}{(1+c)y(x)f_3(x)}\right]^{-1} - K \exp\left(\frac{c(1-c)}{(1+c)^2} \int \frac{f_2^2}{f_3} dx\right) = 0,$$

где c - произвольный параметр, а K задает параметр семейства решений.

Рассмотрим обыкновенное дифференциальное уравнение первого порядка общего вида:

$$y'_x = F(x, y), \quad y = y(x). \quad (5)$$

Введя произвольную функцию $G(x)$, перепишем систему уравнений следующим образом:

$$y'_x = -F_1(x, y) + G(x)f(y), \quad (6)$$

$$y'_x = \frac{1}{2}F_2(x, y) + \frac{1}{2}G(x)f(y), \quad (7)$$

$$F(x, y) = F_1(x, y) + F_2(x, y), \quad (8)$$

где f - известная непрерывная функция от y .

Предложение 1 Если существует функция $G(x)$ такая, что уравнения (6) и (7) имеют общее решение, то это решение также удовлетворяет уравнению (5).

Доказательство. Предложение 1 доказывается просто, так как, объединяя (6) и (7), и учитывая (8), мы получаем (5).

Подставляя теперь x вместо y и $y(x)$ как $p(y)$, мы переписываем (6) и (7) в соответствии с (3) (или (2), если $n = 1$ и $g_2 = 0$), как:

$$p'_y = [G(y) - g_0]p^3 - g_1p^2 - g_2p,$$

$$p'_y = \frac{G(y)}{2}p^3,$$

где

$$F_1(y, p) = g_0p^3 + g_1p^2 + g_2p, \quad F_2(y, p) = 0, \quad f(p) = p^3.$$

Тогда, используя Предложение 1, мы доказываем следующую теорему.

Теорема 2. Нелинейное автономное уравнение (1) имеет следующее точное однопараметрическое семейство решений:

$$\int \frac{dy}{\sqrt{B - \int G(y) dy}} = \pm x + A.$$

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Решения в замкнутой форме уравнения Абеля первого рода

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В работе получены как неявное решение, так и связанное достаточное условие для определенных дифференциальных уравнений класса Абеля первого рода, посредством подходящих преобразований.

Рассмотрим уравнение Абеля первого рода:

$$y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x), \quad y = y(x), \quad (1)$$

Если известно частное решение y_0 , то замена $y = y_0 + 1/u(y)$ приводит к :

$$uu'_x = -(3f_3y_0^2 + 2f_2y_0 + f_1)u^2 - (3f_3y_0 + f_2)u - f_3. \quad (2)$$

Далее, с помощью подстановки

$$u(x) = w(x)/E, \quad E = - \int (3f_3y_0^2 + 2f_2y_0 + f_1) dx,$$

уравнение (2) сводится к:

$$ww'_x = F_1(x)w = F_0(x),$$

где F_0, F_1 - конкретные выражения для функций f_1, f_2, f_3, y_0 и E . В этой работе рассматривается случай, когда "свободный" член f_0 в уравнении (1) равен нулю. В этом случае, с помощью преобразования

$$y(x) = t(x)q[z(x)], \quad (1a)$$

$$z(x) = \int f^2(x) dx, \quad t(x) = \exp \left(\int f_1(x) dx \right), \quad (1b)$$

предложенного Камке [2, Ch. A, уравнение 4.10.4d], уравнение (1) (при $f_0 = 0$) приводится к виду

$$q'_z = \frac{3s}{f^2}q^3 + q^2, \quad z = x(z). \quad (3)$$

Дополнительно рассмотрим "произвольное" уравнение [1, 1.4.1.47 (при $c = -a/b$)]:

$$\omega'_x = [\omega^2 - (f + g)\omega + fg] \left(\omega - \frac{g - cf}{1 - c} \right) h + \frac{f'}{f - g} \omega + \frac{f'g' - gf'}{f - g}, \quad (4)$$

где c - параметр, а $f = f(x)$, $g = g(x)$, $h = h(x)$ - произвольные функции от x . Это последнее уравнение имеет общее решение в замкнутой форме, а именно:

$$|\omega - f|^c |\omega - g|^{1-c} = K \exp \left[\frac{c}{1-c} \int (f - g)^2 h dx \right], \quad (5)$$

где K - постоянная интегрирования. Затем, определяя

$$g(x) = cf(x), \quad c \neq \pm 1$$

уравнение (4) приводится к виду

$$\omega'_x = h\omega^3 - (1+c)fh\omega^2 + \left(cf^2h + \frac{f'}{f}\right)\omega, \quad (6)$$

где могут быть применены преобразования Камке (1a) и (1b). Более точно, положим

$$\omega(x) = s(x)p[r(x)], \\ r(x) = -(1+c) \int fhs \, dx, \quad s(x) = f \exp \left(c \int f^2 h \, dx \right),$$

тогда уравнение (6) сводится к виду

$$p'_r = -\frac{1}{1+c} \frac{s^3}{f} p^3 + p^2, \quad x = x(r). \quad (7)$$

Утверждение. Уравнения (3) и (7) идентичны тогда и только тогда, когда выполняются следующие соотношения:

$$\left(\frac{f_3}{f_2}\right)_x = \frac{c}{(1+c)^2 f_2} f_2 - \frac{f_3 f_1}{f_2}. \quad (8)$$

$$f^2 h = \frac{1}{(1+c)^2} \frac{f_2^2}{f_3} \quad (9)$$

На самом деле, как следует из доказательства следующей теоремы, именно уравнение (8) представляет собой достаточное условие для получения аналитического (в замкнутом виде) решения уравнения Абеля (1) при $f_0 = 0$, :

Теорема Если выполняются равенства (8) и (9), то уравнение Абеля (1) при $f_0 = 0$ и $f_2 \neq 0$ имеет общее неявное решение в виде

$$|(1+c)v(x)y+1|^c \cdot |(1+c)v(x)y+c|^{-1} - K \cdot |(1+c)v(x)y|^{c-1} \exp \left[\frac{c(1-c)}{(1+c)^2} \int \frac{f_2}{v} dx \right] = 0,$$

где $v(x) = \frac{f_3(x)}{f_2(x)}.$

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TO'RTINCHI TARTIBLI DIFFERENSIAL TENGLAMA UCHUN TO'G'RI TO'RTBURCHAK SOHADA CHEGARAVIY MASALA

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Ma'lumki, aralash turdagi tenglamalarni o'rganish va ularni tahlil qilish usullarini yaratishda juda ko'p olimlar, shu jumladan T.D.Jo'rayev, M.S.Salohiddinov va ularning shogirdlari katta hissa qo'shishgan. Mazkur tezis ham yuqori tartibli differensial tenglamalarni o'rganishga bag'ishlangan.

Ushbu

$$\frac{\partial^2}{\partial x^2} \Delta U + c(x, y)U = f(x, y) \quad (1)$$

tenglamani $D = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ sohada qaraymiz. Bu yerda

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Laplas operatori.

Masala. D sohada (1) tenglamaning quyidagi shartlarni qanoatlantiruvchi $U(x, y)$ yechimi topilsin:

- 1) $U(x, y) \in C^4(D) \cap C^2(\overline{D})$;
- 2) $U(x, y)$ quyidagi chegaraviy shrtlarni qanoatlantirsin:

$$\begin{aligned} U(0, y) = \varphi_0(y), 0 \leq y \leq 1 & \quad U(x, 0) = \psi_0(x), 0 \leq x \leq 1 \\ U(1, y) = \varphi_1(y), 0 \leq y \leq 1 & \quad U(x, 1) = \psi_1(x), 0 \leq x \leq 1 \end{aligned} \quad (2)$$

$$U(x, -x + 1) = \psi_2(x), 0 \leq x \leq 1 \quad (3)$$

$$U_x(0, y) = \varphi_3(y), 0 \leq y \leq 1 \quad (4)$$

Bu yerda $\varphi_i(y)$, $\psi_i(x)$, $i = \overline{0, 3}$ berilgan funksiyalar bo'lib, ular yechimni ko'rsatilgan sinfdan bo'lishini ta'minlovchi quyidagi kelishuv shartlari o'rinli:

$$\varphi_0(0) = \psi_0(0)$$

$$\begin{aligned} \varphi_0(1) &= \psi_1(0) & \psi_1(0) &= \psi_2(0) \\ \varphi_1(0) &= \psi_0(1) & \psi_0(1) &= \psi_2(1) \\ \psi'_0(0) &= \varphi_3(0) & \psi'_1(0) &= \varphi_3(1) \end{aligned}$$

Teorema 1. Agar $c(x, y) \geq 0$, $(x, y) \in D$ bo'lsa, u holda masalaning bittadan ortiq yechimi mavjud emas.

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To'rtinchi tartibli tenglama uchun bir nolokal shartli masalaning yechilishi haqida

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Geofizikanning, okeanologiyaning birnecha masalalari, texnikada kriogen suyuqliqlarni xususan, stratifitsirlangan suyuqliqlarni foydalanish masalalari, birjinsli sterjenning yoki balkaning tebranishi masalalarini yechish, to'rtinchi tartibli tenglamalar uchun qo'yilgan aralash yoki chegaraviy masalalarni yechishga keltiriladi.

Faraz qilaylik, $\Omega = \{(x, t) : 0 < x < p, 0 < t < q\}$ sohada

$$Lu(x, t) = u_{ttt} + u_t - u_{xxxx} - b^2 u = 0,$$

tenglama berilgan bo'lsin, bunda $b = \text{const}$.

Masala. Ω sohada quyidagi shartlarni qanoatlantiradigan $u(x, t)$ funksiyani topish kerak:

$$u(x, t) \in C_{x,t}^{2,1}(\overline{\Omega}) \cap C_{x,t}^{4,3}(\Omega); \quad (1)$$

$$Lu(x, t) = 0, \quad (x, t) \in \Omega; \quad (2)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = u_t(x, q) + \psi(x), \quad u(x, q) = \eta(x), \quad 0 \leq x \leq p; \quad (3)$$

$$u(0, t) = u(p, t) = 0, \quad u_{xx}(0, t) = u_{xx}(p, t) = 0, \quad 0 \leq t \leq q, \quad (4)$$

bunda $\varphi(x), \psi(x), \eta(x)$ - berilgan uzluksiz funksiyalar va $\varphi^{(i)}(0) = \varphi^{(i)}(p) = 0$, $i = 0, 2$; $\psi(0) = \psi(p) = 0$, $\eta(0) = \eta(p) = 0$.

t bo'yicha uchinchi tartibli hosila qatnashgan to'rtinchi tartibli tenglamalar uchun aralash-chegaraviy masalalar [1-3] da qaralgan. Bu masalalarning yechimining mavjudligi, yagonaligi va turg'unligi ko'rsatilgan.

(1)–(4) masalaning regular yechimi deb, (1)–(4) shartlarni qanoatlantiradigan $u(x, t) \in C_{x,t}^{4,3}(\Omega) \cap C_{x,t}^{2,1}(\overline{\Omega})$ funksiyani tushunamiz.

(1)–(4) masalaning yechimining mavjudligi o'zgaruvchilarni ajratish usuli bilan ko'rsatildi. Yechim qator ko'rinishida topildi. Bu funksional qatorning va uning mos hosilalaridan iborat yechimlarning masala yechilayotgan yopiq sohada tekis yaqinlashishi ko'rsatildi. Masalaning yechimining yagonaligi xos funksiyalarning to'liqligidan kelib chiqadi. Bu masalaning yechimi uchun L_2 va $C(\Omega)$ fazolarda baholar olindi. Bu baholardan foydalanib masalaning yechimining turg'unligi ko'rsatildi.

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APPLICATION OF THE CAUCHY PROBLEM FOR FIRST ORDER LINEAR LOADED DIFFERENTIAL EQUATIONS

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The differential equation is usually called an uploaded differential equation, if a function of unknown function and its derivatives with values at one or more points are involved [1]. We learn problem of Cauchy for the first-order linear-uploaded simple differential equation in this article.

Just imagine, $P(x)$, $Q_m(x)$, $\overline{m} = \overline{1, n}$ are determined and continuous functions in $[a, b]$ ($-\infty < a < b < +\infty$), $x_1, x_2, \dots, x_{n-1}$ are given real numbers, let it be $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. In this case, if $y = y(x)$ is unknown function, that equality

$$y' + P(x)y = \sum_{m=1}^n y(x_m)Q_m(x), \quad x \in (a, b) \quad (1)$$

is called the first-order linear-uploaded simple differential equation.

The problem of Cauchy for equation (1) is explained as follows: find the solution which is satisfied $y(a) = k$ condition and determined, continuous in $[a, b]$ section, here k is a given real number.

This problem is resolved as follows. It is known, if we assume that the right side of (1) is a known function, there is only one solution which satisfies $y(a) = k$ condition and can be found in the following formula [2]:

$$y(x) = e^{-\int_a^x P(z)dz} \left[\int_a^x e^{\int_a^t P(z)dz} \cdot \sum_{m=1}^n y(x_m)Q_m(t)dt + k \right]. \quad (2)$$

Rewrite (2) slightly:

$$y(x) = \sum_{m=1}^n y(x_m) \int_a^x Q_m(t) e^{\int_x^t P(z)dz} dt + k e^{-\int_a^x P(z)dz}.$$

This formula follows from the following equations at points $x = x_j$, $j = \overline{1, n}$:

$$y(x_j) = \sum_{m=1}^n y(x_m) \int_a^{x_j} Q_m(t) e^{\int_{x_j}^t P(z)dz} dt + k e^{-\int_a^{x_j} P(z)dz}, \quad j = \overline{1, n}. \quad (3)$$

(3)- $y(x_j)$, $j = \overline{1, n}$ is a system of n linear algebraic equations with n unknowns. So that, there are 3 possible situations:

1. The main determinant of (3) $\Delta \neq 0$. In this case, there is only one solution, if we put $y(x_j)$, $j = \overline{1, n}$ in (2), we have only solution. 2. $\Delta = 0$ and determinants $\Delta_j = 0$, $j = \overline{1, n}$ are formed when the columns of Δ are replaced turn by turn $k e^{-\int_a^{x_j} P(z)dz}$, in this case, system (3) has infinite solutions. We can obtain an optional real number. We put

them in (2), we have infinite solutions of problem. 3. $\Delta = 0$ and $\Delta_s \neq 0$ for any $j = s$. In that, there is no solution in (3) system. So, the problem of Cauchy has no solution.

Now let's look at this simple problem. Find the solution that

$$y' + P(x)y = y^2(x_1)Q(x), \quad (4)$$

equation which is satisfied $y(a) = k$ condition and determined, continuous in $[a, b]$ section, here $x_1 \in (a, b)$, $P(x)$ and $Q(x)$ are given continuous functions, k is a given real number.

Obviously, the solution which satisfies the $y(a) = k$ condition of (4)

$$y(x) = y^2(x_1) \int_a^x Q(t) e^{\int_x^t P(z) dz} dt + k e^{-\int_a^x P(z) dz} \quad (5)$$

is determined by the formula. If we say $x = x_1$, we have the quadratic equation with respect to $y(x_1)$:

$$y^2(x_1) \int_a^{x_1} Q(t) e^{\int_{x_1}^t P(z) dz} dt - y(x_1) + k e^{-\int_a^{x_1} P(z) dz} = 0. \quad (6)$$

If we enter that

$$\Delta = 1 - 4k e^{-\int_a^{x_1} P(z) dz} \int_a^{x_1} Q(t) e^{\int_{x_1}^t P(z) dz} dt$$

substitution, in that based on the theory of quadratic equation, that (6) equation has 2 different solutions, one solution or no solution, it depends on $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$. If $\Delta > 0$ or $\Delta = 0$, we put the $y(x_1)$ which is found from (6) in (5), we have a solution to the problem under study.

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ЗАДАЧА КОШИ ДЛЯ НЕОДНОРОДНОГО ВЫРОЖДАЮЩЕГОСЯ УРАВНЕНИЯ КОЛЕБАНИЙ ПЛАСТИНЫ В ДВУХМЕРНОМ ПРОСТРАНСТВЕ

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Известно [1], что решение линейного неоднородного уравнения

$$L(u) = u_{yy} + \frac{2\beta}{y}u_y + \Delta^2 u + \lambda^2 u = f(x, y), n = 2, \quad (1)$$

удовлетворяющее однородным начальным условиям

$$u(x, 0) = 0, \quad u_y(x, y) = 0, \quad x \in R^2, \quad (2)$$

может быть представлено в виде (второй принцип Дюамеля)

$$u(x, y) = \int_0^y U(x, y, \tau) d\tau,$$

где $\beta, \lambda \in R$, причем $0 < 2\beta < 1$, $\Delta^2 = \Delta\Delta$ – бигармонический оператор, а Δ – многомерный оператор Лапласа, а $f(x, t), \varphi(x), \psi(x)$ – заданные функции. Здесь $U(x, t, \tau)$ – решение вспомогательной задачи для однородного уравнения

$$L(U) = 0, \quad x \in R^2, \quad y > \tau,$$

удовлетворяющее полуоднородным начальным условиям

$$U(x, \tau, \tau) = 0, \quad x \in R^2, \quad U_y(x, \tau, \tau) = f(x, \tau), \quad x \in R^2.$$

Последняя задача называется сопутствующей к исходной задаче $\{(1), (2)\}$.

Однако, если уравнение (1) по переменной y вырождается или имеет сингулярный коэффициент при младшем члене, то невозможно применить второй принцип Дюамеля. В таких случаях предлагаем использовать оператор преобразования Эрдейи-Кобера.

Предположим, что решение задачи $\{(1), (2)\}$ существует. Это решение ищем в виде

$$u(x, y) = J_\lambda^{(y)}(-1/2, \beta)v(x, y), \quad (3)$$

где $\eta = -1/2$, $\alpha = \beta$, $v(x, y)$ – неизвестная, четырежды непрерывно дифференцируемая функция, $J_\lambda(\eta, \alpha)v(x, y)$ – обобщенный оператор Эрдейи-Кобера, которая имеет вид

$$J_\lambda(\eta, \alpha)f(x) = \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x (x^2 - t^2)^{\alpha-1} \bar{J}_{\alpha-1} \left(\lambda \sqrt{x^2 - t^2} \right) t^{2\eta+1} f(t) dt, \quad (4)$$

где $\alpha, \eta \in R$, $\alpha > 0$, $\eta \geq -(1/2)$, $f(x) \in L_1(0, b)$, $b > 0$, $\Gamma(\alpha)$ - гамма-функция, $J_\nu(z)$ - функция Бесселя первого рода порядка ν . Основные свойства оператора (4) можно найти в работе [2, стр. 246].

Оператор, обратный к оператору (4), при $0 < \alpha < 1$ имеет вид

$$J_\lambda^{-1}(\eta, \alpha)f(x) = \frac{2x^{-2\eta}}{\Gamma(p-\alpha)} \left(\frac{1}{2x} \frac{d}{dx} \right)^p \int_0^x \frac{\bar{I}_{p-\alpha-1}(\lambda\sqrt{x^2-s^2})}{(x^2-s^2)^{\alpha-p+1}} s^{2(\eta+\alpha)+1} f(s) ds, \quad (5)$$

где $p = [\alpha] + 1$. Известно [2], что оператор (5), когда $\alpha > 0$, $\eta \geq -(1/2)$, $f(x) \in C^{2\ell}(0, b)$, $b > 0$, функции $x^{2\eta+1}[B_\eta^x]^k f(x)$ интегрируемы в нуле и

$$\lim_{x \rightarrow 0} x^{2\eta+1} \frac{d}{dx} [B_\eta^x]^k f(x) = 0, \quad k = \overline{0, \ell-1}.$$

действует как оператор преобразования по формуле

$$[B_{\eta+\alpha}^x + \lambda^2]^\ell J_\lambda(\eta, \alpha)f(x) = J_\lambda(\eta, \alpha)[B_\eta^x]^\ell f(x), \quad (6)$$

где $B_\eta^{(x)} = \frac{d^2}{dx^2} + \frac{2\eta+1}{x} \frac{d}{dx}$ - оператор Бесселя. В случае многих переменных, верхний индекс в операторах означает переменную, по которой действует этот оператор. Подставляя (3) в уравнение (1) и начальные условия (2), а затем используя равенство (6), получим задачу нахождения решения $v(x, \eta)$ уравнения

$$\frac{\partial^2 v}{\partial \eta^2} + \Delta^2 v = F(x, \eta), \quad x \in R^2, \quad \eta > 0, \quad (7)$$

удовлетворяющее начальным условиям (2), где

$$F(x, \eta) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial \eta} \int_0^\eta (\eta^2 - s^2)^{-\beta} \bar{I}_\beta(\lambda\sqrt{\eta^2 - s^2}) s^{2\beta} f_0(x, s) ds. \quad (8)$$

Далее, применяя второй принцип Дюамеля к задаче $\{(7), (2)\}$ и пользуясь решением сопутствующей задачи, решенный в работе [3], а затем подставляя найденное решение $v(x, \eta)$ в равенство (3), после некоторых вычислений, находим решения задачи $\{(1), (2)\}$.

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АСИМПТОТИКА РЕШЕНИЯ НАЧАЛЬНОЙ ЗАДАЧИ С ПОГРАНИЧНЫМ И ВНУТРЕННИМ СЛОЯМИ

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Исследуем сингулярно возмущенное обыкновенное дифференциальное уравнение первого порядка с начальным условием

$$\varepsilon y'(x) + (x - \alpha)^2 p(x) y(x) = f(x), \quad 0 < x \leq T, \quad y(0) = y^0 = \text{const.} \quad (1)$$

где $0 < \varepsilon \ll 1$ — малый параметр, $0 < \alpha < T$, α, T, y^0 — известные постоянные; $p(x), f(x)$ — заданные функции, $\forall x \in [0, T] : 0 < p(x)$, $p, f \in C^\infty[0, T]$; $y(x)$ — неизвестная функция.

Подобные задачи возникают в реальных прикладных задачах очень часто, особенно когда нужно описать переходные процессы, резкие изменения или пограничные эффекты. Например, в теории колебаний и резонансов (описывает медленное прохождение через резонанс в нелинейных колебательных системах. Параметр ε регулирует "скорость изменения" амплитуды. При $x = \alpha$ происходит резонансное взаимодействие, где $(x - \alpha)^2$ обращается в ноль — типичная "точка поворота"; электрические цепи (аналогичные уравнения описывают переходные процессы в RC — и RL — цепях при малой емкости или индуктивности (ε — малая постоянная времени). Быстрая фаза $y'(x)$ соответствует токам или напряжениям, изменяющимся на малых временах; оптика, акустика, квантовая механика (возникают в методе ВКБ, при анализе волновых уравнений с медленно меняющимися коэффициентами. $(x - \alpha)^2$ играет роль потенциального барьера, через который волна "туннелирует" или отражается; в химической кинетике уравнения с малым параметром описывают реакции с быстрой и медленной стадиями, где $y(x)$ — концентрация вещества, а ε — отношение временных масштабов [1]–[4].

Доказаны следующие теоремы

Теорема 1. Внешнее решение начальной задачи (1) на отрезке $x \in (\alpha + \sqrt[3]{\varepsilon}, T]$ представимо в виде

$$v(x, \varepsilon) = \frac{1}{(x - \alpha)^2} (\tilde{v}_0(x) + \frac{\varepsilon}{(x - \alpha)^3} \tilde{v}_1(x) + \dots + \frac{\varepsilon^n}{(x - \alpha)^{3n}} \tilde{v}_n(x) + \dots$$

где $\tilde{v}_0(x) = \frac{f(x)}{p(x)}$, $\tilde{v}_n(x) = (x - \alpha)^{3n} \frac{v'_{n-1}(x)}{p(x)}$, $\tilde{v}_k \in C^\infty[0, T]$, $k = 0, 1, \dots$

Теорема 2. Асимптотическое разложение решения начальной задачи (1) на отрезке $x \in [0, T]$ представимо в виде

$$y(x, \varepsilon) = \sum_{k=0}^n \varepsilon^k u_k(x) + \frac{1}{\mu^2} \sum_{k=0}^{3n+2} \mu^k w_k(t) + \sum_{k=0}^n \varepsilon^k \pi_k(\tau) + O(\varepsilon^n), \quad \varepsilon \rightarrow 0,$$

где $u_k \in C^\infty[0, T]$, $w_k \in C^\infty(R)$; $\pi_k \in C^\infty[0, \infty)$, $\lim_{t \rightarrow \pm\infty} w_k(t) = 0$, $\lim_{\tau \rightarrow \infty} \pi_k(\tau) = 0$, $\tau = x/\varepsilon$, $t = (x - \alpha)/\mu$, $\mu = \sqrt[3]{\varepsilon}$.

Здесь $u_k(x)$ — члены регулярного внешнего решения, $w_k(t)$ — члены внутреннего слоя в окрестности сингулярной точки $x = \alpha$, и $\pi_k(\tau)$ — члены классического пограничного слоя в окрестности начальной точки $x = 0$.

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GEOMETRIK GRAF QIRRALARIDA QUVISH-QOCHISH O'YINLARI

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Bizga Γ chekli geometrik graf berilgan bo'lsin. Grafning barcha qirralarining uzunligi 1 birlikga teng bo'lsin. Γ grafda ikkita o'yinni qaraylik. Bu m ta $\mathbf{P} = \{P_1, P_2, \dots, P_m\}$ nuqtalarni harakatini boshqaruvchi quvuvchi va Q nuqtani harakatini boshqaruvchi qochuvchi ishtirokidagi quvish-qochish o'yini.

Birinchi o'yinda quvuvchi va qochuvchi nuqtalar Γ grafning uchlarinini ketma-ket tanlab harakat qilishadi. Bunda o'yinchilar o'zi turgan uchdan qo'shni uchga o'tishi yoki joyida qolishi mumkin. Agar qaysidir qadamda biror quvuvchi nuqta qochuvchi nuqta bilan ustma-ust tushsa u holda o'yinda quvuvchi yutadi, agar quvuvchi nuqtalar hech qaysi qadamda qochuvchi nuqta bilan ustma-ust tushmasa u holda o'yinda qochuvchi yutadi. Bunday o'yin bilan ko'plab matematiklar shug'illanishgan va salmoqli natijalar olishgan [3-5, 7].

Γ grafdagi quvish-qochish o'yinida qochuvchini tutadigan quvuvchilarning eng kichik sonini $N^*(\Gamma)$ bilan belgilaylik. Agar $N^*(\Gamma) \leq m$ bo'lsa o'yin $P_1, P_2, \dots, P_m$ quvuvchi foydasiga, $N^*(\Gamma) > m$ bo'lsa o'yin Q qochuvchi foydasiga hal bo'ladi.

Ikkinchi o'yinda quvuvchi va qochuvchi nuqtalar Γ grafning qirralarida harakatlanadi. $P_1(t), P_2(t), \dots, P_m(t), Q(t)$ nuqtalarning trayektoriyalari, $\rho_1, \rho_2, \dots, \rho_m, \sigma$ esa ularning maksimal tezliklari bo'lsin. Barcha nuqtalarning maksimal tezliklari $\rho_1 = \rho_2 = \dots = \rho_m = \sigma = 1$ bo'lsin. Bunday o'yin haqidagi asosiy ta'rif va tushuncha [1-2, 6, 8-10] ishlarda keltirilgan va ko'plab natijalar olingan.

Agar o'yin ixtiyoriy vaziyatdan boshlanda ham chekli vaqtda biror quvuvchi nuqta qochuvchi nuqta bilan ustma-ust tushsa u holda o'yinda quvuvchi yutadi, agar o'yin biror vaziyatdan boshlanib qochuvchi nuqta cheksiz vaqt davomida hech qaysi quvuvchi nuqtalar bilan ustma-ust tushmasa u holda o'yinda qochuvchi yutadi.

Γ grafdagi quvish-qochish o'yinida qochuvchini tutadigan quvuvchilarning eng kichik sonini $N(\Gamma)$ bilan belgilaylik.

Endi Γ grafdagi ikkita o'yinning natijalarini solishtiraylik.

1-teorema. *Ixtiyoriy Γ geometrik grafda $N^*(\Gamma) = m$ bo'lsa, u holda $N(\Gamma) \geq m + 1$ o'rinli bo'ladi.*

2-teorema. *Ixtiyoriy Γ geometrik grafda $N^*(\Gamma) = m$ bo'lsa, u holda $N(\Gamma) < m$ o'rinli bo'ladi.*

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A EVASION DIFFERENTIAL GAME

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Let a controlled object P called the Pursuer, chase another object E called the Evader in the space $\mathbb{R}^n$. Denote by x a state of the Pursuer and denote by y a state of the Evader in $\mathbb{R}^n$. In the present we study the Pursuit problem when the objects move in accordance with equations

$$P : \quad \dot{x} + \lambda x = u, \quad x(0) = x_0, \quad (1)$$

$$E : \quad \dot{y} + \lambda y = v, \quad y(0) = y_0, \quad (2)$$

where $\lambda \neq 0$, $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$; x_0, y_0 are the initial positions of the objects P and E , respectively; it is assumed that $x_0 \neq y_0$; the parameter u is the velocity vector of the Pursuer and here the temporal variation of u is considered to be a Lebesgue measurable function $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$. We denote by $\mathbb{U}_{exp}^-$ the set of all measurable functions $u(\cdot)$ satisfying the geometric constraint (briefly, the G_{exp}^- -constraint)

$$|u(t)| \leq \rho e^{-kt} \quad \text{for almost every } t \geq 0, \quad (3)$$

where ρ, k are positive numbers.

Similarly, the parameter v is the velocity vector of the Evader and here the temporal variation of v is considered to be a Lebesgue measurable function $v(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$. We denote by $\mathbb{V}_{exp}^-$ the set of all measurable functions $v(\cdot)$ such that

$$|v(t)| \leq \sigma e^{-kt} \quad \text{for almost every } t \geq 0, \quad (4)$$

where σ is a positive number.

Since the problem (1)-(4) for the case $k = 0$ was studied, here we will study the case where k is the positive number.

From the physical viewpoint, the constraints (3) and (4) indicate that the maximal values of velocities of both objects will damp in duration time of the considered game.

Definition 1. The measurable function $u(\cdot) \in \mathbb{U}_{exp}^-$ is called *the admissible control function* of the Pursuer, respectively in the game (1)-(4), or in brief, the G_{exp}^- -Game.

Definition 2. For the pair of $(x_0, u(\cdot))$, $u(\cdot) \in \mathbb{U}_{exp}^-$ the solution of equation (1), that is

$$x(t) = x_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} u(s) ds \quad (5)$$

is called *a trajectory of the Pursuer* in the interval $[0, +\infty)$.

Definition 3. For the pair of $(y_0, v(\cdot))$, $v(\cdot) \in \mathbb{V}_{exp}^-$ the solution of equation (2), that is

$$y(t) = y_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} v(s) ds \quad (6)$$

is called *a trajectory of the Evader* in the interval $[0, +\infty)$.

In the G_{exp}^- -Game, Pursuer aims to catch Evader, i.e. to realize equality $x(t) = y(t)$ for some $t > 0$, and the aim of Evader is to keep relation $x(t) \neq y(t)$ for all $t, t \geq 0$.

Let $z = x - y$, $z_0 = x_0 - y_0$. Then from (1) and (2) we have the unique equation

$$\dot{z} = u - v, \quad z(0) = z_0.$$

To solve the Problem of Evasion, assume that at each current time t , the Pursuer is allowed to know the initial states x_0, y_0 , the constants λ, ρ, σ, k , the current time t .

Definition 4. We call the control function

$$v^*(t) = -\sigma e^{-kt} \xi_0, \quad t \geq 0, \quad (7)$$

where $\xi_0 = \frac{z_0}{|z_0|}$ the strategy of the Evader in the (1)-(4) Game.

Definition 5. We say that the control (7) guarantees winning for E if, for any $u(\cdot) \in \mathbb{U}_{exp}^-$, the relation $x(t) \neq y(t)$ holds for every $t \geq 0$.

Theorem. If $\sigma \geq \rho$, then $v^*(t)$ strategy for Evader is winning in the interval $[0, +\infty)$.

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4-SHO‘BA: Matematikmodellash va Dinamik sistemalar.

АЛГОРИТМЫ И ПРОГРАММЫ ДЛЯ СОБСТВЕННЫХ ЗНАЧЕНИЙ

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Собственные значения играют исключительные значения в задачах устойчивости динамических систем [1]. Рассмотрим задачу о собственных значениях матриц [1]: $A * x = \lambda * x$, $x \neq 0$, $\lambda = \lambda(A)$, где A - матрица порядка $n * n$. В этом уравнении неизвестные: n -скалярных величин λ и n собственных векторов. Эта задача однородная система линейных уравнений по x , поэтому ненулевое решение существует тогда и только тогда, когда матрица системы имеет детерминант, равного нулю.

$$\det(A - \lambda E) = (-1)^n (\lambda^n - p_1 \lambda^{n-1} - \dots - p_{n-1} \lambda - p_n) = 0,$$

$$p_1 = s_1, k = 2 \dots n : p_k = (s_k - \sum_{i=1}^k \frac{p_i s_{k-i}}{k}), s_k = \text{tr}(A^k) = a_{1,1}^k + \dots + a_{n,n}^k.$$

Это уравнение называется характеристическим уравнением (кратко ХУ).

Теорема 1. (Гамильтона-Келли). Матрица удовлетворяет своему характеристическому уравнению. Краткое доказательство:

$$|\det(A - \lambda E)_{\lambda=A}| = |\det(A - AE)| = |\det(0)| = 0.$$

Задача нахождения $\lambda = \lambda(A)$ достаточно трудоемкая. Для поиска собственных значений симметрических матриц разработаны итерационные методы вращений Якоби, Гивенса. Для произвольных матриц имеются методы Леверрье, Фаддеева, Крылова, Микеладзе, Кублановской-Френсиса QR, Данилевского, Хессенберга. В математических системах типа Mathcad и Python задачи решаются с помощью укрупненного алгоритма решения задачи. Mathcad, Python выступают в качестве исполнителя укрупненного алгоритма [3], а пользователь играет роль создателя алгоритма. Под укрупненным алгоритмом (the largest algorithm) мы понимаем последовательность математических формул, и команд математической системы, дающую решение задачи. В зарубежной литературе имеется подобный термин: «programming-in-the-large». Эти команды математической системы и являются крупной командой, например, функция вычисления степеней матрицы и их следов, решение системы линейных уравнений, вычисления собственных значений, определителя, обратной матрицы, интеграла, решение дифференциального уравнения, и т.д. Нами построены укрупненные алгоритмы 10 методов, упомянутых выше. В качестве примера приведём укрупненные алгоритмы методов Леверрье и QR и результаты вычислений собственных значений.

1) Программа классического метода. Метод состоит из двух шагов:
1-шаг. Построение характеристического уравнения:

$$D(\lambda) = |\lambda E - A| = (\lambda^n - \sum_{i=1}^n p_i \lambda^i$$

2-шаг. Определение собственных значений и векторов:

$$D(\lambda_i) = |\lambda_i E - A| = 0, \quad i = 1..n$$

```
import numpy as np
from numpy.linalg import LinAlgError
A=np. array([[1,2,3,4],[2,3,4,1],[3,4,1,2],[4,1,2,3]])
n=A.shape[0];p=np.poly(A);
r=np.roots(p) # характеристический полином и собственные значения
print(p) # вывод коэффициентов характеристического полинома
print(λ); λ =[-2.828 -2 2.828 10] # вывод собственных значений
2) Программа метода Левеерье-Фаддеева
# Method of Leverrye-Faddeev
import numpy as np
A = np.array([[1, 2, 3, 4], [2, 3, 4, 1], [3, 4, 1, 2], [4, 1, 2, 3]])
n=4; E=np. eye(n); B=copy(A); p=np. zeros(n+1); p[0]=1;p[1]=-np. trace(B)
for k in range(2, n+1):
    B=A@(B+p[k-1]*E);
    p[k]=-np.trace(B)/k
# Computing p
print("p = "p); r=np.roots(p); print ('eigenvalue=', r)
eigenvalues= [-2.828 -2 2.828 10]
3) Программа метода QR
# Method of QR
import numpy as np
A=np.array([[1,2,3,4], [2,3,4,1], [3,4,1,2], [4,1,2,3]])
S=[10,20] # iteraty
# QR decompozisiya
for k in range(40):
    Q,R=np.linalg.qr(A)
    A=np.dot(R,Q) # update of A
    if i+1 in S:
        print (f'iteration:', i+1, -R)
10: [10. 2.8270 -2.8270 2.0019] # собственные значения при числе итерации 10
20: [10. 2.8284 -2.8284 2.0000] # собственные значения при числе итерации 20
Видно сходимость итераций, при увеличении количества итераций.
```

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MATERIALNING MUSTAHKAMLANISHINI HISOBGA OLGAN VA OLMAGAN HOLDA SPLO'SH VA ICHI BO'SH VALLARNING ELASTOPLASTIK HOLATINI MATEMATIK MODELASHTIRISH

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Val detallarining deformatsion javobini aniqlashda elastoplastik modellar mexanik yuklama va issiqlik ta'sirini birgalikda hisobga oladi. Bunday modellash tirish mexanik mustahkamlikni baholash va qoldiq kuchlanishlarning paydo bo'lish mexanizmini tahlil qilish imkonini beradi. Tadqiqotlar shuni ko'rsatadiki, mustahkamlanish jarayonini hisobga olish mexanik holatni yanada realroq ifodalaydi va issiqlik ta'siri natijasida hosil bo'ladigan kuchlanishlar taqsimotiga sezilarli ta'sir ko'rsatadi [1], [2].

Matematik model quyidagi muvozanat va plastiklik tenglamalari asosida tuziladi:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} = 0, \quad \sigma_t - \sigma_r = \sqrt{3} \sigma_T(\varepsilon_p, T),$$

bu yerda σ_r bI'Y radial, σ_t bI'Y tangensial kuchlanishlar, σ_T bI'Y oqish chegarasi bo'lib, u plastik deformatsiya ε_p va harorat T ga bog'liq. Materialning mustahkamlanish xususiyati chiziqli shaklda ifodalanadi:

$$\sigma_T = \sigma_{T0} + k \varepsilon_p,$$

bu yerda k - mustahkamlanish koeffitsienti, σ_{T0} - boshlang'ich oqish chegarasi. Ushbu bog'lanish deformatsiya chuqurligini kamaytirib, plastik zona radiusini o'zgartiradi va termomexanik jarayonni stabillashtiradi [2], [3].

Teorema 1. Faraz qilaylik, materialning oqish chegarasi plastik deformatsiyaga nisbatan chiziqli mustahkamlanish qonuni bilan aniqlanadi:

$$\sigma_T = \sigma_{T0} + k \varepsilon_p.$$

U holda, val tashqi sirtida maksimal tangensial kuchlanish $\sigma_{\max}$ hosil bo'lgan holatda, plastik zona radiusi quyidagi bog'lanish bilan topiladi:

$$r_p = r_o \left[\frac{\sigma_{T0}}{\sigma_{\max} - k \varepsilon_p} \right]^{1/2},$$

bu yerda r_o - valning tashqi radiusi. Teorema natijasiga ko'ra, $k > 0$ bo'lganda, ya'ni material mustahkamlanish effekti mavjud holatda, plastik zona torayadi va deformatsiya faqat sirtga yaqin qatlamda saqlanadi. Bu mexanik mustahkamlikni oshiradi va issiqlik yuklamalariga barqarorlikni kuchaytiradi [1].

Analitik tahlillar shuni ko'rsatadiki, mustahkamlanishni hisobga olgan modellar vallarning issiqlik-mexanik yuklamalar ostidagi haqiqiy deformatsion holatini aniq ifodalaydi. Issiqlik o'tish tezligi yuqori bo'lganda radial kuchlanish gradienti keskinlashadi, ammo mustahkamlanish effekti bu notekislikni kamaytiradi. Mustahkamlanishsiz (ideal plastiklik) model esa yuqori haroratli yuklamalarda real natijadan og'adi.

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SILINDRIK VALLARDA HARORAT TA'SIRIDA KUCHLANISHLARNING TARQALISHINI MATEMATIK MODEL

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Issiqlik ta'siri ostida ishlaydigan metall vallarda haroratning notekis taqsimlanishi natijasida ichki deformatsiyalar va termik kuchlanishlar paydo bo'ladi. Bunday holatlarda kuchlanishlarning fazoviy taqsimlanishini oldindan aniqlash konstruktsiyaning mustahkamligi va bardoshlilikini baholashda muhim ahamiyatga ega.[1] Po'latdan yasalgan silindrik val uchun haroratning vaqt bo'yicha o'zgarishi issiqlik o'tkazuvchanlik tenglamasi orqali ifodalanadi [2]:

$$\frac{\partial T}{\partial t} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right),$$

bu yerda $T(r, t)$ - harorat funksiyasi, $a = \lambda/(\rho c)$ - issiqlik o'tkazuvchanlik koeffitsienti, λ - issiqlik o'tkazuvchanligi, ρ - zichlik, c - issiqlik sig'imi. Valning termik deformatsiyasi elastiklik nazariyasining asosiy bog'lanishlari orqali aniqlanadi:

$$\sigma_r = E \left(\frac{\partial u}{\partial r} - \alpha T \right), \quad \sigma_\theta = E \left(\frac{u}{r} - \alpha T \right),$$

bu yerda E - Yung moduli, α - chiziqli kengayish koeffitsienti, $u(r, t)$ - radial siljish funksiyasi.

Teorema 1. Faraz qilaylik $T(r, t)$ - radius r va vaqt t ga bog'liq silliq funksiya bo'lib, u silindrik vallarda issiqlik tarqalishini ifodalovchi issiqlik o'tkazuvchanlik tenglamasini qanoatlantirsin:

$$\frac{\partial T}{\partial t} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right),$$

bu yerda $a = \lambda/(\rho c)$ - issiqlik o'tkazuvchanlik koeffitsienti. U holda, val ichidagi radial va aylana (tangensial) kuchlanishlar orasidagi farq

$$\sigma_r - \sigma_\theta = E\alpha \left[T(r, t) - \frac{1}{r} \int_0^r T(\xi, t) d\xi \right]$$

ko'rinishda aniqlanadi.

Natijalar. Analitik yechimlar asosida quyidagi xulosalar olindi:

- Haroratning markazdan tashqi sirtga pasayishi radial kuchlanishlarning kamayishiga olib keladi;
- Aylana kuchlanishlar σ_θ tashqi sirtlarda maksimal qiymatga ega bo'lib, bu nuqtalarda termik yoriqlar paydo bo'lish ehtimoli yuqori;
- Olingan analitik natijalar sonli modelashtirish (FEM) orqali tekshirilib, farq 3 foiz dan oshmadi, bu esa modelning ishonchlilikini tasdiqlaydi.

Taklif etilgan model issiqlikka bardoshli metall vallarning termik yuklamalar ostida deformatsion holatini baholash, material tanlash va optimal geometrik shakllarni aniqlashda amaliy ahamiyatga ega.

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ОШИБКИ МНОГОМЕРНОЙ ФОРМУЛ ИНТЕРПОЛЯЦИИ

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Задача интерполяции считается одной из древнейших и основных задач вычислительной математики. Задача многомерной интерполяции в хаотической сетке ставится следующим образом. Пусть

$$|x| = \sqrt{(x, x)} = \sqrt{(x^1)^2 + \dots + (x^m)^2}$$

норма в евклидовом пространстве R^m точек $x = (x^1, \dots, x^m)$, где задана конечная область Ω и сетка хаотических точек (узлов) $\Delta_n = \{x_i, i = 0, \dots, n\}$, а также значения $y_i = f(x_i), i = 0, \dots, n$, некоторой непрерывной гладкой функции $y = f(x)$. Требуется определить интерполяционную формулу

$$I_n(x) = I_n(f, x) = \sum_{i=0}^n c_i \varphi_i(x), I_n(x_i) = f(x_i), i = 0, \dots, n.$$

До 1960 года рассматривались функции от одной переменной. Классические формулы интерполяции Ньютона, Лагранжа имеют вид:

$$N_n(x) = f(x_0) + \sum_{k=1}^n f[x_0, \dots, x_k] \omega_{k-1}(x), \omega_n(x) = (x - x_0) \dots (x - x_n),$$

$$L_n(x) = \sum_{i=0}^n f(x_i) \varphi_i(x), \varphi_i(x) = \prod_{j=0, j \neq i}^n (x_i - x_j)^{-1} (x - x_j), (x),$$

где

$$f[x_0, \dots, x_k] = (f[x_0, \dots, x_{k-2}, x_k] - f[x_0, \dots, x_{k-1}]) / (x_k - x_{k-1})$$

$$R_n(x) = f(x) - L_n(x) = \omega_n(x) f[x_0, \dots, x_n, x] = \omega_n(x) f^{(n+1)}(\xi) / (n+1)!, \xi \in \Omega.$$

В настоящей статье интерполяционные формулы Ньютона и Лагранжа и остаточный член обобщаются для многомерных областей с хаотическими узлами. Обобщенная интерполяционная формула Ньютона [1]:

$$N_n(x) = a_0 + \sum_{k=1}^n a_k \omega_{k-1}^\alpha(x), \omega_k^\alpha(x) = (x) = |x - x_0|^\alpha \dots |x - x_k|^\alpha, k \geq 1,$$

$f[x_0, \dots, x_k] = (f[x_0, \dots, x_{k-2}, x_k] - f[x_0, \dots, x_{k-1}]) / |x_k - x_{k-1}|^\alpha, k \geq 1$ — разделенные разности.

2) Обобщенная формула интерполяции Лагранжа [1]:

$$L_n(x) = \sum_{i=0}^n f(x_i) \varphi_i(x), \varphi_i(x) = \prod_{j \neq i} c_{ij} |x - x_j|^\alpha, c_{ij} = |x_i - x_j|^{-\alpha}.$$

Лемма 1. Справедливы равенства: $a_i = f[x_0, \dots, x_i], i = 0..n$.

Теорема 1. Верно многомерное многоточечное разложение функции:

$$f(x) = N_n(x) + R_n(x),$$

$$R_n(N_n, x) = f(x) - N_n(x) = \omega_n^\alpha(x) f[x_0, \dots, x_n, x].$$

$$\omega_n^\alpha(x) = \prod_{j=0}^n |x - x_j|^\alpha.$$

Следствие 1. Остаточный член $u(x) : u(x_i) = 0, i = 0..n$, имеет вид

$$u(x) = \omega_n^\alpha(x) u[x_0, \dots, x_n, x].$$

Теорема 2. Формулы $N_n(x), L_n(x)$ являются интерполяционными.

Теорема 3. Пусть, $f(x) \in C^k[\Omega], k \geq 1, \alpha \geq 1$. Тогда существуют точки $\xi_i = x_i + c_i(x - x_i)$ и единичные направления e_k такие, что справедливы равенства:

$$f[x_0, \dots, x_k] = h_1^{\alpha-1} \dots h_n^{\alpha-1} D_{e_k} \dots D_{e_1} f(\xi_k) = h_1^{\alpha-1} \dots h_n^{\alpha-1} f^{(k)}(\xi_k) e_1 \dots e_k, h_i = |x_i - x_{i-1}|,$$

Это равенство есть связь между производными Гато и Фреше [1].

Общая формула остаточных членов многомерной интерполяции

Теорема 4 Безу для функций. 1) Если $f(x_0) = 0$, то $f(x)$ делится на $x - x_0$ (в частности, $f(x) = P_n(x) = a_0 + a_1x + \dots + a_nx^n$):

$$f(x) = g_0(x)(x - x_0), g_0(x) = f'(\xi_0(x)), \xi_0(x) = x_0 + \theta_0(x - x_0), 0 < \theta_0 < 1.$$

2) Если $f(x) \in C^1[\Omega], f(x_0) = 0$, то справедливо равенство

$$f(x) = g_0(x)(x - x_0) = (f' \circ \xi_0(x))(x - x_0), f'(x) = \text{grad} f(x) = [f_{x_i}(x), i = 1..n]^T.$$

Теорема 5. (Об общем виде остатка формул интерполяции). Если $f(x_0) = f(x_1) = \dots = f(x_n) = 0, f(x) \in C^{n+1}[\Omega]$, то существует дифференциал $n + 1$ -го порядка такая, что $f(x) = (g_n(x), (x - x_0) \dots (x - x_n))$.

Следствие 2. Если $r(x_0) = r(x_1) = \dots = r(x_n) = 0, r(x) \in C^{n+1}[\Omega]$, то существует точка $\xi(x) \in \Omega$ такая, что имеем $r(x) = r^{(n+1)}(\xi(x))(x - x_0) \dots (x - x_n)$.

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REGRESSIYA TENGLAMALARINI TAQRIBIY YECHISH

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Statistik tahlil va matematik modellashirishda regressiya tenglamalari ma'lumotlar orasidagi bog'liqlikni ifodalovchi asosiy vositadir. Biroq, amaliyotda kuzatiladigan ko'plab jarayonlar chiziqli bo'lmagan xususiyatga ega bo'lganligi sababli, ularni aniq analitik shaklda yechish imkoniyati cheklangan. Shu bois, regressiya tenglamalarini taqribiy yechish usullarini ishlab chiqish muhim hisoblanadi.

Regressiya modelining umumiy ko'rinishi. Faraz qilaylik, eksperimental ma'lumotlar to'plami quyidagicha berilgan:

$$(y_i, x_{i1}, x_{i2}, \dots, x_{im}), \quad i = 1, 2, \dots, n$$

va regressiya modeli quyidagicha ifodalanadi:

$$y_i = f(x_{i1}, x_{i2}, \dots, x_{im}) + \varepsilon_i,$$

bu yerda ε_i - tasodifiy xatolikni bildiradi.

Model quyidagicha berilgan bo'lsin:

$$y_i = f(x_{i1}, x_{i2}, \dots, x_{im}; \beta_1, \beta_2, \dots, \beta_m) + \varepsilon_i$$

bu yerda β_j - noma'lum parametrlar, ε_i - tasodifiy xatolik.

Model parametrlari $\beta = (\beta_1, \dots, \beta_m)$ quyidagi maqsad funksiyasini minimallashtirish orqali aniqlanadi:

$$S(\beta) = \sum_{i=1}^n (y_i - f(x_i, \beta))^2$$

Agar $f(x_i, \beta)$ chiziqli bo'lmasa, unda $S(\beta)$ funksiyani to'g'ridan-to'g'ri differensiallash orqali nolga tenglashtirish murakkab bo'ladi. Shu sababli, iteratsion taqribiy metodlar (masalan, Gauss-Newton, Levenberg-Marquardt) qo'llaniladi.

Teorema. Agar $f(x, \beta)$ funksiyasi β parametrlar bo'yicha ikkinchi tartibli uzluksiz differensiallanuvchi bo'lsa va boshlang'ich yaqinlashuv $\beta^{(0)}$ yetarlicha yaqin tanlansa, unda Gauss-Newton iteratsion jarayoni

$$\beta^{(k+1)} = \beta^{(k)} + (J^T J)^{-1} J^T (y - f(\beta^{(k)}))$$

mahalliy konvergensiyaga ega bo'ladi.

1. Boshlang'ich taxmin $\beta^{(0)}$ tanlanadi;
2. Jakobian $J(\beta^{(k)})$ hisoblanadi;
3. Yangi qiymat $\beta^{(k+1)}$ yuqoridagi formula orqali topiladi;
4. $\|\beta^{(k+1)} - \beta^{(k)}\| < \varepsilon$ bo'lguncha jarayon takrorlanadi.

Analiy misol. Masalan, eksperimental ma'lumotlar uchun model:

$$y = \beta_1 e^{\beta_2 x} + \varepsilon$$

bo'lganda, $f(x, \beta)$ nolinar shaklga ega. Iteratsion yondashuv orqali parametrlar:

$$\beta_1 \approx 2.13, \quad \beta_2 \approx 0.47$$

qiymatlarda yaqinlashadi. Hisoblash natijasi Python (NumPy) kutubxonasi yordamida tekshirililib, xatolik darajasi 0.0035 ni tashkil etdi.

Regressiya tenglamalarini taxminiy yechish usullari, ayniqsa chiziqli bo'lmagan modellar uchun, aniq yechim topishning yagona amaliy yo'lidir. Gauss–Newton va Levenberg–Marquardt kabi iteratsion metodlar konvergensiya tezligi va hisoblash barqarorligi jihatidan samarali hisoblanadi. Ushbu yondashuv eksperimental iqtisodiy, fizik va texnik jarayonlarni modellashtirishda qo'llanilishi mumkin.

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DEPENDENCE STRUCTURE ESTIMATION VIA THE BIVARIATE MARSHALL–OLKIN LIFETIME MODEL

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Modeling dependence structures in lifetime data has become an essential task in reliability engineering, risk management, and survival analysis. Classical exponential models often assume independence between components, which may not reflect real-world systems where failures are correlated. The Bivariate Marshall–Olkin (BMO) distribution, introduced by Marshall and Olkin (1967), provides a mathematically elegant framework to model such dependencies [1].

The BMO model is a key example of a multivariate exponential distribution that can represent both joint and individual failures through a dependence-inducing parameter. Its applications extend to credit risk analysis, reliability networks, and actuarial science (see, [3,4]).

This paper focuses on estimating the parameters of the BMO lifetime model via Maximum Likelihood Estimation (MLE) and analyzing the resulting dependence structure through its parameters and joint survival functions [5].

Let (X_1, X_2) denote the lifetimes of two dependent components. The joint survival function of the BMO model is given by

$$\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = \exp[-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_{12} \max(x_1, x_2))], \quad (1)$$

where $\lambda_1, \lambda_2, \lambda_{12} > 0$ are parameters representing individual and common shock rates.

The joint probability density function (pdf) of (X_1, X_2) is

$$f(x_1, x_2) = \begin{cases} \lambda_1 \lambda_{12} e^{-(\lambda_1 + \lambda_{12})x_1 - \lambda_2 x_2}, & \text{if } x_1 < x_2, \\ \lambda_2 \lambda_{12} e^{-(\lambda_2 + \lambda_{12})x_2 - \lambda_1 x_1}, & \text{if } x_1 > x_2. \end{cases} \quad (2)$$

Given a random sample $\{(x_{1i}, x_{2i})\}_{i=1}^n$, the likelihood function is

$$L(\lambda_1, \lambda_2, \lambda_{12}) = \prod_{i=1}^n f(x_{1i}, x_{2i}). \quad (3)$$

Taking the logarithm

$$\ln L = \sum_{i=1}^n \ln f(x_{1i}, x_{2i}). \quad (4)$$

The MLEs $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_{12})$ satisfy

$$\frac{\partial \ln L}{\partial \lambda_j} = 0, \quad j = 1, 2, 12. \quad (5)$$

Closed-form solutions are not generally available, but iterative numerical methods (e.g., Newton–Raphson) provide accurate estimates.

Theorem 1. Let $\{(x_{1i}, x_{2i})\}_{i=1}^n$ be an i.i.d. sample from the Bivariate Marshall–Olkin distribution with parameters $\lambda_1, \lambda_2, \lambda_{12} > 0$. If all observations are distinct and each

component has positive support, then the log-likelihood function $\ln L(\lambda_1, \lambda_2, \lambda_{12})$ is strictly concave in $(\lambda_1, \lambda_2, \lambda_{12})$. Consequently, the Maximum Likelihood Estimator (MLE) $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_{12})$ exists and is unique.

Proof. Since $f(x_1, x_2)$ is differentiable and $\ln L$ is a sum of strictly concave exponential terms, the Hessian matrix of $\ln L$ is negative definite for $\lambda_j > 0$. Therefore, $\ln L$ attains a unique global maximum in $\mathbb{R}_+^3$, ensuring the existence and uniqueness of $\hat{\boldsymbol{\lambda}}$.

Theorem 2. Under standard regularity conditions for exponential families, the vector of MLEs $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_{12})$ is consistent and asymptotically normal

$$\sqrt{n}(\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}) \xrightarrow{d} N(0, I^{-1}(\boldsymbol{\lambda})), \quad (6)$$

where $I(\boldsymbol{\lambda})$ denotes the Fisher Information Matrix.

Proof. The likelihood belongs to the exponential family; hence, regularity conditions (continuity, differentiability, and identifiability) hold. By standard asymptotic theory, the score function has zero mean and the information equality applies, implying asymptotic normality by the Cramér-Rao theorem.

The parameter λ_{12} measures the intensity of common shocks affecting both components. If $\lambda_{12} = 0$, then X_1 and X_2 are independent. As λ_{12} increases, the dependence between X_1 and X_2 strengthens. The corresponding Kendall's τ for the BMO model is given by [2]

$$\tau = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}. \quad (7)$$

A Monte Carlo simulation for $n = 5000$ and true parameters $(\lambda_1, \lambda_2, \lambda_{12}) = (0.5, 0.7, 0.4)$ confirms that bias and MSE decrease at rate $1/n$, validating asymptotic efficiency.

The Bivariate Marshall–Olkin model captures dependence via a single interpretable parameter λ_{12} that controls the joint failure behavior. Compared to copula-based models, the BMO framework offers analytic tractability and physical interpretability. Its exponential tail makes it particularly suitable for reliability and actuarial applications. Extensions of the model to higher dimensions and hybrid copula–Marshall–Olkin structures can yield even richer dependence representations [3,5].

This study developed a Maximum Likelihood framework for estimating the dependence structure in the Bivariate Marshall–Olkin lifetime model. The theoretical results establish existence, uniqueness, and asymptotic normality of the estimators. Simulation evidence supports the statistical efficiency of the MLE method. Future work will extend the approach to higher-dimensional and copula-integrated lifetime models.

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EXISTENCE OF DOUBLE CLOSED TRAJECTORIES IN QUADRATIC DYNAMICAL SYSTEMS

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Abstract. A quadratic dynamical system exhibiting two distinct closed trajectories is investigated. The DN-tracking method is used to establish their existence and numerical stability, confirming orbit multiplicity and providing a concrete system for further nonlinear analysis.

Keywords: quadratic dynamical system, closed trajectory, error bound, DN-tracking method.

Consider the quadratic dynamical system

$$\begin{cases} \dot{z}_1 = 1 + z_1 - z_2^2, \\ \dot{z}_2 = -z_2 + z_1^2, \\ \dot{z}_3 = -z_1 - z_3 + z_3^2, \end{cases} \quad (1)$$

where $z = (z_1, z_2, z_3)$.

Computer analyses show that there exist two distinct closed trajectories with the same period, for which $z(0) \approx z(T)$, where $T \approx 4.3$ with their initial point.

A point on the closed trajectory C_1 : $(x_0, y_0, z_0) = (-0.40238, 0.95169, 0.79991)$.

And on the C_2 : $(x_0, y_0, z_0) = (1.24020, 2.34985, -0.88096)$.

The existence of these closed orbits, and consequently the following result, can be established through the use of the DN-tracking method, which serves as a numerical approach for tracing trajectories and verifying their recurrence [1-3].

Theorem. System (1) has two closed trajectories, one in the domain

$$\Pi_1 = \{(z_1, z_2, z_3) \mid -0.48 \leq z_1 \leq 2.22, 0.19 \leq z_2 \leq 2.42, 0.71 \leq z_3 \leq 1.91\}$$

and the other

$$\Pi_2 = \{(z_1, z_2, z_3) \mid -0.48 \leq z_1 \leq 2.22, 0.19 \leq z_2 \leq 2.42, -0.95 \leq z_3 \leq 0.25\}.$$

For these trajectories the following bounds hold:

$$\begin{aligned} \max_{z \in \Pi_1} |f(z)| &< 6.3, & \max_{z \in \Pi_2} |f(z)| &< 6.2, \\ \max_{z \in \Pi_1} \left\| \frac{\partial f(z)}{\partial z} \right\| &< 7.4, & \max_{z \in \Pi_2} \left\| \frac{\partial f(z)}{\partial z} \right\| &< 7.4, \\ \max_{z \in \Pi_1} \left\| \frac{\partial^2 f(z)}{\partial z^2} \right\| &< 3.5, & \max_{z \in \Pi_2} \left\| \frac{\partial^2 f(z)}{\partial z^2} \right\| &< 3.5. \end{aligned}$$

Furthermore, DN-tracking provides the error bounds

$$|z_1(t) - \zeta_{n_1}| < \varepsilon_1 = 5.1 \times 10^{-5} \quad \text{for the first orbit,}$$

$$|z_2(t) - \zeta_{n_2}| < \varepsilon_2 = 5.2 \times 10^{-5} \quad \text{for the second one.}$$

These inequalities imply the accurate tracking of both trajectories, $z_1(t)$ and $z_2(t)$, using distinct, computable, and storable vector sequences ζ_{n_1} and ζ_{n_2} , with corresponding accuracy levels of ε_1 and ε_2 .

A detailed analysis of a quadratic dynamical system reveals the existence and persistence of two stable closed trajectories. The validity of these findings is supported by numerical verification, and the structure of the proof is adapted from the existence analysis presented in [4]. This work highlights the complex dynamics possible in relatively simple nonlinear systems and provides a justified basis for future investigations of similar phenomena.

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Trajectory of the Canonical Form of a Volterra-Type Cubic Stochastic Operator

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Let $E = \{1, 2, \dots, m\}$. By the $(m - 1)$ -simplex we mean the set

$$S^{m-1} = \left\{ \mathbf{x} \in \mathbf{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}$$

Each element $\mathbf{x} \in S^{m-1}$ is a probability measure on E , and so it may be looked upon as the state of a biological (physical and so on) system of m elements.

A *cubic stochastic operator* is a mapping $W: S^{m-1} \rightarrow S^{m-1}$ of the form

$$W: x'_l = \sum_{i,j,k=1}^m p_{ijk,l} x_i x_j x_k = 1, \quad l = 1, \dots, m, \quad (1)$$

where $p_{ijk,l}$ are coefficients of heredity such that

$$p_{ijk,l} \geq 0, \quad \sum_{l=1}^m p_{ijk,l} = 1, \quad i, j, k, l = 1, \dots, m, \quad (2)$$

and the coefficients $p_{ijk,l}$ do not change for any permutation of i, j and k if the types are not related with sex.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$ the trajectory $\mathbf{x}^{(n)}, n = 0, 1, 2, \dots$, of an initial point $\mathbf{x}^{(0)}$ under the action of the CSO is defined by $\mathbf{x}^{(n+1)} = W(\mathbf{x}^{(n)})$, where $n = 0, 1, 2, \dots$.

One of the main problems in mathematical biology consists of the study of the asymptotical behaviour of the trajectories. Note that the main problem is open even in two-dimensional case.

A Volterra CSO is defined with the additional assumption

$$p_{ijk,l} = 0 \quad \text{if } l \notin \{i, j, k\}.$$

Let W be an operator defined on simplex S^{m-1} . A point $\mathbf{x} \in S^{m-1}$ is called a *periodic* point of W if there exists an n so that $W^n(\mathbf{x}) = \mathbf{x}$. The smallest positive integer n satisfying the above is called the prime period or least period of the point $\mathbf{x}$. A period-one point is called a *fixed* point of W . Denote by $\text{Fix}(W)$ the set of all fixed points of the operator W , i.e.

$$\text{Fix}(W) = \{\mathbf{x} \in S^2 : W(\mathbf{x}) = \mathbf{x}\}.$$

Let $DW(\mathbf{x}^*) = (\partial W_i / \partial x_j)(\mathbf{x}^*)$ be a Jacobian of W at the point $\mathbf{x}^*$.

Definition 1.[3] A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DW(\mathbf{x}^*)$ has no eigenvalues on the unit circle in $\mathbb{C}$.

Definition 2.[3] A hyperbolic fixed point $\mathbf{x}^*$ is called:

- i) attracting if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are in the unit disk;

- ii) repelling if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are outside the closed unit disk;
- iii) a saddle otherwise.

We consider the following the Canonical Form of a Volterra-Type Cubic Stochastic Operator $W : S^2 \rightarrow S^2$

$$W : \begin{cases} x'_1 = x_1 (1 + x_1^2 - (x_1 + x_2 + x_3)), \\ x'_2 = x_2 (1 + x_2^2 + 2x_1 - (x_2 + x_3)), \\ x'_3 = x_3 (1 + x_3^2 + 2(x_1 + x_2) - x_3 - 3x_1x_2). \end{cases} \quad (3)$$

Theorem. The dynamical system generated by operator (3) has the following assertions hold

1. $Fix(W) = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$;
2. e_1 is a repelling; e_2 is a saddle; e_3 is an attracting fixed points;
3. $\lim_{n \rightarrow \infty} W^n(\lambda) = e_3$ for any $\lambda = (x, y, z) \in S^2 \setminus Fix(W)$.

Adabiyotlar

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MATHEMATICAL MODELING AND ANALYSIS OF DYNAMIC PROCESSES IN TECHNICAL SYSTEMS

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Mathematical modeling serves as a fundamental tool in the design, analysis, and optimization of technical systems. Dynamic systems, which change their states over time, represent complex objects widely applied in fields such as automation, electrical engineering, mechanics, and information technology. The primary objectives of such systems are to ensure control, stability, sensitivity, and efficiency.

In the process of mathematical modeling of a technical system, the physical laws governing the system are first identified and then expressed in mathematical form. For instance, consider a simple electrical circuit. According to Ohm's law and Kirchhoff's circuit laws, the system can be described by the following differential equation:

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t) dt = u(t),$$

where L is the inductance, R is the resistance, C is the capacitance, $i(t)$ is the current flowing through the circuit, and $u(t)$ is the applied voltage. This model enables the analysis of the system's time response, oscillatory behavior, and stability.

A similar modeling approach can be applied to mechanical systems. For example, if a body with mass m is attached to an elastic spring with stiffness k and subjected to a damping force with coefficient b , its motion, based on Newton's second law, can be modeled as follows:

$$m \frac{d^2x(t)}{dt^2} + b \frac{dx(t)}{dt} + kx(t) = F(t),$$

where $x(t)$ represents displacement and $F(t)$ is the external force acting on the system. This model allows determining the system's natural frequency, damping coefficient, and forced vibration behavior.

In practice, such models are analyzed using simulation software such as MATLAB/Simulink, Scilab, or Python. Based on simulation results, system operating conditions can be optimized, control algorithms can be developed, and safety levels can be enhanced [1].

The mathematical modeling of dynamic systems provides the following key advantages in engineering applications:

- Predicting system behavior and detecting potential faults at early stages;
- Enhancing energy efficiency through optimal parameter selection;
- Establishing a rigorous mathematical foundation for the development of control algorithms;
- Reducing experimental requirements, thereby saving time and costs.

Theorem. Consider a linear time-invariant dynamic system described by the differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t),$$

where $y(t)$ is the system output, $f(t)$ is the external input, and a_i are constant parameters.

If the roots λ_i of the characteristic polynomial

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

satisfy the condition

$$\operatorname{Re}(\lambda_i) < 0, \quad i = 1, 2, \dots, n,$$

then the system is asymptotically stable, i.e.,

$$\lim_{t \rightarrow \infty} y(t) = y_p(t),$$

where $y_p(t)$ is the particular (forced) solution of the system.

In conclusion, mathematical modeling enables a deep understanding of the operating principles of technical systems, facilitates their optimization, and improves their efficiency. Consequently, it holds significant scientific and practical importance in the advancement of modern technologies [2].

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$W_2^{(5,4)}(0,1)$ FAZOSIDA QURILGAN SARD MA'NOSIDAGI OPTIMAL KVADRATUR FORMULA NORMASI BAHOSI

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Quyidagi ko'rinishdagi kvadratur formulani qaraymiz

$$\int_0^1 \varphi(x) dx \cong \sum_{\beta=0}^N (C_0[\beta]\varphi(h\beta) + C_1[\beta]\varphi'(h\beta) + C_3[\beta]\varphi'''(h\beta) + C_4[\beta]\varphi^{(4)}(h\beta)), \quad (1)$$

bu yerda

$$\begin{aligned} C_0[0] &= \frac{h}{2}, & C_1[0] &= \frac{h^2}{12}, & C_3[0] &= -\frac{h^4}{720}, \\ C_0[\beta] &= h, \beta = \overline{1, N-1}, & C_1[\beta] &= 0, \beta = \overline{1, N-1}, & C_3[\beta] &= 0, \beta = \overline{1, N-1}, \\ C_0[N] &= \frac{h}{2}, & C_1[N] &= -\frac{h^2}{12}, & C_3[N] &= \frac{h^4}{720}, \end{aligned}$$

$C_4[\beta]$ -(1) kvadratur formulaning no'malum koeffitsientlari, $h = \frac{1}{N}$, N - natural son. (1) kvadratur formulaning xatolik funksionali quyidagicha $\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N (C_0[\beta]\delta(x-h\beta) - C_1[\beta]\delta'(x-h\beta) - C_3[\beta]\delta^{(3)}(x-h\beta) + C_4[\beta]\delta^{(4)}(x-h\beta))$ ifodalanadi. Bu yerda $\varepsilon_{[0,1]}(x)$ - $[0,1]$ oraliqning xarakteristik funksiyasi, $\delta(x)$ -Dirakning delta-funksiyasi, $\varphi(x)$ - $W_2^{(5,4)}(0,1)$ Gilbert fazosiga tegishli funksiya. $W_2^{(5,4)}(0,1) = \{\varphi : [0,1] \rightarrow \mathbb{R} | \varphi^{(4)} - \text{ab. uz.}, \varphi^{(5)} \in L_2(0,1)\}$ va bu fazoda norma $\|\varphi\|_{W_2^{(5,4)}} = \left(\int_0^1 (\varphi^{(5)}(x) + \varphi^{(4)}(x))^2 dx \right)^{1/2}$

ko'rinishida aniqlangan.

$W_2^{(5,4)}(0,1)$ fazosida qurilgan Sard ma'nosidagi (1) optimal kvadratur formula koeffitsientlari quyidagicha aniqlangan [1]

$$\begin{aligned} \mathring{C}_4[0] &= \frac{e^h - 1}{e^h + 1} \left(1 + \frac{h^2}{12} - \frac{h^4}{720} \right) - \frac{h}{2}, \\ \mathring{C}_4[\beta] &= \frac{e^h - 1}{e^h + 1} \left(2 + \frac{h^2}{6} - \frac{h^4}{360} \right) - h, \quad \beta = 1, 2, \dots, N-1, \\ \mathring{C}_4[N] &= \frac{e^h - 1}{e^h + 1} \left(1 + \frac{h^2}{12} - \frac{h^4}{720} \right) - \frac{h}{2}. \end{aligned}$$

Bu koeffitsientlar $\|\ell\|_{W_2^{(5,4)*}}^2$ xatolik funksionalining normasiga minimum beradi.

Teorema 1. $W_2^{(5,4)}(0,1)$ fazosida qurilgan Sard ma'nosidagi (1) optimal kvadratur formula xatolik funksionali normasi quyidagicha

$$\|\mathring{\ell}\|_{W_2^{(5,4)*}}^2 = \frac{1}{47900160} h^6 - \frac{101}{62270208000} h^{12} + O(h^{13}).$$

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ASORATSIZ FINIT OQIMLARNING UMUMIY FORMASI

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Oqim tushunchasi.

Oqim (flow) - bu fazo yoki fazoviy ko'rsatkichlarda vaqtga bog'liq holda uzluksiz harakatlanish jarayonini ifodalaydi.

Matematikada oqim differensial tenglamalar yordamida aniqlanadi.

Finit oqimlar.

Finit oqim – bu chekli vaqt ichida yoki chekli o'lchamdagi sohada aniqlanadigan oqim.

Bunday oqimlar nazariy jihatdan sodda, amaliy qo'llanishlarda qulay modellarni beradi.

Asoratsiz oqim.

Asoratsiz oqim (divergentsiz oqim) deb, fazodagi oqimning har bir nuqtasida massa yoki energiya yo'qolishi yoki hosil bo'lishi kuzatilmagan oqimga aytiladi.

Matematik ifodasi:

$$\nabla \cdot \vec{v} = 0,$$

bu yerda $\vec{v}$ – oqim tezlik vektori.

Umumiy forma.

Asoratsiz finit oqimning umumiy formasi potensial funksiyalar orqali ifodalanadi:

$$\vec{v} = \nabla \varphi \quad \text{yoki} \quad \vec{v} = \nabla \times \vec{A},$$

bu yerda φ – skalyar potensial, $\vec{A}$ – vektor potensial.

Ushbu forma oqimning ichki strukturasi va cheklangan sharoitlarda saqlanishini ko'rsatadi.

Ahamiyati.

Asoratsiz finit oqimlarning umumiy formasi gidrodinamika, aerodinamika, elektrostatika va magnit maydon nazariyalarida asosiy tushuncha hisoblanadi.

Bu forma yordamida oqimlarni modellashtirish, hisoblash va fizik jarayonlarni tushuntirish osonlashadi.

Teorema 1. (Asoratsiz finit oqimning potensial formasi)

Agar $\Omega \subset \mathbb{R}^n$ chekli sohada $\vec{v}(x)$ oqim vektori uzluksiz differensiallanuvchi bo'lsa va

$$\nabla \cdot \vec{v}(x) = 0, \quad \forall x \in \Omega,$$

u holda mavjud bo'lgan **vektor potensial** $\vec{A}(x)$ orqali oqimni quyidagicha yozish mumkin:

$$\vec{v}(x) = \nabla \times \vec{A}(x).$$

Isboti: Divergentsiz shart $\nabla \cdot \vec{v} = 0$ Helmholtz teoremasiga ko'ra vektorni rotordan hosil qilish mumkinligini bildiradi. Cheklangan sohada (Ω finit) bu potensial yagona emas, ammo mavjudligi kafolatlanadi.

Teorema 2. (Asoratsiz finit oqimning ortogonalligi)

Agar $\Omega \subset \mathbb{R}^n$ chekli sohada asoratsiz oqim $\vec{v}(x)$ mavjud bo'lsa, u holda quyidagi ortogonallik sharti bajariladi:

$$\int_{\Omega} \vec{v}(x) \cdot \nabla f(x) dx = 0$$

har qanday silliq $f \in C^1(\Omega)$ funksiyalar uchun.

Isboti: Divergentsiz shartdan foydalanamiz:

$$\int_{\Omega} \vec{v} \cdot \nabla f dx = \int_{\Omega} \nabla \cdot (f\vec{v}) dx - \int_{\Omega} f(\nabla \cdot \vec{v}) dx.$$

Ikkinchi had nol, chunki $\nabla \cdot \vec{v} = 0$. Gauss-Ostrogradskiy formulasiga ko'ra, birinchi integral chegaraviy oqimga teng bo'ladi. Agar $\vec{v}$ finit va Ω yopiq soha bo'lsa, oqim chegaradan chiqmaydi ($\vec{v} \cdot \vec{n} = 0$), natijada butun integral nolga teng.

Xulosa

- **1-teorema** oqimni **vektor potensial** orqali ifodalash mumkinligini ko'rsatadi.

- **2-teorema** asoratsiz finit oqimlarning **gradient maydonlarga ortogonal** ekanini isbotlaydi.

Bu natijalar asoratsiz finit oqimlarni matematik modellashtirishda, xususan, differensial tenglamalar va funksional analizda muhim rol o'ynaydi.

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ESTIMATE LIFE-SPAN TIME OF THE HEAT CONDUCTION PROBLEM WITH A SOURCE AND A NONLINEAR BOUNDARY CONDITION IN MULTIDIMENSIONAL SPACE.

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We study a nonlinear diffusion problem with a source specified by nonlinear boundary conditions in $Q = \{(x, t) | x \in R_+^N, \quad t > 0\}$

$$u_t = \nabla (|\nabla u^m|^{p-2} \nabla u^m) + u^\beta, \quad x \in R_+^N, \quad t > 0, \quad (1)$$

with nonlinear boundary flow and initial condition

$$-|\nabla u^m|^{p-2} \frac{\partial u^m}{\partial x_1} = u^q, \quad x_1 = 0, \quad t > 0, \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in R_+^N, \quad (3)$$

where $R_+^N = \{(x_1, x') | x' \in R^{N-1}, x_1 > 0\}$, $m > 1, p > 1 + 1/m, q > 0$ - are given numerical parameters and is a non-negative bounded and continuous function in $Q \setminus (0, 0)$. Here we mean by a solution, a function $u(x, t)$ is nonnegative and continuous in Q , satisfying (1)-(3) in the distribution sense. In particular, the problem (1)-(3) can be thought of as a model to describe heat propagation with a gradient-dependent thermal conductivity in a medium with chemical reaction and a nonlinear radiation law at the boundary (see [1]). Qualitative properties of solutions of a doubly nonlinear reaction-diffusion, selfsimilar profiles of solutions, global existence, and blow-up solutions studied in [1, 2, 3].

Let us denote

$$T^* = \sup \left\{ T > 0; \sup_{t \in [0, T)} \|u(x, t)\|_\infty < \infty \right\},$$

then T^* is called “the life span of the solution $u(x, t)$ ”. If $T^* = \infty$, then the solution $u(x, t)$ is global in time mean. On the other side, $T^* < \infty$ the solution $u(x, t)$ is called “blow-up in finite time T^* ”.

We denote the maximal existence time (life span) of a weak solution by T^* .

- **Upper Bound:** If $\beta > q$, then solutions blow up in finite time and

$$T^* \leq \frac{C}{\|u_0\|_1^{\beta-1}}.$$

- **Lower Bound:** Solutions exist at least up to time

$$T^* \geq \frac{c}{\|u_0\|_\infty^{\beta-1}}.$$

- **Self-Similar Blow-up:** For $\beta > 1$, self-similar solutions exist that blow up at T , and describe the asymptotic profile near blow-up.

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STABILITY ANALYSIS OF NONLINEAR DIFFERENCE EQUATIONS

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Discrete dynamical systems, or difference equations, permeate almost every branch of pure and applied mathematics. Every dynamical system of the form $x_{n+1} = f(x_n)$ can be represented as a difference equation, and conversely, each difference equation can be interpreted as a dynamical system. In recent years, the study of such equations has gained significant attention due to their wide range of applications in analyzing real-world models arising in biology, economics, probability theory, and genetics.

The theory of difference equations plays a fundamental role in applied analysis. They naturally emerge in numerous disciplines, including biology, ecology, physiology, physics, engineering, and economics. Consequently, the investigation of the qualitative behavior of solutions to nonlinear difference equations-particularly the stability of their fixed and periodic points-has become both important and intellectually stimulating[1-5].

Let us now consider the following difference equation:

$$x_{n+1} = f(x_n) = \frac{\alpha x_n(K - x_n)}{1 + \beta x_n}, \quad (1)$$

where $\alpha, K, \beta > 0$ and x_n denotes the population size (or biomass) at the n -th generation, and x_{n+1} represents the population size at the next generation. The model incorporates both logistic growth and saturation effects. Specifically, α is the intrinsic growth parameter, representing the reproduction or growth intensity of the population, K is the carrying capacity of the environment, reflecting the maximum population that the available resources can sustain; β is the saturation parameter, which accounts for density-dependent limitations such as competition for resources or functional response effects similar to Holling type-II dynamics in predator-prey interactions.

We denote

$$\mathcal{K} = [0, K].$$

Proposition 1. If the parameters satisfy

$$\alpha > 0, \quad \beta > 0, \quad 0 < K \leq \frac{4\alpha}{(\alpha - \beta)^2}, \quad (2)$$

then the function f given in (1) maps the set $\mathcal{K}$ into itself.

Theorem 1. Suppose that (2) holds. For the function defined in (1), the local dynamics around its fixed points are as follows:

- (i) If $K \leq \frac{1}{\alpha}$, then the function (1) has a unique fixed point $x_1^* = 0$. The stability of x_1^* is given by

$$x_1^* \text{ is } \begin{cases} \text{attracting,} & \text{if } K < \frac{1}{\alpha}, \\ \text{non-hyperbolic,} & \text{if } K = \frac{1}{\alpha}, \\ \text{repelling,} & \text{if } K > \frac{1}{\alpha}. \end{cases}$$

(ii) If $K > \frac{1}{\alpha}$ and either $\alpha \geq \beta$ or $\alpha < \beta < 3\alpha$, then the function (1) has two fixed points

$$x_1^* = 0, \quad x_2^* = \frac{\alpha K - 1}{\alpha + \beta}.$$

The fixed point x_2^* is always attracting.

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NAMANGAN VILOYATI IQTISODIYATINI TAXLIL ETISHDA MATEMATIK USULLARNI QO'LLANISHI

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Ishlab chiqarish samaradorligini tahlil qilish va prognozlash iqtisodiyotni boshqarishning asosiy yo'nalishlaridan biri hisoblanadi. Raqamlashtirishning rivojlanishiga ta'sir ko'rsatuvchi asosiy omillardan biri sifatida turli sohalarda matematik modellardan foydalanib tadqiqotlar olib borish turadi. Misol sifatida [1-4] keltirish mumkin.

Kobb-Douglas ishlab chiqarish funksiyasi ushbu tahlil uchun qulay va samarali vosita bo'lib, u ishlab chiqarish hajmi, ish kuchi va kapital o'rtasidagi bog'liqlikni aniqlash imkonini beradi. Namangan viloyati iqtisodiyoti ham Kobb-Douglas modeli yordamida samarali tahlil qilinishi mumkin. Ushbu ma'ruzada Namangan viloyatining asosiy iqtisodiy ko'rsatkichlari, amaliy tahlillar va statistik ma'lumotlar asosida Kobb-Douglas funksiyasining qo'llanilishi yoritiladi.

Kobb-Douglas ishlab chiqarish funksiyasi iqtisodiyotda ishlab chiqarish jarayonini o'rganish uchun ishlatiladigan matematik model hisoblanadi. Ushbu funksiya iqtisodiyotning asosiy ishlab chiqarish omillari - ish kuchi va kapital o'rtasidagi bog'liqlikni ko'rsatadi va ularning ishlab chiqarish hajmiga ta'sirini tahlil qilish imkonini beradi.

Kobb-Douglas ishlab chiqarish funksiyasi quyidagicha ifodalanadi:

$$Y = AK^{\alpha}L^{\beta}, \quad \alpha + \beta = 1,$$

bu yerda Y - ishlab chiqarish hajmi yoki yalpi hududiy mahsulot (YAHM), L - ish kuchi (band aholining soni), K - kapital (investitsiyalar yoki asosiy vositalar qiymati), A - texnologik koeffitsiyent (innovatsion rivojlanish darajasi), α - ish kuchi ulushi, β - kapital ulushi.

Namangan viloyati uchun 2010-2023-yillardagi ko'rsatkichlardan foydalanib Kobb-Douglas funksiyasi quyidagicha tuzildi:

$$Y = 7.39 K^{0.52} L^{0.48}.$$

Teorema. Agar asosiy fondlarni o'tgan yilga nisbatan 25 foiz oshirish va mehnat resurslarini 7 foiz kamaytirish rejalashtirilgan bo'lsa, ishlab chiqarish hajmini prognozlash mumkin.

Buning uchun yangi qiymatlar olinadi:

$$K = 11.5 \times 1.25 = 14.375, \quad L = 760 \times 0.93 = 706.8,$$

va hisoblaymiz:

$$Y_{\text{yangi}} = 7.39 \times 14.375^{0.52} \times 706.8^{0.48}.$$

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DNK MOLEKULARINING TOPOLOGIK XOSSALARI: CFLUGTREANU–WHITE–FULLER TEOREMASI YORDAMIDA HISOBLASH

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Annotatsiya: Maqolada topologiya fanining biologiyadagi qo'llanilishi, xususan DNK molekulalarining murakkab fazoviy tuzilishini tahlil qilishda **Crlugrreanu–White–Fuller** teoremasining ahamiyati yoritiladi. Teorema DNK supero'ramlari, rekombinatsiya jarayonlari va oqsillar bilan o'zaro ta'sir mexanizmlarini model qilishda nazariy asos sifatida xizmat qiladi.

Kalit so'zlar: Topologiya, DNK, oqsil, tugun nazariyasi, linking number, twist, writhe, supero'ram, rekombinatsiya, molekulyar biologiya, gen, spiral, fazoviy tuzilish, invarianti, matematik model, genom, biotexnologiya, struktura, algebraik topologiya, uzluksizlik.

Аннотация: В статье освещается применение топологии в биологии, в частности, анализ сложной пространственной структуры молекул ДНК с использованием теоремы **Crlugrreanu–White–Fuller**. Теорема служит теоретической основой для моделирования супервитков ДНК, процессов рекомбинации и взаимодействий с белками.

Ключевые слова:

Топология, ДНК, белок, теория узлов, linking number, twist, writhe, суперспираль, рекомбинация, молекулярная биология, ген, спираль, пространственная структура, инвариант, математическая модель, геном, биотехнология, структура, алгебраическая топология, непрерывность.

Annotation:

The article discusses the application of topology in biology, particularly the analysis of the complex spatial structure of DNA molecules using the **Crlugrreanu–White–Fuller theorem**. The theorem provides a theoretical basis for modeling DNA supercoiling, recombination processes, and protein interactions.

Keywords: Topology, DNA, protein, knot theory, linking number, twist, writhe, supercoil, recombination, molecular biology, gene, spiral, spatial structure, invariant, mathematical model, genome, biotechnology, structure, algebraic topology, continuity.

Topologiya nafaqat sof matematikaning qiziqarli bo'limi, balki biologiya, kimyo va tibbiyotda ham muhim amaliy qo'llanishlarga ega. Xususan, DNK molekulalari va oqsillarning tuzilishini tahlil qilishda topologik usullar juda muhim rol o'ynaydi. DNK spiral shaklida o'ralgan bo'lib, uning umumiy holatini matematik tarzda ifodalash uchun tugun nazariyasi va bog'lanish soni (linking number) tushunchalari qo'llaniladi. Bu borada eng muhim natijalardan biri Crlugrreanu–White–Fuller teoremasidir.

1 Crlugrreanu–White–Fuller teoremasi.

Teorema shunday deydi: DNKning ikki ipli spiral tuzilishi uchta topologik kattalik bilan tavsiflanadi: $L_k = T_w + W_r$

Bu yerda: L_k (Linking number) – ikki ipning nechta marta o'ralganligini bildiruvchi butun son; T_w (Twist) – spiralning o'z o'qi atrofida nechta burilish qilganligini ko'rsatadi;

W_r (Writhe) – spiralning umumiy fazoviy egilish yoki tugunlashish darajasi.

Isbot g'oyasi

1. DNK ikki ipli spiral bo'lib, yopiq egri chiziqlar sifatida qaraladi. 2. Tugun nazariyasiga ko'ra, bunday chiziqlarning o'zaro bog'lanishi linking number orqali aniqlanadi. 3. Ushbu bog'lanish ikki qismga ajratiladi: lokal burilish (Tw) va global egilish (Wr). 4. Crlugrreanu (1959), White (1969) va Fuller (1971) ushbu ajratishni matematik asoslagan[1].

2 Biologik qo'llanishi

Ushbu teorema biologiyada juda keng qo'llanadi:- DNK supero'ram (supercoiling) hodisasini tushuntirish;- Genetik rekombinatsiya jarayonlarini modellashtirish;- Oqsillar va DNK o'rtasidagi bog'lanishlarni o'rganish;- Biotexnologik tajribalarda DNK tuzilishini nazorat qilish.

2.1 Misol 1: DNK supero'ram (supercoiling)

Faraz qilaylik, DNK molekulasi yopiq halqa shaklida bo'lib, uning linking numberi $L_k = 20$ ga teng. DNK o'z-o'zidan burilish (Twist) qilsa, $Tw = 18$ bo'ladi. Shunda writhe (Wr) qiymatini aniqlaymiz:

$$L_k = T_w + W_r \Rightarrow 20 = 18 + W_r \Rightarrow W_r = 2$$

Tahlil: Bu degani DNK molekulasi fazoda 2 marta “supero'ram” yoki global egilish bilan o'ralgan. Bu topologik xossani biologlar gen ekspressiyasi va DNK siqilishining boshqarilishi uchun o'rganadilar[3].

3 Xulosa

Crlugrreanu–White–Fuller teoremasi DNK molekulalarining murakkab topologik xossalarni matematik jihatdan aniqlash imkonini beradi. Teorema yordamida linking number, twist va writhe kabi parametrlar aniqlanib, molekulaning fazoviy konfiguratsiyasini tushunishga yordam beradi. Shuningdek, u DNK supero'ramlari, rekombinatsiya jarayonlari va oqsillar bilan o'zaro ta'sirning topologik natijalarini modellashtirishda muhim vositadir. Natijada, teorema biologiya, molekulyar genetika va biotexnologiyada nazariy va amaliy jihatdan qo'llaniladigan aniq matematik instrument sifatida xizmat qiladi.

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Constraining a Hyperbolic Cosmological Model with Observational Data

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We investigate a spatially flat FLRW universe whose scale factor is parameterized hyperbolically,

$$a(t) = \sinh^{1/n}(t), \quad n > 0.$$

Using observational Hubble data from cosmic chronometers (CC), Type Ia supernovae (Pantheon+), and baryon acoustic oscillations (BAO), we constrain the present Hubble constant H_0 and the model parameter n via MCMC χ^2 minimization. We then analyze the deceleration parameter $q(z)$, equation of state parameters $\omega(z)$ and $\omega_{de}(z)$, energy conditions, and diagnostic tools (jerk $j(z)$, statefinder $\{r, s\}$, and $O_m(z)$). Our best fit values are

$$\begin{aligned} \text{CC:} & \quad H_0 = 67.6 \pm 1.7 \text{ km/s/Mpc}, \quad n = 1.25 \pm 0.12, \\ \text{Pantheon+:} & \quad H_0 = 69.5 \pm 1.0, \quad n = 1.441 \pm 0.035, \\ \text{BAO:} & \quad H_0 \approx 70_{-9}^{+10}, \quad n = 1.417 \pm 0.026. \end{aligned}$$

Transition redshifts z_t at $q(z_t) = 0$ are $\{0.74, 0.33, 0.36\}$ and current $q_0 \approx \{-0.38, -0.27, -0.29\}$ for CC, Pantheon+, BAO respectively. Both $\omega(z)$ and $\omega_{de}(z)$ lie in $(-1, -1/3)$, indicating quintessence-like behavior approaching -1 at $z \rightarrow 0$. NEC and DEC are satisfied while SEC is violated, consistent with late time acceleration. The jerk parameter remains positive and tends to unity in the far future; statefinder trajectories lie in the quintessence sector ($r < 1, s > 0$), approaching the Λ CDM fixed point; $O_m(z)$ has a negative slope, also indicating quintessence. Our results show the hyperbolic model is observationally viable and offers a compelling quintessence-like alternative to Λ CDM.

Observations of Type Ia supernovae, CMB, BAO and large scale structure confirm the accelerating expansion of the Universe, commonly attributed to dark energy (DE). The standard Λ CDM model while successful faces conceptual issues (coincidence and cosmological constant problems). Model independent parameterizations of the scale factor or Hubble function provide a phenomenological approach to probe DE. In this work we adopt

$$a(t) = \sinh^{1/n}(t),$$

and confront it with current data.

Hyperbolic FLRW Model We assume a spatially flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2),$$

filled with pressureless matter ($\rho_m, p_m = 0$) and DE (ρ_{de}, p_{de}). In units $8\pi G = c = 1$, Einstein's equations give

$$3H^2 = \rho_m + \rho_{de}, \tag{1}$$

$$2\dot{H} + 3H^2 = -p_{de}. \tag{2}$$

Matter conservation yields $\rho_m = \rho_{m0}(1+z)^3$. Parameterizing

$$a(t) = \sinh^{1/n}(t) \implies H(t) = \frac{\coth(t)}{n},$$

and using $1+z = 1/a$ one finds

$$t(z) = \sinh^{-1}[(1+z)^{-n}], \quad H(z) = H_0 \sqrt{(1+z)^{2n} + 1}, \quad H_0 = \sqrt{2}/n.$$

The deceleration parameter is

$$q(z) = -1 - \frac{\dot{H}}{H^2} = n - 1 - \frac{n}{(1+z)^{2n} + 1},$$

with transition redshift $q(z_t) = 0 \implies z_t = [(n-1)^{-1}]^{1/(2n)} - 1$. The DE equation of state reads

$$\omega_{de}(z) = -\frac{2\dot{H} + 3H^2}{3H^2 - \rho_m}, \quad \omega(z) = -1 + \frac{2}{3n} \left[1 - \frac{1}{(1+z)^{2n} + 1} \right].$$

Observational Constraints We employ three datasets:

- **Cosmic Chronometers (CC):** 31 measurements of $H(z)$, minimizing $\chi_{CC}^2 = \sum_i [H_{\text{obs}}(z_i) - H_{\text{th}}(z_i)]^2 / \sigma^2(z_i)$.
- **Pantheon+ (SNe Ia):** ~ 1700 supernovae with covariance matrix C , $\chi_{SN}^2 = (\mu_{\text{obs}} - \mu_{\text{th}})^T C^{-1} (\mu_{\text{obs}} - \mu_{\text{th}})$.
- **BAO:** Measurements from 6dFGS, SDSS, BOSS; $\chi_{BAO}^2 = X^T C^{-1} X$.

We run an MCMC sampler to obtain best Γ Yfit (H_0, n) for each dataset.

Results and Diagnostics Our best Γ Yfit (H_0, n) appear in the Abstract. Key implications:

- *Deceleration:* Transition $z_t \approx \{0.74, 0.33, 0.36\}$, present $q_0 \approx \{-0.38, -0.27, -0.29\}$.
- *EoS parameters:* $\omega(z), \omega_{de}(z) \in (-1, -1/3)$, approaching -1 as $z \rightarrow 0$ (quintessence Γ Ylike).
- *Energy conditions:* NEC and DEC satisfied; SEC violated at late times.
- *Jerk parameter:* $j(z) > 0$, $j \rightarrow 1$ as $z \rightarrow -1$; today $j_0 \neq 1$.
- *Statefinder:* $(r, s) = (j, \frac{r-1}{3(q-1/2)})$ lies in quintessence sector ($r < 1, s > 0$), moving toward $(1, 0)$ of Λ CDM.
- *$O_m(z)$ diagnostic:* $O_m(z) = [(H/H_0)^2 - 1] / [(1+z)^3 - 1]$ has negative slope quintessence sign.

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5-SHO‘BA:EHTIMOLLARNAZARIYASIVAMATEMATIK STATISTIKA.

Multivariate subexponentiality under Brownian motion

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Let consider an insurer who operates simultaneously d -lines of business and the surplus process can be described by the equation

$$R(k) = x\alpha + k\gamma - \sum_{n=1}^{N(k)} X^{(n)} + B(k), \quad t \geq 0,$$

for any $k \geq 0$, where x represents the initial capital, $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_d)$ represents the allocation of the initial capital to the d -lines of business, namely we demand

$\alpha_1, \alpha_2, \dots, \alpha_d > 0$ and $\sum_{n=1}^d \alpha_n = 1$, $\gamma := (\gamma_1, \gamma_2, \dots, \gamma_d)$ represents the constant premium

rates in the d -lines of business, with $\gamma_1, \gamma_2, \dots, \gamma_d \geq 0$, and $\{Y^{(n)}, n \in \mathbb{N}\}$ are the claim vectors, which represent independent and identically distributed copies of the random vector $Y = (Y_1, \dots, Y_d)$, following distribution F . So, the $Y^{(n)} = (Y_1^{(n)}, \dots, Y_d^{(n)})$ are random. Here X_k are the claim vectors, which represent independent and identically distributed copies of the random vector X following distribution F . So the X_k are random vectors on the non-negative half-axis and they can have some (but not all) zero components.

Let suppose that $E[Y] < (\infty, \dots, \infty)$, in the sense that $E[Y_m] < \infty$ for any $m = 1, \dots, d$ and further the claim vectors $\{Y^{(n)}, n \in \mathbb{N}\}$ arrivals represent a time sequence $\{T_n, n \in \mathbb{N}\}$, with $T_0 = 0$ and the inter-arrival times $\{\theta_n, n \in \mathbb{N}\}$ represent a sequence of independent, identically distributed, positive random variables, such that $\theta_1 = T_1$ and $\theta_n = T_n - T_{n-1}$ for any integer $n \geq 2$.

So we obtain that

$$N(k) = \sup\{l \geq 1 : T_l \leq k\},$$

for any $k \geq 0$, which is a renewal counting process.

Finally, we suppose that the insurer invests, and her/his investments are described by a multivariate Brownian motion

$$\{B(k) = (B_1(k), \dots, B_d(k)), k \geq 0\}$$

with

$$\{B_1(k), k \geq 0\}, \dots, \{B_d(k), k \geq 0\},$$

to be arbitrarily correlated univariate Brownian motions.

Supposition 1. The $\{Y^{(n)}, n \in \mathbb{N}\}$ and $\{B(k), k \geq 0\}$ are mutually independent and each component of $B(k)$ has non-negative mean $\mathbb{E}[B_m(k)] \geq 0$ for any $k \geq 0$.

Supposition 2. Let L be some ruin set, which is open, decreasing, such that $0 \in \partial L$, L^c is convex and for any $x > 0$, it holds $xL = L$.

Lemma 1. Let $Q \in \mathbb{R}$ be some fixed set. If $F_{(L)} \in \mathcal{S}_Q$ then it holds that $F_{(L)}^* \in \mathcal{S}_Q$.

Theorem 1. Let $Q = \alpha - L$, with L some ruin set that satisfies Supposition 2. In the risk model from (1), under Supposition 1 and for $F_{(L)} \in \mathcal{S}_Q$, it holds

$$\psi_{\alpha,L}(x) \sim H(x), \quad x \rightarrow \infty.$$

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Application of probability theory in calculating the reliability of parts and assemblies at manufacturing enterprises.

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The article shows a method for finding the time of uninterrupted operation of certain system elements through the integral distribution function, and clearly demonstrates the probability of failures, where the pattern degenerates into the Poisson law. The article also discusses methods for reserving resources over a limited time interval based on the addition theorem of probabilities and the Bernoulli formula, using mathematical logic, differential equation theory, and linear and dynamic programming.

The software used to automate process control in the machine-building industry is becoming more complex every year. This is primarily due to the fact that more and more complex functions are being assigned to these devices. At the same time, as the control devices become more complex, they are also subject to requirements that are in direct conflict with their system logic, such as high reliability, which means that they must be able to perform their functions without any failures. This is especially true for technical conditions where human intervention is completely eliminated.

Numerous methods are currently being used to improve reliability, including increasing the reliability of individual components, developing special techniques for maintaining the system in working condition, such as redundancy of components and units, duplication of entire units, and preventive maintenance. All general considerations of a theoretical and technical nature concerning the improvement of the reliability of the device and elements are naturally included in the classical theory of reliability. The main mathematical tool of research in this area is probability theory and mathematical statistics.

Observation of the lifetime of elements. Let's use the following definition: reliability is the probability that the performance of a device will remain within certain limits over a specified period of time. The first problem that has to be faced in the theory of reliability is the following: the equipment goes out of working condition, as a rule, due to the fact that some element goes out of working condition, or, as they say, due to the failure of the element. The most relevant question is how much time passes from the moment of inclusion of a new element in operation until its failure. Unfortunately, this question cannot be answered unambiguously. Numerous special tests and observations of the lifetime of elements during their operation have shown that even in a batch of elements manufactured simultaneously, there is a huge variation in the duration of their working period. A single semiconductor diode or microcontroller can operate for several tens of thousands of hours. The usual picture that one has to deal with is as follows: let $F(x)$ denote by the distribution function (integral) the duration of the uninterrupted operation of a certain element, that is, the probability that it will work no more than time X . Then the characteristic form $F(x)$ will be as shown in fig. 1.

It turns out that the first hours of operation of an element are especially dangerous. As they say, during this period, low-quality products "burn out." Then comes a long period when the probability of an element failing remains almost constant and very low. Finally, there is an aging period when the probability of an element failing increases dramatically over time.

Suppose that each failed element is immediately replaced with a new one. This raises the question of what is the probability that exactly t elements will fail over a period of time m . It turns out that if the failures of the elements are independent of each other and each element has only a negligible probability of failure compared to the probability of failure of at least one element, then the probability that the number of failures is exactly m elements over a period of t is approximately equal to,

$$P_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m!}, \quad (1)$$

where λ represents a positive parameter that does not depend on t . The physical meaning of this parameter is that it is the average number of system failures, namely the mathematical expectation per unit of time. The pattern expressed by formula (1) is known as the Poisson distribution in probability theory. A statistical analysis of extensive microelectronics data, which includes observations on thousands of identical devices, shows that the pattern (1) is almost perfectly satisfied for this type of equipment. The reciprocal of $\lambda \frac{1}{2}$ is the mean time between failures of the system, described by the mathematical expectation of the time between failures.

From formula (1) for $m = 0$ it follows that the reliability, i.e. the probability of failure-free operation, follows an exponential law

$$P_0(t) = e^{-\lambda t}. \quad (2)$$

As is known, the exponential reliability law is characterized by the constancy of the mathematical expectation of the frequency of failures or their intensity. Not all electronic equipment has reliability that follows the exponential law, but only high-quality equipment has this law.

However, what is of practical interest is not the probability of the exact number of failures over a fixed time t , but the probability that the number of failures over this time will be greater than or less than some fixed value. If we denote the number of failures by μ , then from formula (1) we find

$$P(\mu \geq c) = \sum_{m=c}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!}, \quad (3)$$

$$P(\mu \leq c-1) = \sum_{m=0}^{c-1} \frac{(\lambda t)^m e^{-\lambda t}}{m!} = 1 - P(\mu \geq c). \quad (4)$$

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МНОГОТИПНЫЙ ВЕРТЯЩИЙСЯ ПРОЦЕСС С ИММИГРАЦИЕЙ (1-СХЕМА)

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Рассмотрим следующие модели ветвящихся процессов с иммиграцией. Процессе иммиграции задаётся производящей функцией плотностей

$$g_i(s) = \sum_{k \geq 0} g_r^i s^k, \quad i = 1, 2, 3, \dots, n$$

(вероятность иммиграции к частиц i - того типа за время $t \rightarrow 0$ равна $g_\gamma^i t + o(t)$ при $k > 0$, $1 + g_0^i t + o(t)$, при $k = 0$ и $\sum_{k \geq 0} g_k^i = 0$). А процесс размножения является процессом Гальтона - Ватсона с производящей функцией

$$F^i(s) = \sum_{k \geq 0} F_k^i s^k \quad i = 1, 2, 3, \dots, n$$

(F_k^i -вероятность непосредственного превращения одной частицы i - того типа в k частиц).

Будем говорить, что начальный период жизни ветвящегося процесса, начинающегося с n частиц с иммиграций, имеет длину τ , если число частиц $\eta^i(0) = n_i$, $\eta^i(t) > 0$ для всех t , $0 \leq t < \tau$, а $\eta^i(\tau) = 0$

$$\gamma_i(t) = P\{\eta^i(t) = 0 / \eta^i(0) = 0\},$$

Теорема. Пусть процесс иммиграции с непрерывным временем, а процесс размножения с дискретным временем и $F'_i = 0$, $F_i(0) > 0$, $g_i(0) = g_0^i < 0$, тогда

$$\gamma_i(t) = c_1^i e^{\alpha^i t} \left(1 - r^i q^i g_i(q^i) \left(c_2^i + \{t\} \beta_i^{[t]} + \delta_i(t) \right) \right)$$

где $\{t\}$ - дробная часть t , $[t]$ -целая чистит c_1^i , c_2^i - некоторые константы. $\delta_i(t) = 0$ (β_i^t), при $t \rightarrow \infty$, $i = 1, \dots, r$.

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ДВУХТИПНЫЙ ВЕРТЯЩИЙСЯ ПРОЦЕСС С ИММИГРАЦИЕЙ (2-схема)

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Рассмотрим следующие модели ветвящихся процессов с иммиграцией.

Процесс иммиграции происходит в целочисленные моменты времени с производящей функцией

$$G^i(s) = \sum_{k \geq 0} G_k^i s^k, \quad i = 1, 2$$

(G_k^i - вероятность того, что иммигрировало k частиц i - того типа), а процесс размножения является ветвящимся процессом с непрерывным времени с производящей функцией плотностей

$$f^i(s) = \sum_{k \geq 0} f_k^i s^k$$

(вероятность превращения одной частицы i - того типа в k частиц за время $t \rightarrow 0$ равна $f_k^i t + o(t)$ когда $k \neq 1$, и $1 + f_1^i t + o(t)$, когда $k = 1$, причем $\sum_{k \geq 0} f_k^i = 0$).

Будем говорить, что начальный период жизни ветвящегося процесса, начинающегося с n частиц с иммиграций, имеет длину τ , если число частиц $\eta^i(0) = n_i$, $\eta^i(t) > 0$ для всех t , $0 \leq t < \tau$, а $\eta^i(\tau) = 0$

$$\gamma_i(t) = P\{\eta^i(t) = 0 / \eta^i(0) = 0\}$$

Теорема. Пусть процесс иммиграции с дискретным временем, а процесс размножения с непрерывным временем, и $f_i' > 0$, $f_i(0) < 0$, $G_i(0) = G_i^0 > 0$, тогда

$$\gamma_i(t) = P\{z_i(t) = 0 | z_i(0) = 0\} = c_1^i(\{t\}) \alpha_i^{[t]} (1 + c_2^i(t) e^{\beta_i t})$$

где $C_1^i(\{t\})$ - непрерывная справа положительная периодическая функция с периодом 1, $|c_2^i(t)| \leq c_2^i - \text{const}$.

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SUPERKRITIK TARMOQLANISH JARAYONIDA SHARTLI TIRIK QOLISH EHTIMOLINING ASIMPTOTIK EKVIVALENTLIGI

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Tarmoqlanish jarayonlari ehtimollar nazariyasining klassik modellaridan biri hisoblanadi. Bu jarayon dastlab biologik populyatsiyalarni tavsiflash uchun ishlab chiqilgan bo'lsa-da, bugungi kunda u iqtisodiyot, demografiya, genetika, epidemiya va hatto axborot texnologiyalari sohalarida ham keng qo'llanadi. Masalan, virusning tarqalish modeli, avlodlar ketma-ketligining rivojlanishi yoki investitsiya loyihalarining rivojlanish jarayoni tarmoqlanish jarayonlari orqali ifodalanishi mumkin.

Modelning mohiyati shundan iboratki, har bir individ mustaqil ravishda yangi avlod hosil qiladi. Har bir individning qancha avlod qoldirishi tasodifiy kattalik bo'lib, bu kattalik ehtimollar taqsimoti bilan aniqlanadi. Agar p_k bitta individning k ta avlod qoldirish ehtimoli bo'lsa, u holda jarayonni ifodalovchi hosil qiluvchi funksiya

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad 0 \leq s \leq 1$$

ko'rinishda yoziladi. Bu funksiya juda katta ahamiyatga ega, chunki u orqali jarayonning ko'plab xossalari aniqlash mumkin.

Jarayonning o'rtacha avlod soni

$$m = f'(1) = \sum_{k=1}^{\infty} k p_k$$

bo'lib, uning qiymati jarayonning asosiy taqdirini belgilab beradi. Uchta asosiy holat farqlanadi: agar $m < 1$ bo'lsa, jarayon *subkritik* bo'lib, populyatsiya deyarli albatta yo'qolib ketadi; agar $m = 1$ bo'lsa, jarayon *kritik* bo'ladi va yo'qolib ketish ehtimoli yana 1 ga teng bo'ladi, ammo bu yo'qolish juda uzoq vaqt davom etishi mumkin; agar $m > 1$ bo'lsa, jarayon *superkritik* bo'ladi va bu holda yo'qolib ketmaslik ehtimoli ham mavjud bo'ladi.

Superkritik holat matematik va amaliy jihatdan eng qiziqarli hisoblanadi. Bu holda populyatsiya ba'zida yo'qolib ketishi mumkin, lekin ma'lum ehtimollik bilan cheksiz davom etadi. Yo'qolib ketish ehtimoli q hosil qiluvchi funksiyaning sobit nuqtasi sifatida aniqlanadi, ya'ni $f(q) = q$ tenglamaning $0 < q < 1$ yechimi hisoblanadi. Shu bilan birga, jarayonning cheksiz davom etishi ehtimoli $1 - q$ ga teng bo'ladi.

Bizning tadqiqotimizning asosiy maqsadi shartli tirik qolish ehtimolini aniqroq o'rganishdan iborat. Ko'plab adabiyotlarda faqatgina umumiy baho beriladi, ya'ni $f_n(s)$ ketma-ket kompozitsiyalari q ga geometrik yaqinlashadi:

$$|f_n(s) - q| \leq C r^n, \quad r = f'(q).$$

Bu baho juda foydali bo'lsa-da, amaliy hisoblashlar uchun yetarli emas. Chunki ko'pincha bizga jarayonning yo'qolmasligi sharti ostida tirik qolish ehtimoli kerak bo'ladi va bunda aniqroq formulalar zarur.

Shu sababli quyidagicha ta'riflaymiz:

$$\delta_n = \frac{f_n(q) - q}{1 - q}, \quad P(Z_n > 0 \mid Z_\infty = \infty) = 1 - \delta_n.$$

Bu yerda δ_n - shartli tirik qolish ehtimolining birdan farqi, ya'ni "kamchilik" qismini bildiradi. Demak, δ_n qanchalik kichik bo'lsa, tirik qolish ehtimoli shuncha katta bo'ladi.

Teorema. Agar $m > 1$ bo'lsa va $q - f(q) = q$ tenglamaning $0 < q < 1$ ildizi bo'lsa, unda quyidagilar bajariladi:

$$\lim_{n \rightarrow \infty} \frac{\delta_{n+1}}{\delta_n} = r, \quad \delta_n = r^n + o(r^n), \quad r = f'(q).$$

Bundan tashqari, har qanday $\varepsilon > 0$ uchun yetarlicha katta n larda

$$(1 - \varepsilon)r^n \leq \delta_n \leq (1 + \varepsilon)r^n$$

bo'ladi. Bu natija shuni anglatadiki, δ_n nafaqat geometrik tartibda kamayadi, balki aniq asimptotik ekvivalentlikka ega. Shunday qilib, shartli tirik qolish ehtimoli 1 ga r^n tezlikda yaqinlashadi va bu tezlik faqatgina hosil qiluvchi funksiyaning q nuqtasidagi hosilasi orqali belgilanadi.

Isbot. O'rtacha qiymat teoremasiga binoan, istalgan $h > 0$ uchun

$$f_n(q + h) - q = h \prod_{j=0}^{n-1} f'(f_j(q + h)).$$

Superkritik holatda $f_j(q + h) \rightarrow q$ geometrik yaqinlashadi va uzluksizlik sababli $f'(f_j(q + h)) \rightarrow f'(q) = r$. Shunday qilib, yetarlicha katta j larda hosilalar r ga juda yaqin bo'ladi. Demak, ko'paytma asimptotik ravishda r^n ga teng bo'ladi. Ayniqsa, $h = 1 - q$ holida

$$\delta_n = \frac{f_n(q + h) - q}{1 - q}$$

uchun yuqoridagi baholardan darhol

$$(1 - \varepsilon)r^n \leq \delta_n \leq (1 + \varepsilon)r^n$$

kelib chiqadi; bundan nisbat chegarasi va $\delta_n \sim r^n$ xulosalari ham olinadi. Teorema isbotlandi. $\square$

Quyida aniq misollar orqali natija tekshiriladi. Birinchi misolda $p_0 = 0.2$, $p_1 = 0.3$, $p_2 = 0.5$. Hosil qiluvchi funksiya $f(s) = 0.2 + 0.3s + 0.5s^2$. O'rtacha avlod soni $m = f'(1) = 0.3 + 2 \cdot 0.5 = 1.3 > 1$, demak jarayon superkritik. Yo'qolib ketish ehtimoli $q = 0.2 + 0.3q + 0.5q^2$ tenglamadan $q = 0.4$ chiqadi. Shunda $r = f'(q) = 0.3 + 2 \cdot 0.5 \cdot 0.4 = 0.7$. Demak, $\delta_n \sim 0.7^n$. Aniqlik uchun $n = 1$ da $f(0.4) = 0.44$ bo'lib, $\delta_1 = \frac{0.44 - 0.4}{0.6} = 0.066\bar{6}$ va $P(Z_1 > 0 \mid Z_\infty = \infty) = 0.933\bar{3}$. $n = 2$ da $f_2(0.4) = 0.496$, shundan $\delta_2 = \frac{0.496 - 0.4}{0.6} = 0.16$ va $P(Z_2 > 0 \mid Z_\infty = \infty) = 0.84$.

Ikkinchi misolda $p_0 = 0.1$, $p_1 = 0.4$, $p_3 = 0.5$. Hosil qiluvchi funksiya $f(s) = 0.1 + 0.4s + 0.5s^3$. O'rtacha avlod soni $m = f'(1) = 0.4 + 1.5 = 1.9 > 1$, demak jarayon superkritik. Yo'qolib ketish ehtimoli $q = 0.1 + 0.4q + 0.5q^3$ tenglamadan sonli hisoblash orqali $q \approx 0.182$ chiqadi. Hosila $r = f'(q) = 0.4 + 1.5q^2 \approx 0.4497$. Demak, $\delta_n \sim 0.45^n$.

$n = 1$ da $f(0.182) \approx 0.223$, ya'ni $\delta_1 = \frac{0.223-0.182}{0.818} \approx 0.050$ va $P(Z_1 > 0 \mid Z_\infty = \infty) \approx 0.95$.
 $n = 2$ da $f_2(0.182) \approx 0.279$, shundan $\delta_2 = \frac{0.279-0.182}{0.818} \approx 0.120$ va $P(Z_2 > 0 \mid Z_\infty = \infty) \approx 0.881$.

Misollar shuni ko'rsatadiki, shartli tirik qolish ehtimolini amaliy hisoblash uchun murakkab iteratsiyalarni bajarishning hojati yo'q. Faqatgina yo'qolib ketish ehtimoli q va hosila $r = f'(q)$ aniqlansa, darhol natijaviy yaqinlashuv topiladi va $P(Z_n > 0 \mid Z_\infty = \infty) \approx 1 - r^n$ ko'rinishidagi qulay baho olinadi. Xulosa qilib aytganda, superkritik tarmoqlanish jarayonida shartli tirik qolish ehtimolining birdan farqi δ_n aniq asimptotik ekvivalentlikka ega ekanligi isbotlandi; bu natija nazariy jihatdan ham, amaliy hisoblashlarda ham muhim ahamiyat kasb etadi.

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(1-p)-foizli qolgan umr muddatini to'liq bo'lmagan ma'lumotlarda baholash**Xolmuradov F.M.¹, Alixanov O.O.², Tillabayev I.N.³**Namangan davlat universiteti
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So'nggi yillarda klinik sinovlar, sug'urta statistikasi va ishonchlilik nazariyasida *kuzatuvlar to'liq bo'lmashligi* (censoring) muammosi keng o'rganilmoqda. Amaliy ma'lumotlarning to'liq bo'lmashligi Y umr muddatlari bilan birga kuzatuv holati ham noaniq bo'lishiga olib keladi. Shu sababli qolgan umurni aniq baholash uchun maxsus metodlar kerak.

Ananaviy ravishda qolgan umr muddatining o'rtacha qiymati

$$\mathbb{E}(X - t \mid X > t)$$

ko'plab ishlar obyekti bo'lgan (Haines va Singpurvalla, 1974; Abdushukurov A.A., 1998). Biroq klinik va sug'urta ma'lumotlarida nisbiy o'lchov *Y foizli qolgan umr*

$$\pi(p; t) = Q(1 - p(1 - F(t))) - t,$$

metrikasi ham muhim, chunki u p foizli yashash chegarasini beradi.

Asosiy nazariy model

Umr muddatining foizli mezonlari: Tasodifiy o'zgaruvchi X Y umr davomiyligi bo'lsin, uning taqsimot funksiyasi

$$F(t) = P(X \leq t), \quad F(0) = 0, \quad F(\infty) = 1.$$

Kvantil funksiyasi

$$Q(y) = F^{-1}(y) = \inf\{t \geq 0 : F(t) \geq y\}, \quad 0 < y < 1.$$

(1-p)-foizli qolgan umr mezonlari:

$$\pi(p; t) = Q(1 - p(1 - F(t))) - t, \quad 0 < p < 1, \quad t \geq 0.$$

Senzurlanish modeli: Ma'lumotlar ikki tomonlama senzurlanish orqali olinadi. Har bir subyekt uchun ikkita kuzatuv vaqti (Y_1, Y_2) mavjud. Kuzatilgan qiymat:

$$Z = \max\{Y_1, \min(X, Y_2)\}.$$

Belgilar:

$$\delta_0 = I(Y_1 > \min(X, Y_2)), \quad \delta_1 = I(Y_1 \leq X \leq Y_2), \quad \delta_2 = I(Y_1 \leq Y_2 \leq X).$$

Demak, ma'lumotlar to'plami

$$\{(Z_i, \delta_{0i}, \delta_{1i}, \delta_{2i})\}_{i=1}^n.$$

Informativ senzurlanishning: Agar senzurlanishning va umr muddatlari mustaqil bo'lmasa (informativ senzurlanish), quyidagi parametrik modelni qo'llaymiz:

$$G_1(t) = P(Y_1 \leq t) = [1 - (1 - F(t))^{\theta+1}]^{\theta}, \quad G_2(t) = P(Y_2 \leq t) = 1 - (1 - F(t))^{\theta},$$

$\beta, \theta > 0$ ʻnomaʻlum parametrlar.

Baxo va ularning xossalari

$$\{(Z_i, \delta_{0i}, \delta_{1i}, \delta_{2i})\}_{i=1}^n.$$

Empirik taqsimot funksiyasi:

$$H_n(t) = \frac{1}{n} \sum_{i=1}^n I(Z_i \leq t),$$

$$P_n^{(0)} = \frac{1}{n} \sum_{i=1}^n \delta_{0i}, \quad P_n^{(1)} = \frac{1}{n} \sum_{i=1}^n \delta_{1i}.$$

Baxo:

$$1 - \hat{F}_n(t) = [1 - H_n(t)]^{\lambda_n \gamma_n}, \quad \lambda_n = \frac{1}{1 - P_n^{(0)}}, \quad \gamma_n = \frac{P_n^{(1)}}{1 - P_n^{(0)}}.$$

Kvantil baxo va qolgan umr bahosi: Kvantil baxo:

$$\hat{Q}_n(y) = \inf\{t \geq 0 : \hat{H}_n(t) \geq y\},$$

(1-p)-foizli qolgan umr estimator:

$$\hat{\pi}_n(p; t) = \hat{Q}_n(1 - p(1 - \hat{F}_n(t))) - t.$$

Asimptotik normal taqsimot

Teorema: Agar F uzluksiz va senzurlanish modeli shartlari bajarilsa,

$$\hat{\pi}_n(p; t) \xrightarrow{\text{a.s.}} \pi(p; t), \quad n \rightarrow \infty.$$

[Asimptotik normallik] Agar shartlar bajarilsa,

$$\sqrt{n}(\hat{\pi}_n(p; t) - \pi(p; t)) \xrightarrow{d} \mathcal{N}(0, \sigma^2(p; t)),$$

bu yerda $\sigma^2(p; t)$ ʻnomaʻlum dispersiya.

Ishonch intervallari uchun:

$$\pi(p; t) \approx \hat{\pi}_n(p; t) \pm z_{1-\alpha/2} \frac{\hat{\sigma}(p; t)}{\sqrt{n}}.$$

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EHTIMOLLAFDAN TUZILGAN QATORLARNING YAQINLASHISH MASALALARI

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Biz $\mathbb{R}$ orqali haqiqiy sonlar to'plamini, $\mathbb{N}$ bilan natural sonlar to'plamini belgilaymiz. Faraz qilaylik, quyidagi d -o'lchovli koordinatalari natural sonlardan iborat vektorlar to'plami berilgan bo'lsin:

$$Z_+^d = \{\vec{n} = (n_1, \dots, n_d), n_i \in \mathbb{N}, i = 1, \dots, d\},$$

bu yerda $d \in \mathbb{N}$. Z_+^d to'plamda qisman tartiblanganlik munosabati quyidagicha kiritamiz: $\vec{m}, \vec{n} \in Z_+^d$ uchun $\vec{m} < \vec{n}$ munosabat $m_i < n_i, i = 1, \dots, d$ ni bildiradi.

Ushbu $\vec{n} = (n_1, \dots, n_d) \in Z_+^d$ uchun

$$|\vec{n}| = \prod_{i=1}^d n_i$$

bo'lsin, agar $n_i \rightarrow \infty$ ha bir $i = 1, \dots, d$ uchun bo'lsa, $\vec{n} \rightarrow \infty$ ni bildiradi. Quyidagi munosabat o'rinli $\alpha \vec{n} = (\alpha n_1, \dots, \alpha n_d), \alpha \in \mathbb{R}$. $\log^+ x = \max(\log x, 0)$ belgilashni kiritamiz.

Kelgusida $\{X(\vec{n}); n \in Z_+^d\}$ orqali bog'liq bo'lmagan, bir xil taqsimlangan tasodifiy miqdorlar to'plamini belgilaymiz. Shunda

$$S(\vec{n}) = \sum_{k < \vec{n}} X(\vec{k}), \quad S(\vec{0}) = 0.$$

X orqali $X(\vec{n}); n \in Z_+^d$ tasodifiy miqdor bilan bir xil taqsimotga ega bo'lgan taqsimotga ega bo'lgan tasodifiy miqdorni belgilaymiz.

Ushbu yo'nalishdagi dastlabki natijalar L.E. Baum va M. Katz [1], hamda A. Gut va A. Spataru [7,8] tomonidan keltirilgan. Ular yuqoridagilarni umumlashtirib: $EX = 0$ va $E[X^2(\log(1 + |X|))^{d-1}] < \infty$, bo'lsa, u holda

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2}{(-2 \log \varepsilon)^{d-1}} \sum_n P(|S_n| \geq \varepsilon |n|) = \frac{\mathbb{E}X^2}{(d-1)!}.$$

Ushbu maqolada quyidagi teorema isbot qilinadi.

Teorema. Faraz qilaylik $EX = 0$, u holda $\mathbb{E}[|X|^r(\log(1 + |X|))^{d-1}] < \infty, r \geq 2$, $\mathbb{E}X^2 = 1$ ni olamiz va N standart normal taqsimotli tasodifiy miqdor bo'lsin. $1 \leq p < 2$ uchun

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{\frac{2p}{2-p}(r/p-1)}}{(-\log \varepsilon)^{d-1}} \sum_n |n|^{r/p-2} P(|S_n| \geq \varepsilon |n|^{1/p}) = \\ = \frac{1}{(d-1)!} \left(\frac{2p}{2-p} \right)^{d-1} p^{r-p} \mathbb{E}|N|^{\frac{\alpha p}{(\alpha-p)}(r/p-1)}. \end{aligned}$$

Natija. Agar $r = 2$ va $\alpha = 2$ bo'lsa, yuqoridagi natijalar klassik holni umumlashtiradi.

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О СХОДИМОСТИ СХЕМЫ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ НЕСТАЦИОНАРНЫХ ЗАДАЧ КОНВЕКЦИИ-ДИФФУЗИИ

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Решение сложных прикладных разработок приводит к созданию более точных численных алгоритмов или совершенствованию существующих. Она проявляется особенно при исследовании сложных нестационарных процессов, например, как решения нестационарных задач конвекции – диффузии. Существует много задач, которые приводит к решению нестационарного дифференциального уравнения конвекции – диффузии [1]

$$\frac{\partial u}{\partial t} + L_1 u + L_2 u = f(x, t), \quad (1)$$

$$(x, t) \in Q_T = \{x = (x_1, x_2, x_3) \in \Omega \subset R^3, t \in (0, T]\},$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \in (0, T], \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (3)$$

Здесь $\Omega = \{x : x = (x_1, x_2, x_3), 0 < x_m < l_m, m = \overline{1, 3}\}$, $\bar{\Omega} = \Omega \cup \partial\Omega$, $L_1 u$ – оператор конвективного переноса, $L_2 u$ – оператор диффузионного переноса. Вместо (2) можно рассматривать другие локальные или нелокальные краевые условия.

К такому классу задач относятся, например математические модели размещения источников загрязнения в водоемах и прибрежных морях, а также линеаризованная математическая модель динамики морских и океанических течений [2]. Кроме того, в виде (1) можно записывать классическую задачу для параболического уравнения.

В данной работе задача (1)-(3) сначала аппроксимируются по пространственным переменным методом конечных элементов Галеркина-Петрова, в результате чего получим задачу Коши для эволюционного уравнения первого порядка (полудискретная задача)

$$D \frac{du_h(t)}{dt} + Au_h(t) = f_h(t), \quad u_h(0) = u_{0,h}, \quad (4)$$

где полудискретная функция u_h аппроксимирует непрерывную функцию u .

Далее для аппроксимации задачу (4) применяются трехпараметрическая векторная разностная схема [3]

$$Dy_t - (\tau^2/12)A\dot{y}_t + Ay^{(0.5)} = \varphi_1, \quad \gamma D\dot{y}_t + \alpha Ay_t + \beta A\dot{y}^{(0.5)} = \varphi_2, \quad (5)$$

$$y^0 = u_{h,0}, \quad \dot{y}^0 = \dot{u}_{h,0},$$

где $\dot{u}_{h,0} = D^{-1}(f_h^0 - Au_{h,0})$, $\phi_1 = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} f(t) dt$, $\phi_2 = \frac{12}{\tau^3} \int_{t_n}^{t_{n+1}} f(t) \left(s_1 \vartheta_2^{(1)} + s_2 \vartheta_2^{(3)} \right) dt$, $s_1 = 15\gamma - 35\alpha/3$, $s_2 = 140\gamma - 350\alpha/3$, $\vartheta_2^{(1)} = t - (t_n + \tau/2)$, $\vartheta_2^{(3)} = \tau(\xi^3 - 3\xi^2/2 + \xi/2)$, $\xi = (t - t_n)/\tau$. Здесь дискретная функция y аппроксимирует u_h .

Схемы (5) подчиняются условию четвертого порядка аппроксимации по времени при $\alpha + \beta = \gamma$; $\alpha, \beta, \gamma = O(\tau^2)$.

Доказана следующая теорема.

Теорема. Пусть выполнены условия аппроксимации $\alpha + \beta = \gamma$; $\alpha, \beta, \gamma = O(\tau^2)$ и устойчивости $\alpha = \tau^2/12$, $\beta > 0$, $\gamma > 0$. Тогда решение разностных схем (5) сходятся к достаточно гладкому решению задачи (1)-(3) и справедлива оценка точности

$$\|y(t) - u(t)\|_1 + \|\dot{y}(t) - \dot{u}_h(t)\|_1 \leq M(h^3 + \tau^4).$$

Приведен алгоритм реализации схемы и на основе тестовых задач проведены численные расчеты, подтверждающие теоретические выводы.

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Геодезические отображения поверхностей положительной постоянной гауссовой кривизной

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Геодезические отображения играют важную роль в римановой геометрии и её приложениях, связывая различные многообразия посредством сохранения геодезических линий. Эти отображения естественным образом возникают в задачах математической физики, теории относительности и даже в машинном обучении, где геодезические потоки используются для оптимизации и анализа данных. Пусть M и N — гладкие многообразия размерности n с аффинными связностями.

Определение 1 [1, с. 70]. Геодезическим отображением f многообразие M на N , называется взаимно однозначное соответствие между их точками, при котором каждая геодезическая линия многообразия M переходит в геодезическую линию многообразия N .

Рассмотрим эти многообразия в общей по отображению f системе координат $x^1, x^2, \dots, x^n$. Обозначим компоненты объектов связности многообразий M и N , в соответствующих точках $M(x)$ и $\overline{M}(x)$ через $\Gamma_{ij}^h(x)$ и $\overline{\Gamma}_{ij}^h(x)$, предполагая их симметричными, и положим

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_i(x) \delta_j^h + \psi_j(x) \delta_i^h \quad (h, i, j = 1, 2, \dots, n) \quad (1)$$

где, P_{ij}^k — тензор деформации.

Замечание Когда вектор $\psi_i(x)$ тождественно равен нулю, то геодезическое отображение f называется тривиальным или аффинным.

Пример 1. Пусть поверхность $\overline{S}$ является образом поверхности S при гомотетии в трёхмерном пространстве и k — коэффициент гомотетии. Тогда метрические тензоры поверхностей связаны соотношением $\overline{g}_{ij} = k^2 \cdot g_{ij}$. Следовательно, символы Кристоффеля совпадают $\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k$ и геодезическая ℓ на S отображается на геодезическую $\overline{\ell}$ на $\overline{S}$, где канонические параметры s на ℓ и $\overline{s}$ на $\overline{\ell}$ связаны соотношением $\overline{s} = k \cdot s + \text{const}$.

Имея возможность использовать теорию геодезических отображений римановых многообразий, мы обратим внимание на специальные типы римановых пространств постоянной кривизны.

Пусть f — геодезическое отображение и F поверхность вращения, которое имеют метрику вида

$$ds^2 = a(\omega) d\omega^2 + b(\omega) dt^2, \quad (2)$$

где $a(\omega)$ и $b(\omega)$ — ненулевые функции, а $\omega \in [\omega_1, \omega_2]$, $t \in [0, 2\pi]$ координаты. Параметры ω_1 и ω_2 могут достигать до значений $\pm\infty$. Обозначим через F_1 образ поверхности вращения F при отображении f . Предположим геодезические линии $l = l(t)$, $t \in R$ из F отображаются в геодезические F_1 и метрика поверхности F_1 имеет следующий вид:

$$d\overline{s}^2 = \frac{pa(\omega)}{(1 + qb(\omega))^2} d\omega^2 + \frac{pb(\omega)}{1 + qb(\omega)} dt^2 \quad (3)$$

где, p, q - положительные параметры, $1 + qb(\omega) \neq 0$, $qb'(x) \equiv 0$ и ω — тот же параметр, что и на поверхности F и $t \in [0, 2\pi)$.

Теорема . Если поверхности вращения положительной постоянной гауссовой кривизной допускают нетривиальное геодезическое отображение, а полученные поверхности имеют постоянную положительную гауссову кривизну.

Пример 2. Рассмотрим поверхности положительной постоянной кривизной с метрикой

$$ds^2 = d\omega^2 + \cos^2 \frac{\omega}{R} dt^2$$

где, R - радиус вращения и $\omega \in [\omega_1, \omega_2]$, $t \in [0, 2\pi)$. Параметры ω_1 и ω_2 могут достигать до значений $\pm\infty$.

По формуле (3), метрика поверхностей вращения положительной постоянной гауссовой кривизной имеет следующий вид:

$$ds^2 = \frac{p}{(1 + q\cos^2 \frac{\omega}{R})^2} d\omega^2 + \frac{p\cos^2 \frac{\omega}{R}}{1 + q\cos^2 \frac{\omega}{R}} dt^2 \quad (4)$$

Согласно [3, с.553], гауссова кривизна двумерного многообразия определяется следующим образом:

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{\left(\frac{\partial \Gamma_{11}^i}{\partial t} - \frac{\partial \Gamma_{21}^i}{\partial \omega} + \sum_j \Gamma_{2j}^i \cdot \Gamma_{11}^j - \sum_j \Gamma_{1j}^i \cdot \Gamma_{21}^j \right) \cdot g_{i2}}{g_{11}g_{22} - g_{12}^2} \quad (5)$$

где, Γ_{ij}^k — символы Кристоффеля, R_{1212} компонента тензора Римана.

Для полученной метрики (4) Γ_{ij}^k — символы Кристоффеля имеет вид

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{R} \cdot \frac{q \sin 2\frac{\omega}{R}}{1 + q\cos^2 \frac{\omega}{R}}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \quad \Gamma_{22}^1 = \frac{1}{2R} \sin 2\frac{\omega}{R} \\ \Gamma_{11}^2 &= 0, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{1}{R} \cdot \frac{\sin \frac{\omega}{R}}{\cos \frac{\omega}{R} (1 + q\cos^2 \frac{\omega}{R})}, \quad \Gamma_{22}^2 = 0. \end{aligned}$$

Обозначим гауссову кривину поверхностей вращения положительной постоянной гауссовой кривизной через K_+ и используя формулу (5) имеем

$$K_+ = \frac{1}{R^2} \cdot \frac{1+q}{p}$$

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ЧИСЛЕННОЕ РЕШЕНИЕ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ОДНОЙ НЕЛИНЕЙНОЙ ЗАДАЧИ КВАЗИСТАЦИОНАРНЫХ ПРОЦЕССОВ В ПОЛУПРОВОДНИКАХ

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В области

$$\bar{Q}_T = \{(x, t) : x = (x_1, x_2, x_3) \in \bar{\Omega} = [0 \leq x_\alpha \leq l_\alpha, \alpha = 1, 2, 3], t \in [0, T]\}$$

рассмотрена следующая нелинейная начально-краевая задача:

$$\frac{\partial}{\partial t}(\Delta_3 u - u) + u^3 = 0, \quad (x, t) \in Q_T, \quad (1)$$

$$u(x, 0) = u_0(x), \quad x \in \bar{\Omega}, \quad (2)$$

$$u|_{\partial\Omega} = 0, \quad t \in (0, T], \quad (3)$$

где $\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ - трёхмерный оператор Лапласа,

$Q_T = \Omega \times (0, T]$, $\Omega = \{0 < x_\alpha < l_\alpha, \alpha = 1, 2, 3\}$.

Уравнение (1) описывает квазистационарные процессы в полупроводниках при наличии стационарного распределения источников тока свободных зарядов [1].

Условие (3) означает, что граница полупроводниковой среды является заземленной и идеально проводящей.

Построим подпространство $\overset{\circ}{H}_h$, аппроксимирующий H - гильбертово пространство со скалярным произведением (u, v) и нормой $\|\cdot\|$. Здесь $\overset{\circ}{H}_h$ - пространство дискретных функции обращающихся в нуль на граничных точках. Сначала аппроксимируем уравнение (1) методом конечных разностей по пространственным переменным, в результате получаем задачу Коши для системы нелинейных обыкновенных дифференциальных уравнений первого порядка

$$D \frac{du_h}{dt} = F(u_h), \quad u_h(0) = u_{h,0}, \quad (4)$$

где $D = \sum_{m=1}^3 \Lambda_m - E$ линейный постоянный оператор действует из $\overset{\circ}{H}_h \rightarrow \overset{\circ}{H}_h$,

$$\Lambda_m = \frac{1}{h_m^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & \cdots & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Здесь $D^* = D > 0, \forall t \geq 0, F(u_h) = -u_h^3, u_h = u_h(t) \in H_h, f(u_h) \in H_h, \Lambda_m u_h = u_{h,x_m \bar{x}_m}, m = 1, 2, 3, u_h$ - значение функции в фиксированном узле $x = (i_1 h_1, i_2 h_2, i_3 h_3)$.

Разностный оператор (матрица) D аппроксимирует дифференциальный оператор $\Delta_3 u - u$ со вторым порядком, т.е. $O(|h|^2)$, $|h| = \sqrt{h_1^2 + h_2^2 + h_3^2}$. Кроме того, отметим, что оператор D невырождена, следовательно, существует обратный оператор D^{-1} .

Далее рассмотрим вопросы повышения аппроксимации дифференциального оператора $\Delta_3 u - u$. Для этого воспользуемся разложением ряда Тейлора, т.е. $\Lambda_m u = \Lambda_m u + \frac{h^2}{12} \Lambda_m u + O(h_m^4)$, $m = \overline{1, 3}$. Следовательно, оператор $\bar{D} = \left(1 + \frac{h^2}{12}\right) \sum_{m=1}^3 \Lambda_m - E$ аппроксимирует $\Delta_3 u - u$ с четвертым порядком, т.е. $O(|h|^4)$. Тогда задача (4) имеет вид

$$\bar{D} \frac{du_h}{dt} = F(u_h), \quad u_h(0) = u_{h,0}. \quad (5)$$

Для дискретизации задачи Коши (5) используются методы Рунге-Кутты первого (схемы Эйлера), второго и четвертого порядков, а также А-устойчивые методы Бобкова [3] различных порядков. На основе результатов работы [2] доказаны теоремы об устойчивости и точности численных алгоритмов. Приведены алгоритмы реализации схемы. На основе тестовых задач проведены численные расчеты, которые подтверждают теоретические выводы.

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